Probability Limit Theorems, II, Homework 2

(1) Suppose that $d\mu(x)$ is a probability measure with

$$\int x^n d\mu(x) = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2} dx$$

for all positive $n$. Show then that $d\mu(x) = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2} dx$. Hint: Applying Tchebychev’s inequality to appropriate moments yields

$$\Pr(|X| \geq R) \leq Ce^{-R^p}$$

for some $p > 1$.

This implies that the Fourier transform (characteristic function)

$$\hat{\mu}(\xi) = E_{\mu}[e^{i\xi X}]$$

is an entire analytic function of $\xi = \xi + i\eta$. The moment conditions then imply that

$$\hat{\mu}(\xi) = e^{-\xi^2/2}.$$

Now, for any interval, $(a, b)$, express $\mu((a, b))$ as an integral involving $\hat{\mu}(\xi)$.

(2) Let $D$ be a bounded open set and $q \in \partial D$. For any $x \in D$ and $R > |x - q|$, define

$$\tau = \inf \{t \mid W(t) + x \in \partial D \text{ or } |W(t) + x - q| \geq R\}.$$

Define $h(x, R) = \Pr(W(\tau) + x \in \partial D)$, and $p(\alpha, R) = \inf_{|x - q| = \alpha} h(x, R)$, and $l(R) = \liminf_{\alpha \to 0} p(\alpha, R)$. Show that if $l(R) > 0$ then $l(R) = 1$.

(3) As in problem (2), let $E$ be the complement of $\overline{D}$, the closure of $D$. For any set $A$, $|A|$ will be the Lebesgue measure of $A$. For $q \in \partial D$, let

$$v(q) = \limsup_{R \to 0} \frac{|E \cap B_R(q)|}{|B_R(q)|}.$$

Use problem (2) to help show that if $v(q) > 0$ then $q$ is a regular point. In particular, show that if $q$ satisfies the exterior cone condition then $q$ is a regular point. Remember that if $W(t) + x \in E$, then $\tau < t$. Try to show that $\Pr(W(t) + x \in E) > 0$. 