Derivative Securities

Class 2
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Lecture outline

latest correction: none yet

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Outline

Arbitrage, pricing, risk neutral probabilities

- General abstract discrete model
- Definition of arbitrage
- The geometry
- "No arbitrage" is equivalent to "there exist risk neutral probabilities"
- Complete market -- a new instrument can be replicated
- The one period binomial model, the $\Delta$ hedge
- The multi-period binomial model, the binomial tree
- Rebalancing and dynamic replication
General abstract discrete model

- $N$ instruments, $i = 1, \ldots, N$
- $C_i =$ price today of instrument $i$
- Prices may be positive or negative
- $M$ possible states of the world “tomorrow”, $j = 1, \ldots, M$
- $V_{ij} =$ price tomorrow of instrument $i$ in state $j$
- $\Pi =$ portfolio purchased today
- $W_i =$ weight of instrument $i$ in $\Pi$
- Weights may be positive or negative
- Cost/value of $\Pi$ today is $\Pi_0 = \sum_{i=1}^{N} W_i C_i$
- Cost/value of $\Pi$ in state $j$ tomorrow is $\Pi_{T,j} = \sum_{i=1}^{N} W_i V_{ij}$

$\Pi$ is an abstract arbitrage if:

- $\Pi_0 = 0$
- $\Pi_{T,j} \geq 0$ for all $j$
- $\Pi_{T,j} > 0$ for some $j$

Axiom: the model is arbitrage free -- no such $\Pi$ exists
Geometry and linear algebra

- Cash flow vector: $\Pi_T = (\Pi_{T,1}, \Pi_{T,2}, \ldots, \Pi_{T,M}) \in \mathbb{R}^M$
- $\mathcal{P} \subseteq \mathbb{R}^M$ = the set of all cash flow vectors achievable by portfolios
  - A linear subspace -- may add portfolios, and scalar multiply
- $\mathcal{L} \subseteq \mathcal{P}$ = the set of all portfolios with cost = $\Pi_0 = 0$
  - A linear subspace of $\mathcal{P}$ -- may add zero cost portfolios, and scalar multiply
- There may be more than one set of weights that gives the same $\Pi_T$
- Lemma: If there is no arbitrage, then the cost, $\Pi_0$, is the same for any portfolio with the same output vector, $\Pi_T$.
  - Proof: otherwise, buy the cheap way (the cheaper set of weights) and sell the more expensive version (the other set of weights). That is an arbitrage.
- Thus, the cost is a linear function of $\Pi_T$
- Let $n$ be a vector normal to $\mathcal{L}$ inside $\mathcal{P}$
- $\Pi_0 = C (n \cdot \Pi_T)$
  - Two linear functions that vanish together
“No Arbitrage” and “Risk Neutral Pricing”

- \( A \) = the set of portfolios with \( \Pi_{T,j} \geq 0 \) for all outcomes \( j = 1, \ldots, M \)
- “No Arbitrage” means that \( L \) does not intersect \( A \), except at 0.
- In that case -- see figure -- n is inside \( A \).
- This means that the \( n_j \geq 0 \) for all outcomes \( j = 1, \ldots, M \).
- Define risk neutral probabilities \( P_j = C n_j \)
  - \( P_j \geq 0 \) for all \( j \), \( P_1 + P_2 + \cdots + P_M = 1 \) (through choice of \( C \))

\[ \Pi_0 = \text{Portfolio cost} \]
\[ = C \left( n \cdot \Pi_T \right) \]
\[ = C_1 \left( P_1 \Pi_{T,1} + P_2 \Pi_{T,2} + \cdots + P_M \Pi_{T,m} \right) \]
\[ = C_2 \mathbb{E}_P \left[ \Pi_T \right] \]
\[ \Pi_0 = C_2 \mathbb{E}_{RN} \left[ \Pi_T \right] \]

Price = discounted (\( C_2 < 1 \)) expected value
Complete market and replication

- A market is *complete* if $P = R^M$
- An *option* is a contract that pays $U_j$ in state $j$ at time $T$
- In a complete market, there is a portfolio, $\Pi$, with $\Pi_T = U$
- *Replication*: $\Pi_{T,j} = U_j$ for all states of the world, $j = 1, \ldots, M$
- In a complete market, any option can be replicated.
- In a complete market without arbitrage, the price of the replicating portfolio is uniquely determined by its payout structure, $U$
- If the option is traded at time 0, it is part of the market
- *Theorem*: assume that
  - The market with the option is arbitrage free
  - The market without the option is complete
- *Then*:
  - The option may be replicated
  - All replicating portfolios have the same price
  - That price must be the market price of the option
  - That price is the discounted expected payout in the risk neutral measure

$$\text{Price( option )} = C \ E_P[ \text{ option payout } ]$$
Complete market and replication, comments

• The risk neutral probabilities are determined by the complete market without the option -- they are the same for every extra option.
• If the market is complete, the risk neutral probabilities are uniquely determined by the market -- the direction of a normal to a hyperplane of dimension M-1 is unique.
• If the market is not complete, the normal direction within P is unique -- there are unique risk neutral probabilities for any option that can be replicated.
• If the option cannot be replicated, then there is a range of prices that do not lead to arbitrage.
• Real markets have market frictions that prevent arbitrarily small arbitrage transactions.
  – Transaction costs: portfolios with equivalent values at time T may have different costs at time 0.
  – Limited liquidity: the cost to buy n “shares” of asset i may not be proportional to n -- move the market.
• This material often is described differently, using linear programming.
• Keith Lewis told me it was easier to do it geometrically, as it is here.
Utility, risk neutral pricing

- Let $X$ be an investment whose value in state $j$ is $X_j$.
- Let $Q_j$ be the *real world* probability of state $j$, possibly subjective.
- The real world expected value is
  \[ M = E_{Q}[X] = X_1Q_1 + X_2Q_2 + \cdots + X_MQ_M \]
- Fundamental axiom of finance: $\text{Price}(X) \leq M$
- If variance($X$) > 0, a *risk averse* investor has $\text{value}(X) < M$
- A risk neutral investor has $\text{value}(X) = M$
- The difference $M - \text{value}(X)$ is the *risk premium* of $X$ for that investor
- The difference $M - \text{price}(X)$ is the *risk premium* of the market
- Risk premia depend on personal psychology and needs
- The market risk premium is determined by interactions between investors. It should be positive but is hard to predict quantitatively
- In this setup, it is hard to predict $\text{price}(X)$ from first principles

- Risk neutral pricing says that there are risk neutral probabilities $P \neq Q$ so that $\text{price}(X) = C E_P[X]$, if $X$ is an option payout in a complete market
- Since $X$ can be replicated, $\text{value}(X)$ is the same for every investor, and is equal to $C E_P[X]$.
- Can find prices of options without psychology.
Binary “one period” model

- The market has two instruments, *stock* and *cash* (also called *bond*)
- There are $M = 2$ states of the world “tomorrow”, called “up” and “down”
- The value of “cash” today is 1
- The value of “cash” tomorrow is $e^{rT}$, $r$ being the risk free rate
- The value of “stock” today is $S_0$
- The value of stock tomorrow is
  - $u S_0$ in state “up”
  - $d S_0$ in state “down”
  - Assume $u > d$
- This market is complete (check)
Risk neutral probabilities for the binary model

• With $M = 2$, the cost free portfolios form a one line
• $W_s = \text{weight of stock} = a$
• $W_c = \text{weight of cash} = -aS_0$ (to be cost free)
• Portfolio values at time $T$
  - $\Pi_{T,u} = aS_0( u - e^{rT} )$
  - $\Pi_{T,d} = aS_0( d - e^{rT} )$
  - Opposite sign (no arbitrage) if $d < e^{rT} < u$
• Normal: $(x,y) \Rightarrow (-y,x)$
• Normal to L: $(u - e^{rT}, d - e^{rT}) \Rightarrow ( e^{rT} - d, u - e^{rT} )$, both positive
• Normalize to get probabilities:
  - $n_u + n_d = u - d$
  - $n_u/(u - d) = p_u = (e^{rT} - d)/(u - d)$
  - $n_d/(u - d) = p_d = (u - e^{rT})/(u - d)$
  - Discount factor = $e^{-rT}$, otherwise risk free cash is an arbitrage
• If $V$ is an option that pays ($V_u, V_d$), then the price of $V$ today is

$$\text{price}(V) = e^{-rT}E_p[V_T] = e^{-rT} \left( V_u (e^{rT} - d) + V_d (u - e^{rT}) \right) / (u - d)$$
Binary model, Delta hedging

• A derivatives desk is asked to hold an option but does not want risk
• Short a *replicating portfolio*, $\Pi$, of stock and cash
• The total portfolio has zero value and zero risk.
• Make a profit from commissions.
• Replicating portfolio = $\Pi = \Delta$ Stock + C Cash,
• $\Pi_T = V_T$, both up and down
• $\Pi_0 = \Delta S_0 + C$
• $\Pi_{T,u} = \Delta u S_0 + e^{rt}C = V_u$
• $\Pi_{T,d} = \Delta d S_0 + e^{rt}C = V_d$
• Solve: $\Delta = \left( \frac{V_u - V_d}{u S_0 - d S_0} \right) = \text{(change in V) / (change in S)}$
• $\left( V - \Delta S \right)_u = \left( V - \Delta S \right)_d$
• $\Delta$ hedged portfolio value at time $T$ is not random, risk free
• Equivalent to cash, value known at time 0
Binomial multi-period model

• Times 0 = t₀, t₁, ..., tₙ = T, tₖ = kδt
• Cash increases by e^{rδt} between tₖ and tₖ+1
• S₀ = present spot price = known
• Sₖ₊₁ = uSₖ or Sₖ₊₁ = dSₖ
• S₁ = uS₀, or S₁ = dS₀, as before
• S₂ = u²S₀, or S₂ = udS₀, or S₂ = d²S₀
• ud = du -- the binomial tree is recombining (diagram)
• N+1 possible values of Sₙ = Sₜ, 2ᴺ if not recombining
• State j has j up steps and k - j down steps: Sₖⱼ = uᵢdᵏ⁻ⱼ S₀
• European style option pays Vₙⱼ at time tₙ = T in state j
• Vₖⱼ = price/value of option at state j at time k
• Vₖⱼ is determined by Vₖ₊₁,j and Vₖ₊₁,j₊₁ as before
• Work backwards:
  – Given all Vₙⱼ values, calculate all Vₙ₋₁,j values
  – Given all Vₙ₋₁,j values, calculate all Vₙ₋₂,j values
  – Eventually, reach V₀
Dynamic hedging, rebalancing in the binomial tree model

• At time $t_k$ in state $j$, there is a hedge ratio $\Delta_{kj} = \frac{(V_{k+1,j+1} - V_{k+1,j})}{S_k (u-d)}$
• This is how many shares of stock you own before you leave time $t_k$
• At time $t_{k-1}$, you probably had a different number of shares:
  - $\Delta_{k-1,j-1}$ or $\Delta_{k-1,j}$, neither one equal to $\Delta_{kj}$
• When you arrive at time $t_k$, you have to replace the old number of shares with the correct number, $\Delta_{kj}$. This is *rebalancing*.
• You pay for the new shares by spending your cash, this requires more borrowing if the cash position is negative.
• This is *dynamic hedging*, or
• *Dynamic replication*: $\Pi_T = V_T$ for any state at time $T$
• The dynamic hedging strategy produces a portfolio of stock and cash worth exactly $V_{Tj}$, if $S_{Tj}$ is the state at time $T$.
• It is *self financing*. You generate the cash you need to buy stock. You keep the proceeds from selling stock.