1. Consider a two-worker bucket brigade (BB) of the kind we are studying. Worker 1 takes 3 hours to produce a widget, worker 2 takes 1 hour. Assuming the length of the production line is 1 and time is in hours, the velocities of the workers are \( v_1 = \frac{1}{3}, v_2 = 1 \). Let \( x_2^{(1)} = \frac{1}{2} \) at the initial reset. Draw and \( x-t \) diagram of the BB including the first three resets after the start. Compute the values of \( x_2^{(k)}, k = 2, 3, 4 \). Obtain the difference equation for \( x_2^{(k)} \), solve it, and compare with your diagram. Verify that the optimal production rate of \( P = v_1 + v_2 \) widgets per hour is obtained at the balanced equilibrium.

2. For the example of problem 1, switch the order of the workers, so that \( v_1 = 1, v_2 = \frac{1}{3} \). Draw the \( x-t \) diagram for the production over the first 8 hours. Verify the sub-optimal production rate in this case.

3. Show that, in the case of problem 1 above, once the balanced equilibrium is obtained, both workers spend the same amount of time working on each widget, and determine this time. Prove that this equipartitioning of time is obtained in any balanced equilibrium in the 2-worker case, and give a formula for the time in terms of \( v_1, v_2 \). (Note: worker 2 takes over the widget from worker 1 after a reset.)

4. Consider a three-worker line with \( v_1 = 2, v_2 = 1, v_3 = 3 \). Note that this is not a balanced ordering (i.e. \( v_1 < v_2 < v_3 \)). Use

\[
x_3^{(k+1)} = \min[x_2^{(k)} + \frac{v_2}{v_3}(1 - x_3^{(k)}), 1], \quad x_2^{(k+1)} = \min[\frac{v_1}{v_3}(1 - x_3^{(k)}), x_3^{(k+1)}]
\]

and start with \( x_2 = \frac{1}{4}, x_3 = \frac{2}{3} \). Your calculations should suggest a convergence to \( (x_2^*, x_3^*) = (\frac{1}{3}, \frac{1}{2}) \). Verify that this is an equilibrium. Sketch the \( x, t \)-diagram over one cycle, and verify that the optimal production rate \( v_1 + v_2 + v_3 \) is obtained. How much time does each worker spend on a given widget?

5. In the example of problem 4, the iterates satisfy

\[
x_3^{(k+1)} = x_2^{(k)} + \frac{1}{3}(1 - x_3^{(k)}), \quad x_2^{(k+1)} = \frac{2}{3}(1 - x_3^{(k)})
\]

Express these equations as a single second-order difference equation for \( x_2^{(k)} \) or \( x_3^{(k)} \) and solve. In this way verify the convergence of the system to the equilibrium.