1. (Ex. 4.2.1 of John’s notes) Find a solution of the Bessel equation of order 0:

\[ z'' + \frac{1}{x}z' + z = 0, \]

satisfying \( z = 1, \ z' = 0 \) for \( x = 0 \), by assuming a power series representation. Use the functional relation obtained by the Wronskian to find another (linearly independent) solution of the equation. Show that this solution becomes infinite as \( x \to 0 \) like \( \log x \).

2. (Ex. 4.2.2) Observe that the differential equation

\[ L[y] = y''' - \frac{2}{x}y'' + y' - \frac{2}{x} y = 0 \]

has the two solutions \( \cos x, \sin x \).

(a) Find the general solution by rewriting the equation as an equation for \( y'' + y \).

(b) Find the general solution using instead by writing the scalar equation as a system of three first-order equations and using the formal procedure from class.

(c) Solve the inhomogeneous equation \( L[y] = x^2 \) in any way.

3. (Ex. 4.2.3) Show that if \( w(x) \) is a vector such that \( w^T(x)y(x) = \text{constant} \) for every solution \( y(x) \) of \( y' = Ay \), then \( w(x) \) is a solution of the adjoint system \( w' = -A^T w \).

4. (Method of integral transforms) Consider an integral with respect to the complex variable \( s \) of the following form:

\[ w = \int_C G(s)e^{sz+F(s)} ds, \]

where \( C \) is a piecewise smooth curve in the \( s \)-plane which begins and ends at \( \infty \), by tending asymptotically to two straight lines \( R_1, R_2 \) extending from the origin to \( \infty \).

We seek a solution of Airy’s equation \( Lw \equiv w'' - zw = 0 \), having the above form.

By differentiating under the integral sign, show that the functions \( G, F = 1, -\frac{1}{6}s^3 \) yields a perfect differential, i.e.

\[ Lw = \int_C G(s)L e^{sz+F(s)} ds = \int_C dH(s, z), \]

and determine choices of \( R_1, R_2 \) such that

\[ \lim_{s \to \infty} H(s, z) = 0 \]

on \( R_1, R_2 \). Show that one solution of Airy’s equation is given by the real integral

\[ A_i(z) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{1}{3} t^3 + tz \right) dt. \]

(Hint: Choose \( R_i \) so that \( ds = (i - \epsilon) dt \) on \( R_1 \), extending downward, and \( ds = (i + \epsilon) dt \) on \( R_2 \), extending upward, where \( t \) is a real variable. Then take the limit \( \epsilon \to 0 \).)

5. Ex. 5.1.1. (Hint: The existence of a \( C \), such that \( C^{-1} AC \) has Jordan Normal Form \( \Lambda \), is assumed. Find a \( D \) such that \( D^{-1} B_k D = \tilde{B}_k \), where \( \tilde{B}_k \) is a block in \( \Lambda \), as described in the problem.)

6. Ex. 5.1.2.