1. Let $f(x)$ have Fourier transform $\hat{f}(k)$, that is $\hat{f}(k) = \mathcal{F}(f)$. Show that, if $\hat{f}(k) \to 0$ as $k \to \pm \infty$, that $\mathcal{F}^{-1}(i\hat{f}'(k)) = -ix\mathcal{F}^{-1}(\hat{f})$. From this deduce that $xf(x)$ has the Fourier transform $-i\hat{f}'(k)$. Use this to find the Fourier transform of $xe^{-\alpha|x|}$, $\alpha > 0$.

2. Find the Fourier transforms with respect to $x$ of the following differential equations:

(a) $\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} - \frac{\partial^3 f}{\partial x^3} = 0$,

(b) $\frac{\partial f}{\partial t} + x\frac{\partial f}{\partial x} = 0$.

In (b) make use of problem (1) above.

3. From the derivation of the fundamental solution (with $k = 1$) $T = \frac{1}{\sqrt{t}} F(\eta), \eta = x/\sqrt{t}$, $F = \frac{1}{2\sqrt{\pi}} e^{-\eta^2/4}$, we arrived at the equation

$$ F_\eta + \frac{1}{2} \eta F = C = \text{constant}. $$

Show that, if the constant is not taken equal to zero, there is another solution

$$ S(x, t) = \frac{1}{\sqrt{\eta}} G(\eta), G(\eta) = e^{-\eta^2/4} \int_0^\eta e^{u^2/4} du. $$

Dividing $\int_0^\eta$ into $\int_0^M + \int_M^\eta$ with $0 < M < \eta$, and using integration by parts, show how the $G$ behaves as $\eta \to \infty$, and in particular that it decays algebraically like $\frac{1}{\eta}$, not exponentially.

4. Use the Fourier transform to find the fundamental solution of the partial differential equation

$$ \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0. $$

Express you answer as an inverse Fourier transform. Show that the answer has the form $u = t^{-1/4} f(x/t^{1/4})$. Derive a differential equation for $f$ from this form and show that it reduces to $\eta f - 4f\eta = 0$, given that $F$ and its derivatives vanish at infinity. Show that the Fourier integral solution satisfies this differential equation.

5. Find the solution of the following inhomogeneous IBVP for the heat equation:

$$ T_t - kT_{xx} = 1, 0 < x < \pi, $$

$$ T(0, t) = 1, T(\pi, t) = 3, T(x, 0) = 0. $$

Your solutions will be in the form

$$ T = f(x) + \sum_{n=1}^\infty c_n e^{-n^2kt} \sin nx, $$

where the $c_n$ should be given explicitly.

6. Suppose that the IVP for the 1D heat equation on $-\infty < x < +\infty$ is solved with $u(x, 0) = f(x), f(-x) = -f(x)$. Show that the solution may be written in the form

$$ u(x, t) = \int_0^{+\infty} f(\xi)[U(x - \xi, t) - U(x + \xi, t)]d\xi, $$

where $U$ is the fundamental solution. From this deduce that $u$ is the solution to the IVP for the half interval $x > 0$, with initial values $f(x)$ on this interval, subject to the boundary condition $u(0, t) = 0$. This is an example of the reflection method.