1. (Reading: Batchelor 543-545). Consider axisymmetric motion, with velocity \( \mathbf{u} = (u_r, u_\theta, u_z) \), of a fluid of constant density. The equation of continuity in cylindrical polar coordinates is
\[
\frac{\partial u_z}{\partial z} + \frac{1}{r} \frac{\partial ru_r}{\partial r} = 0.
\]
(a) Show that this equations is satisfied if
\[
u_z = \frac{1}{r} \frac{\partial \psi}{\partial r}, u_r = -\frac{1}{r} \frac{\partial \psi}{\partial z},
\]
for some function \( \psi \), and verify that
\[
\omega_\theta = -\frac{1}{r} L(\psi), \quad L = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}.
\]
(b) Assume the flow is steady, i.e. \( \partial \mathbf{u}/\partial t = 0 \). Show that if
\[
\mathbf{u} \cdot \nabla Q = u_r \frac{\partial Q}{\partial r} + u_z \frac{\partial Q}{\partial z} = 0
\]
Then \( Q \) is a function of \( \psi \) alone, \( Q = F(\psi) \).
(c) Applied to Bernoulli’s expression
\[
\mathbf{u} \cdot \nabla \left( \frac{p}{\rho} + \frac{1}{2} q^2 \right) = 0
\]
show that
\[
\frac{p}{\rho} + \frac{1}{2} (u_r^2 + u_\theta^2 + u_z^2) = H(\psi)
\]
for some function \( H \). Applied to the equation for \( u_\theta \), show that \( ru_\theta = C(\psi) \) for some function \( C \). Show that this is a special case of Kelvin’s theorem.

2. Continuing problem 1, we showed in class that
\[
(u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z})(\omega_\theta/r) = \frac{2}{r^2} u_\theta \frac{\partial u_\theta}{\partial z}.
\]
From this show (using results of problem 1) that \( \psi \) satisfies
\[
L(\psi) = -CC_\psi + r^2 f(\psi)
\]
for some functions \( C, f \). Finally, show from the momentum equation that \( f = H_\psi \) where \( H \) is defined in problem 1.

3. (Reading: Milne-Thomson p. 89, Batchelor p. 384) (a) Prove Kelvin’s minimum energy theorem: In a simply-connected domain \( V \) let \( \mathbf{u} = \nabla \phi, \nabla^2 \phi = 0 \), with \( \partial \phi/\partial n = f \) on the boundary \( S \) of \( V \). (This \( \mathbf{u} \) is unique in a simply-connected domain). If \( \mathbf{v} \) is any differentiable vector field satisfying \( \nabla \cdot \mathbf{v} = 0 \) in \( V \) and \( \mathbf{v} \cdot \mathbf{n} = f \) on \( S \), then
\[
\int_V |\mathbf{v}|^2 dV \geq \int_V |\mathbf{u}|^2 dV.
\]
(Hint: Let \( \mathbf{v} = \mathbf{u} + \mathbf{w} \), and apply the divergence theorem to the cross term.)