Continuous Time Finance: 
Homework Problems as of Feb 17 2003

1. Using data from H.15 (Federal Reserve Statistical Release), apply the Fama-Bliss bootstrapping technique to construct an instantaneous forward rate curve based on current “on-the-run” Treasury securities.

2. Using historical data for 6m Eurodollar rates from H.15, estimate the linear autorregressive model of the type for the “short term rate”

\[ r_n = a + br_{n-1} + \varepsilon_n. \]

Deduce reasonable values for the parameters \((r_0, \kappa, \theta, \sigma)\) of a Vasicek-type short rate model that fits the data. (Note: Chen and Scott find approximately \(\kappa = 0.25, \theta = 6.4\%, \sigma = 0.01\), using historical data from the 80’s and early 90’s. What do you find? Discuss.)

3. Let \(X(t)\) be an Ornstein-Uhlenbeck process satisfying the SDE

\[
\begin{align*}
\frac{dX}{X(0)} &= -\kappa X dt + \sigma dW, \\
X(0) &= 0.
\end{align*}
\]

Verify that the following equation holds:

\[
\mathbb{E}\left\{ e^{-\int_0^T X(s) ds} \right\} = \exp\left\{ \frac{\sigma^2}{2\kappa^2} \left(1 - e^{-\kappa T}\right)^2 \right\}.
\]

4. Assume a given instantaneous forward rate curve, \(f(t), t \leq T_{max}\). Using the “fudge factor technique” and the results of (3), find a function \(b(t), t \leq T_{max}\), such that

\[
\mathbb{E}\left\{ e^{-\int_0^T f(x) + b(s)) ds} \right\} = \exp\left( -\int_0^T f(s) ds \right),
\]

where \(X(s)\) is the O-U process in Ex. 3. Show that the calibrated risk-neutral short-rate process \(r(t) = X(t) + b(t)\) satisfies an SDE of the form

\[
\frac{dr(t)}{r(t)} = -\kappa (\theta(t) - r(t)) dt + \sigma dW(t)
\]

and give an expression for the function \(\theta(t)\) in terms of \(f(t), \kappa\) and \(\sigma\).

5. In general, assume that \(r(t)\) represents a short rate process, i.e. the rate of return on a theoretical bank deposit. Justify the following statement: “Under any risk-neutral measure, a zero-coupon bond paying $1 at time T has fair value
\[
Z(T) = \mathbb{E} \left( e^{-\int_0^T r(s) ds} \right)
\]

6. Suppose that, under a risk-neutral measure, an asset has price dynamics
\[
\frac{dX}{X} = \sigma(t) dW + \mu(t) dt
\]
where \(\sigma(t)\) is a deterministic function of time and \(\mu(t)\) is a given stochastic process (deterministic or random). Let \(F_T\) denote the forward price of the asset for delivery at time \(T\). Let \(Z(T)\) denote the PV of a dollar payment at time \(T\). Derive a formula for the value of a call-option on the asset with maturity date \(T\) and strike price \(K\).

7. Assume that the 2-year and 2.5-year discount yields are respectively 3.00% and 3.25%. Using the Hull-White-Vasicek model with parameters \(\kappa = 0.2\), and \(\sigma = 0.02\), compute the fair values of 2-year call-options on the six-month rate (paid in arrears) with strikes \(K = \alpha F\), where \(F\) is the forward rate from 2 to 2.5 years and \(\alpha = 0.8, 0.85, 0.90, 0.95, 1.00, 1.05, 1.10, 1.15, 1.20\). Next, calculate the Black-76 implied volatilities for these options. Deduce that the HWV model gives rise to a “volatility skew” if we now look at interest-rate options from the Black-Scholes perspective (with the rate as lognormal asset). What effect does the choice of the parameter \(\kappa\) have (if any) on the shape of the volatility skew?

8. Assume that the discount yield curve is linear, with a 6-month yield of 1.25% and a 10-year yield of 4.50%. Assume that the 6-month ATM caplet volatility curve is given by

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>6y</th>
<th>7y</th>
<th>8y</th>
<th>9y</th>
<th>10y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Caplet Vol</td>
<td>18</td>
<td>20</td>
<td>19</td>
<td>18</td>
<td>17</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

and furthermore that, for intermediate maturities, caplet variances are interpolated linearly. Using this data and a mean-reversion parameter \(\kappa = 0\), calibrate the volatility function and the drift for a 1-factor Hull-White-Vasicek model extending to 10.5 years. (Note: this is the same as a Ho-Lee model with time-dependent volatility). For the short rate volatility function, use a piecewise-constant function \(\sigma(t)\) which takes constant values in the intervals \((0.5n, 0.5(n + 1))\), \(n = 0, 1, ..., 20\). Using the calibrated model, price European-style 5×5 payer swaptions with strikes between 80% and 120% of the 5×5 forward swap rate, in increments of 5%. Calculate the corresponding Black-76 volatilities and graph the corresponding swaption volatility skew curve as a function of strike rate. Optional: re-do the exercise with \(\kappa = 0.25\) and find out if there is a difference between the two volatility skews.