From SABR to Geodesics

A systematic approach for modeling volatility curves with applications to option market-making and pricing multi-asset equity derivatives

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Fitting Volatility Skews

SPX JUN04 (PRICING DATE MAY 22)

Blue line= average implied vol (puts/calls)
Pink line= fitted parabola
If you zoom into the region of interest, the parabolic fit is seen as clearly inadequate. Reason: the out-of-the-money options "lift" the curve. Parabolic fits are not consistent with arbitrage-free pricing.
Parabolic fitting requires Delta Truncation!

Fit only volatilities such that $-0.2 < x < +0.2$

$$\sigma_{\text{parabolic}}(x) = 0.16 - 0.34x + 4.45x^2 = 0.16 \times \left(1 - 2.1x + 27.33x^2\right)$$
Truncated Parabolic Fit: a look at the full curve

Out of the money options are not guaranteed to be well-fitted
Using a better spline to fit the data (from SABR)

\[ \sigma_{\text{imp}}(x) \approx \frac{\kappa |x|}{\ln \left( \kappa \frac{1 - e^{-\beta x}}{\sigma_0 \beta} \right) + \sqrt{1 + \kappa^2 \left( \frac{1 - e^{-\beta x}}{\sigma_0 \beta} \right)^2}} \]

\[ x = \ln \left( \frac{K}{F_0} \right) \]

\[ \gamma \equiv \text{slope} \left( x = 0 \right) = \frac{\beta}{2} \]

Sigma, beta and kappa are adjustable parameters

Formula is derived from a stochastic volatility model so it does not violate arbitrage conditions
Fitting a SABR-like spline to the SPX front-month curve
parabolic of optiondata.aspx on '05Oct04'd

Green = parabolic
Red = SABR
Blue = Mid-Market
# Differential Geometry and Implied Volatility Modeling

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Factor Models and Diffusion Kernels

\[ x(t) = (X_1(t), \ldots, X_n(t)) \]
\[ w(t) = (W_1(t), \ldots, W_m(t)) \]

\[ dX_{i} = \sum_{k=1}^{m} \sigma_{jk}^k dW_k + b_i dt, \quad i = 1, 2, 3, \ldots, n \]

\[ \pi(x, t; y, T) = \text{Prob.}\{x(T) = y | x(t) = x\} \]

\[ E\{F(x(T)) | x(t) = x\} = \int_{y \in \mathbb{R}^n} F(y) \pi(x, t; y, T) dy \]
### Fokker-Planck Equation and Dimensionless Time

The Fokker-Planck equation is given by:

\[
\frac{\partial \pi}{\partial t} + \frac{1}{2} \sum_{ij=1}^{n} a_{ij} \frac{\partial^2 \pi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \pi}{\partial x_i} = 0
\]

where \( \pi(x, T; y, T) = \delta(x - y) \).

The covariance matrix of state variables is:

\[
a_{ij} = \sum_{k=1}^{m} \sigma_i^k \sigma_j^k
\]

\( \sigma \) is the volatility of S&P = 0.15, \( t = 1 \) yr. corresponds to \( \tau = 0.0225 << 1 \).

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Varadhan Asymptotics for the Diffusion Kernel

\[ \lim_{t \to 0} \tau \ln \pi(x,0; y,T) = -\frac{L^2(x,y)}{2} ; \quad \tau = (\sigma^2)T, \]

\[ L(x,y) = \text{geodesic distance between } x \text{ and } y \]

\[ L(x,y) = \inf_{\gamma} \int_{0}^{1} \left\| \frac{d\gamma}{dt} \right\| dt, \]

\[ \|v\|_x^2 = \sum_{i,j=1}^{n} g_{ij}(x)v_iv_j \]

\[ g_{ij} = (\sigma^2)(a^{-1})_{ij} \]

Dimensionless Riemann tensor
Heuristically: Diffusion Kernels “resemble” Gaussian Kernels with $|x-y|$ replaced by $L(x,y)$

$$\pi(x,0; y, T) \approx c(\tau) \cdot e^{-\frac{(L(x,y))^2}{2\tau}} \quad \tau \ll 1$$

$$(dL)^2 = \sum_{ij=1}^{n} g_{ij}(x) dx_i dx_j$$

We shall use this approximation to compute option prices and implied volatilities assuming tau is small
Example 1: Local volatility model

\[
\frac{dF_t}{F_t} = \sigma(F_t, t) dW_t \quad x = \ln \left( \frac{F}{F_0} \right) \\
 dx_t = \sigma(x_t, t) dW_t + (...) dt \\
(dL)^2 = \frac{(\sigma)^2}{(\sigma(x,0))^2} dx^2 = \frac{dx^2}{(\sigma(x,0))^2} = \left( \frac{dx}{\sigma(x)} \right)^2 \\
L(x, y) = \left| \int_x^y \frac{du}{\sigma(u)} \right| = |G(y) - G(x)|
\]

1-dimensional distances are always 'trivial'
Special solvable 2-D case: the CEV Model

\[ \sigma(F,t) = \sigma\left(\frac{F}{F_0}\right)^\beta \]

\[ \sigma(x,t) = \sigma e^{\beta x} \]

\[ \sigma(x) = e^{\beta x} \]

Distance = area under the curve

Negative beta for Equities (leverage)

Distance = area under the curve
Stochastic Volatility Models

\[ \frac{dF_t}{F_t} = \sigma_t dW_t \]
\[ d\sigma_t = \kappa dZ_t \]
\[ E\{dW_t dZ_t\} = \rho dt \]

\[ \frac{d\sigma_t}{\sigma_t} = \beta \frac{dF_t}{F_t} + \varepsilon \]

Forward price
Stochastic vol.
Leverage
Beta= regression coefficient of vol on stock returns
Equivalent Model with Independent Brownian Motions (SABR)

\[ \sigma_t = \sigma_t^{(0)} \exp(\beta x_t) \quad x_t = \ln \left( \frac{F_t}{F_0} \right) \]

``Parametric leverage'' SV for tails

``CEV'' with stochastic independent volatility is equivalent to SV model with correlated volatility, from the Riemann viewpoint
Riemann Metric for SV / SABR: The Poincare Upper Half-Space Model

\[ \eta \equiv \kappa \left( \frac{1 - e^{-\beta x}}{\beta} \right), \quad \sigma \equiv \sigma^{(0)} \]

\[ dL^2 = \frac{\sigma^2}{\kappa^2} \cdot d\eta^2 + d\sigma^2 \]

Geodesics are half-circles with center on the horizontal axis

\[ L(P, Q) = \frac{\sigma}{\kappa} \int_{\theta_P}^{\theta_Q} \frac{d\theta}{\sin \theta} \]
Using the asymptotics to compute option prices

\[ F_T = F(x_T) \quad \quad P_0 = F(0) < K \]

\[
\text{CALL} = \int_{R^n} \max(F(y) - K, 0) \pi(0,0; y, T) d^n y
\]

\[
\approx c \int_{R^n} \max(F(y) - K, 0) e^{-\frac{L^2(0,y)}{2\tau}} d^n y
\]

\[
\approx c \int \left\{ y: F(y) > K \right\} e^{-\frac{L^2(0,y)}{2\tau}} d^n y
\]

\[
\approx c \int \left\{ y: F(y) > K \right\} \left[ -\ln \left( \frac{1}{F(y) - K} + \frac{L^2(0,y)}{2\tau} \right) \right] d^n y
\]
Steepest-descent approximation for computing implied volatilities

\[
\int e^{-\left[ \ln\left( \frac{1}{F(y) - K} \right) + \frac{L^2(0,y)}{2\tau} \right]} \, d^n y \approx e^{-\min_{y : F(y) > K} \left[ \ln\left( \frac{1}{F(y) - K} \right) + \frac{L^2(0,y)}{2\tau} \right]}
\]

\[
\min_{y : F(y) > K} \left[ \ln\left( \frac{1}{F(y) - K} \right) + \frac{L^2(0,y)}{2\tau} \right] = \frac{1}{\tau} \min_{y : F(y) > K} \left[ \tau \ln\left( \frac{1}{F(y) - K} \right) + \frac{L^2(0,y)}{2} \right]
\]

\[
\approx \frac{1}{2\tau} \min \{ L^2(0,y) | y : F(y) > K \}, \quad \tau \ll 1
\]
Equate formulas for OTM calls with Black-Scholes...

\[ L^*(K) = \min \{ L(0,y) | y : F(y) > K \} \]

\[ \ln \text{CALL} \approx \frac{(L^*(K))^2}{2\tau} \]

\[ \ln \text{CALL} \approx \frac{(\ln(K/F_0))^2}{2\sigma_{imp}^2(K)T} = \frac{(\ln(K/F_0))^2}{2\left(\frac{\sigma_{imp}^2(K)}{(\bar{\sigma})^2}\right)\tau} \]

Minimum distance from 0 to the region \{F(y)>0\}

Small-tau asymptotics (model)

Small-tau asymptotics (Black-Scholes)
Approximation for Implied Volatility for general diffusion model

\[ \sigma_{\text{imp}}(K) = \bar{\sigma} \frac{\left| \ln \left( \frac{K}{F_0} \right) \right|}{\min \{ L(0, y) | y : F(y) > K \}} \]

\[ = \frac{\left| \ln \left( \frac{K}{F_0} \right) \right|}{\min \{ L_1(0, y) | y : F(y) > K \}} \]

\[ L_1(x, y) \equiv \min_{\gamma(0)=x, \gamma(1)=y} \int_0^1 \sqrt{\sum_{i,j=1}^n (a^{-1})_{ij} \gamma_i(t) \gamma_j(t)} \, dt \]
Example 1: Local Volatility Model

\[
\sigma_{\text{imp}}(K) \approx \frac{\ln(K / F_0)}{\ln(K/F_0)} = \left( \frac{1}{x} \int_0^x \frac{du}{\sigma(u,0)} \right)^{-1}
\]

\[
x = \ln \left( \frac{K}{F_0} \right)
\]

Implied Volatility = Harmonic Mean of Local Volatility

Berestycki, Busca and Florent, 2001
Example 2: Constant Elasticity of Variance

\[ \sigma(x, t) = \sigma_0 e^{\beta x} \]

CEV

\[ \sigma_{\text{imp}}(x) \approx \sigma_0 \left| \frac{\beta x}{1 - e^{-\beta x}} \right| \]

Implied volatility

![Graph showing implied volatility vs. x = ln(K/F)]
Example 3: Stochastic Volatility / SABR

\[
\sigma_{\text{imp}}(x) \approx \frac{\kappa|x|}{\ln \left[ \frac{1-e^{-\beta x}}{\sigma_0 \beta} \right] + \sqrt{1+\kappa^2 \left( \frac{1-e^{-\beta x}}{\sigma_0 \beta} \right)^2}}
\]

\[x = \ln \left( \frac{K}{F_0} \right)\]
Minimizing the distance to the line 
eta = const. in the Poincare plane

\[ \eta = \kappa \left(1 - e^{-\beta x}\right) \]

\[ R = \sqrt{\sigma_0^2 + \eta} \]

\[ \bar{\sigma} = R \]

\[ L^* = \frac{1}{\kappa} \int_{\theta_i}^{\pi/2} d\theta \frac{1}{\sin \theta} \]

\[ \sigma_{\text{implied}}(x) \approx \frac{|x|}{L^*} \]
Example 3bis: Stochastic Volatility / Hull-White

\[
\frac{dS}{S} = \sigma dW, \quad x = \ln(S / S_0)
\]

\[
\frac{d\sigma}{\sigma} = \kappa dZ, \quad E(dWdZ) = \rho dt
\]

\[
dL^2 = \frac{1}{\kappa^2 (1 - \rho^2)} \cdot \frac{\kappa^2 dx^2 - 2 \rho \kappa \sigma dx + d\sigma^2}{\sigma^2}
\]

\[
z = \frac{\kappa x - \rho \sigma}{\sqrt{1 - \rho^2}}
\]

\[
dL^2 = \frac{1}{\kappa^2} \cdot \frac{dz^2 + d\sigma^2}{\sigma^2} \quad \text{Poincare plane after change of variables}
\]
Geodesic Distance

\[ P = \left( \frac{-\rho \sigma}{\sqrt{1 - \rho^2}}, \sigma \right) \]

\[ R = \left( \frac{\kappa}{\sqrt{1 - \rho^2}}, 0 \right) \]
Approximation for Implied Volatility Curves

\[ L^* = d(P, Q) = \frac{1}{\kappa} \int_{\varphi_p}^{\varphi_0} \frac{du}{\sin u} \]

\[ L^* = \frac{1}{\kappa} \ln \left( \frac{\kappa x + \rho \sigma + \sqrt{\left(\kappa x + \rho \sigma\right)^2 + \sigma^2(1 - \rho^2)}}{\sigma(1 + \rho)} \right) \]

\[ \sigma_{imp}(x) \approx \frac{\kappa |x|}{\ln \left( \frac{\kappa x + \rho \sigma + \sqrt{\left(\kappa x + \rho \sigma\right)^2 + \sigma^2(1 - \rho^2)}}{\sigma(1 + \rho)} \right)} \]
Auto-calibration of SABR and Heston

\[ \sigma_0 = 20\% \]

\[ \beta = -4 \]

\[ \kappa_{\text{sabr}} = 0.5 \]

\[ \kappa_{\text{Heston}} = 2\sigma_0\kappa_{\text{sabr}} = 0.2 \]
Example 4: the Heston Model
A variant of the Poincare Half-Space

\[ \frac{dS_{t}}{S_{t}} = \sqrt{V_{t}} \, dW_{t} \]

\[ dV_{t} = \kappa \sqrt{V_{t}} \, dZ_{t} \quad E(dW_{t} \, dZ_{t}) = \rho \, dt \]

\[ dL_{t}^{2} = \frac{1}{\kappa^{2}} \left( d\xi^{2} + dV^{2} \right) / V \]

\[ \xi = \frac{\kappa \left( 1 - e^{-\beta x} \right)}{\beta} \]

Note: \( V \), not \( V \) squared
Closed-form solution for geodesics

\[ \xi = \kappa \left( \frac{1 - e^{-\beta x}}{\beta} \right) \]

\[ dL^2 = \frac{d\xi^2 + dV^2}{\kappa^2 V} \]

\[ \xi(\theta) = \frac{R^2}{2} (\theta - \sin \theta \cos \theta) + \xi(0) \]

\[ V(\theta) = R^2 \sin^2 \theta \quad 0 \leq \theta \leq \pi \]

\[ dL = \frac{2R^2}{\kappa} \sin \theta d\theta \]

Geodesics are cycloids.
Implied volatility curve for Heston model is obtained as an algebraic system

\[ \xi = \frac{\sigma_0^2}{\sin^2 \theta_{\text{init}}} \left( \frac{\pi}{2} - \theta_{\text{init}} + \sin \theta_{\text{init}} \cos \theta_{\text{init}} \right) \]

\[ \sigma(\xi) = \frac{\kappa |\xi| \sin^2 \theta_{\text{init}}}{2 \sigma_0^2 |\cos \theta_{\text{init}}|} \]

Given \( x_i \), solve for \( \theta_{\text{init}} \), and substitute in the second equation.
<table>
<thead>
<tr>
<th>Multi-Asset Derivatives</th>
</tr>
</thead>
</table>

| QQQQ | 05 Oct 38.00 (QQQ JL) | 05 Oct 39.00 (QQQ JM) | 05 Oct 40.00 (QQQ JN) | Zero Call | 05 Nov 37.00 (QQQ KK) | 05 Nov 38.00 (QQQ KL) | 05 Nov 39.00 (QQQ KM) | 05 Nov 40.00 (QQQ KN) | Zero Call | 05 Dec 37.00 (QQQ LK) | 05 Dec 38.00 (QQQ LL) | 05 Dec 39.00 (QQQ LM) | 05 Dec 40.00 (QQQ LN) | 05 Dec 41.00 (QQQ LO) | Zero Call | 06 Jan 36.00 (QQQ L1) | 06 Jan 36.25 (QQQ L2) | 06 Jan 37.00 (QQQ AK) | 06 Jan 37.625 (YIZ A) | 06 Jan 38.00 (QQQ AL) | 06 Jan 38.625 (YIZ A) | 06 Jan 39.00 (QQQ AM) | 06 Jan 39.625 (YIZ A) | 06 Jan 40.00 (QQQ AN) | 06 Jan 40.625 (YIZ A) | 06 Jan 41.00 (QQQ AO) | 06 Jan 41.625 (YIZ A) | 06 Jan 42.00 (QQQ AP) | 06 Mar 36.00 (QQQ CJ) | 06 Mar 37.00 (QQQ CK) |
|------|------------------------|------------------------|------------------------|-----------|------------------------|------------------------|------------------------|------------------------|-----------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|-----------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 4550 |                       |                        |                        |           |                       |                        |                        |                        |           |                       |                        |                        |                        |                        |                        |                        |           |                       |                        |                        |                        |                        |                        |                        |                        |                        |                        |                        |                        |                        |                        |
Multi-Asset Derivatives: Index Options, Rainbows

Derive index volatility skew from **single-stock skews** and **correlation matrix**

\[ dx_i = \sigma(x_i, t) dW_i, \quad i = 1, 2, \ldots, n \]

\[ E(dW_i dW_j) = \rho_{ij} dt \]

N equations for the index components

\[ I = \sum_{i=1}^{n} w_i S_i = \sum_{i=1}^{n} w_i S_i(0) e^{x_i} \quad \bar{x} = \ln\left(\frac{F_l}{I(0)}\right) \]
### BBH: ETF of 20 Biotechnology Stocks

**(Components of IBH)**

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<th>Ticker</th>
<th>Shares</th>
<th>ATM ImVol</th>
<th>Ticker</th>
<th>Shares</th>
<th>ATM ImVol</th>
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<td>BBH</td>
<td>-</td>
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</table>
Implied Volatility Skews
Multiple Names, Multiple Expirations

AMGN

BGEN

MEDI
What is the `fair value' of the index volatility reconstructed from the components?
Riemannian metric for the multi-D local vol model

If correlations are constant, the metric is "flat": it is Euclidean metric after making the change of variables $x \rightarrow y$.

Geodesics are straight lines in the $y$-coordinates.
Steepest Descent=Most Likely Stock Price Configuration

Replace conditional distribution by “Dirac function” at most likely configuration
Exact solution: Euler-Lagrange Equations

$$\sigma_{\text{impl.}i}(x) = \frac{\left| x \right|}{\sqrt{\sum_{ij=1}^{n} (\rho^{-1})_{ij} \int_{0}^{x^*_i} du \frac{\sigma_i(u,0)}{\sigma_{\text{impl.,}i}(x^*_i)} \int_{0}^{x^*_j} du \frac{\sigma_j(u,0)}{\sigma_{\text{impl.,}j}(x^*_j)}}}}$$

Euler - Lagrange equations

$$\int_{0}^{x^*_i} du \frac{\sigma_i(u,0)}{\sigma_{\text{impl.}}(u,0)} = \Lambda \sum_{j=1}^{n} \rho_{ij} p_j(x^*_j) \sigma_j(x^*_j,0) \quad i = 1, 2, \ldots, n$$
Approximate solution: introduce the stock betas

\[ x_i = \beta_i \bar{x} + \varepsilon_i \]

Regression relation between stock and index returns

\[ x_i^* = \beta_i \bar{x} \]

Approximate formula for the optimal stock configuration

\[ \frac{1}{\sigma_{\text{imp},i}(\bar{x})} \approx \sqrt{\sum_{ij=1}^{n} \frac{(\rho^{-1})_{ij} \beta_i \beta_j}{\sigma_{\text{imp},i}(\bar{\beta}_i \bar{x}) \sigma_{\text{imp},i}(\bar{\beta}_i \bar{x})}} \]

\[ \sigma_{\text{imp},i}(\bar{x}) \approx \sqrt{\sum_{ij=1}^{n} \rho_{ij} p_i p_j \sigma_{\text{imp},i}(\bar{\beta}_i \bar{x}) \sigma_{\text{imp},i}(\bar{\beta}_i \bar{x})} \]

Performs well in the range -0.2<x<+0.2
DJX: Dow Jones Industrial Average

T=1 month

DJX Nov 02   Pricing Date: 10/25/02

Vol

Delta
T = 2 months
T=3 months

DJX Jan 03  Pricing Date: 10/25/02

Vol

88 84 79 75 69 62 55 48 40 33 26 15

Delta

BidVol

AskVol

SDA
T = 5 months
T=7 months

DJX June 03  Pricing Date: 10/25/02

Vol vs Delta

- BidVol
- AskVol
- SDA
BBH: Biotechnology HLDR

$T = 1$ month
$T = 2$ months

**Graph:**

- **Title:** BBH Dec 02 Date: Oct 25 02

- **Axes:**
  - **X-axis:** Delta
  - **Y-axis:** Vol

- **Graph Lines:**
  - BidVol (red diamonds)
  - AskVol (black squares)
  - SDA (green squares)

- **Data Points:**
  - Vol values range from 60 to 30 at various Delta values from 88 to 23.
Is dimensionless time too long? (Error bars: Juyoung Lim)

Is correlation causing the discrepancy?
S&P 100 Index Options
(Quote date: Aug 20, 2002)
S&P 100 Index Options
(Quote date: Aug 20, 2002)
S&P 100 Index Options
(Quote date: Aug 20, 2002)
S&P 100 Index Options
(Quote date: Aug 20, 2002)
Implied Correlation: a single correlation coefficient consistent with index vol

\[
(\sigma_{impl}^2)^2 = \sum_{i=1}^{N} p_i^2 (\sigma_{i,impl}^2)^2 + \rho \sum_{i \neq j} p_i p_j \sigma_{i,impl} \sigma_{j,impl}
\]

\[
\rho = \frac{(\sigma_{impl}^2)^2 - \sum_{i=1}^{N} p_i (\sigma_{i,impl}^2)^2}{\sum_{i=1}^{N} p_i \sigma_{i,impl} \sigma_{j,impl}} - \frac{\left(\sum_{i=1}^{N} p_i \sigma_{i,impl}^2\right)^2 - \sum_{i=1}^{N} p_i^2 (\sigma_{i,impl}^2)^2}{\sum_{i=1}^{N} p_i \sigma_{i,impl} \sigma_{j,impl}}
\]

Approximate formula:

\[
-\rho \approx \left(\frac{\sigma_{impl}}{\sum_{i=1}^{N} p_i \sigma_{impl}^2}\right)^2
\]

Implied correlation can be defined for different strikes, using SDA.
Dow Jones Index

- Historical Correlation
- 3 month Implied Correlation
- Index Price
Dow Jones Index: Correlation Skew

Quote Date 9/1/1998 Spot price = 78.26

Implied Correlation

Strike
Quote Date 12/10/2001 Spot=99.21

- 12/22/2001
- 1/19/2002
- 2/16/2002
- 3/16/2002
- 6/22/2002
- 9/21/2002

implied correlation vs. strike
A model for "Correlation skew": Stochastic Volatility Systems

\[
\frac{dS_i}{S_i} = \sigma_i dW_i \\
E(dW_i dW_j) = \rho_{ij} dt \\
\frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i \\
E(dW_i dZ_j) = r_{ij} dt
\]  

\[-x = \frac{dI}{I}, \quad x_i = \frac{dS_i}{S_i} \quad y_i = \frac{d\sigma_i}{\sigma_i}\]

Look for most likely configuration of stocks and vols \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) corresponding to a given index displacement \(x\)
Most likely configuration for stocks moves and volatility moves, given the index move

\[ x_i^* = \beta_i \bar{x} \]
\[ \beta_i = \frac{\sigma_i \rho_{il}}{\sigma_i} \]
\[ y_i^* = \gamma_i \bar{x} \]
\[ \gamma_i = \frac{\kappa_i r_{il}}{\sigma_i} \]

\[ \sigma_{I,\text{loc}}^2(x, t) \equiv \sum_{ij=1}^{n} p_i p_j \sigma_i(0, t) \sigma_j(0, t) e^{\gamma_i \bar{X}} e^{\gamma_j \bar{X}} \rho_{ij} \]
Method I: Dupire & Most Likely Configuration for Stock Moves

- Step 1: Local volatility for each stock consistent with options market
- Step 2: Find most likely configuration for stocks
Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities

N-dimensional Equity market

\[ \sigma_{I,\text{loc}}(\bar{x}, t) \]

- Only one step: compute the most likely configuration of stocks and volatilities at the same time
Methods I and II are not `equivalent’

\( \sigma_{i, \text{loc}}(x_i, t) \approx \sigma_i(0, t) e^{\sigma_i x_i} \)

\( \bar{\sigma}_i = \frac{\kappa_i r_{ii}}{\sigma_i} \)

Index vol., Method I

\( \sigma_{I, \text{loc}}^2(x, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\sigma_i \beta_i x} e^{\sigma_j \beta_j x} \)

Index vol., Method II

\( \sigma_{I, \text{loc}}^2(x, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} \)

Dupire local vol. for single names
Stochastic Volatility Systems give rise to Index-dependent correlations

\[ \sigma_{l,\text{loc}}^2(x, t) \approx \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} \]

Method II

\[ \approx \sum_{ij} p_i p_j \sigma_i(0, t) e^{\beta_i \overline{\sigma} x} \sigma_j(0, t) e^{\beta_j \overline{\sigma} x} \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} e^{-\beta_i \overline{\sigma} x} e^{-\beta_j \overline{\sigma} x} \]

\[ \approx \sum_{ij} p_i p_j \sigma_{i,\text{loc}}(\beta_i \overline{x}, t) \sigma_{j,\text{loc}}(\beta_j \overline{x}, t) \rho_{ij}(\overline{x}) \]

\[ \rho_{ij}(\overline{x}) \equiv \rho_{ij} e^{(\gamma_i + \gamma_j - \beta_i \overline{\sigma} - \beta_j \overline{\sigma}) \overline{x}} \]
Equivalence holds only under additional assumptions on stock-volatility correlations

\[ \sigma_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \frac{\sigma_i \rho_{il}}{\sigma_l} = \frac{\kappa_i r_{ii} \rho_{il}}{\sigma_l} \quad \text{Method I} \]

\[ \gamma_i = \frac{\kappa_i r_{il}}{\sigma_l} \quad \text{Method II} \]

\[ r_{il} = r_{ii} \rho_{il} \]

\[ r_{ij} = r_{ii} \rho_{ij} \]

Conditions under which both methods give equivalent valuations
Open (and very doable) problems

- Apply this technology for pricing swaptions based on the volatility skew of LIBOR rates or forward rates.
- If we use a Local Volatility model (e.g. BGM with square-root volatility), the answer is identical to the previous formula.
- The "full" SABR multi-asset model gives rise to a complicated Riemannian metric:

\[
\begin{align*}
   dL^2 &= \sum_{ij=1}^{n} g_{ij} \frac{d\eta_i}{\sigma_i} \frac{d\eta_j}{\sigma_j} + \sum_{i=1}^{n} \frac{(d\sigma_i)^2}{\kappa_i^2 \sigma_i^2} \\
   \sigma &= \text{(some parameters)}
\end{align*}
\]
- Credit default models for pricing CDOs are amenable to the same approach, especially copula-type models. I am not aware of any solutions.
**Epilogue: Structural Credit Model**

Let \( x = (x_1, \ldots, x_n) \) be a vector of firm values. Firm \( i \) defaults before time \( T \) if \( x_i(T) < \alpha_i \).

Equal weighted CDO: loss of \( m \) dollars if

\[
\mathbf{x}(T) \in \Omega_m = \bigcup_{\text{card}(I) \geq m} \bigcap_{i \in I} \{ x : x_i < \alpha_i \}
\]

Solve

\[
\inf \{ L(0, x) : x \in \Omega_m \}
\]