Derivative Securities: Lecture 4
Introduction to option pricing

Sources:
J. Hull, 7\textsuperscript{th} edition
Avellaneda and Laurence (2000)
Yahoo!Finance & assorted websites
Option Pricing

• In previous lectures, we covered forward pricing and the importance of cost-of-carry.

• We also covered Put-Call Parity, which can be viewed as relation that should hold between European-style puts and calls with the same expiration.

• Put-call parity can be seen as pricing conversions relative to forwards on the same underlying asset.

• What other relations exist between options and spreads on the same underlying asset?
Call Spread

**Call Spread:** Long a call with strike $K$, short a call with strike $L$ ($L > K$)

Since the payoff is non-negative, the value of the spread must be positive

$$K < L \quad \Rightarrow \quad Call(K, T) > Call(L, T)$$

$$CS(K, L, T) = Call(K, T) - Call(L, T) > 0$$

Spread makes money if the price of the underlying goes up.
**Put Spread**

**Put Spread**: Long a put with strike $L$, short a put with strike $L$ ($L>K$)

Since the payoff is non-negative, the value of the spread must be positive

$$K < L \quad \Rightarrow \quad Put(K,T) < Put(L,T)$$

$$PS(K,L,T) = Put(L,T) - Put(K,T) > 0$$

Spread makes money if the price of the underlying goes down
**Butterfly Spread**

**Butterfly spread:** Long call with strike $K$, long call with strike $L$, short 2 calls with strike $(K+L)/2$

Since the spread has non-negative payoff, it must have positive value

$$B(K, (K + L)/2, L, T) = Call(K, T) + Call(L, T) - 2Call\left(\frac{K + L}{2}, T\right) > 0$$

Butterflies make $\$ if the stock price is near $(K+L)/2$ at expiration.
Straddle: Long call and long put with the same strike

Straddles make money if the stock price moves away from the strike and ends far from it.
**Strangle**

**Strangle**: Long 1 put with strike $K$, long 1 call with strike $L$, $L > K$.

Strangle is also non-directional, like a straddle, but makes $ only if the stock moves very far away.

Straddles and strangles are often used to express views about volatility of the underlying stock and are non-directional.
**Risk-reversal**

Risk-reversal: Long 1 put with strike K, short 1 call with strike L, L>K.

Directional spread. Can be seen as financing a put by selling an upside call.
Calendar Spread

**Calendar Spread:** Short 1 call with maturity T1, Long 1 call with maturity T2, T1<T2
Same strike

- If the underlying pays no dividends between T1 and T2, then the longer maturity call is above intrinsic value at time T1. Calendars have positive value.

- For American options, calendars always have positive value
Reconstructing Call prices from Butterfly Spreads

Assume for simplicity a countable number of strikes, $K_n = n\Delta K$, $n = 0,1,2,3,\ldots$ and that the stock price can only take values on the lattice $S_T = m\Delta K$, $m = 1,2,3,\ldots$

$$Call(K_n, T) = \sum_{j \geq n} \left(Call(K_j, T) - Call(K_{j+1}, T)\right) = \sum_{j \geq n} CS(K_j, K_{j+1}, T)$$

- A call can be viewed as a portfolio of call spreads
- A call spread can be viewed as a portfolio of butterfly spreads

$$CS(K_j, K_{j+1}, T) = \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T)$$
Calls as super-positions of butterfly spreads

\[ Call(K_n,T) = \sum_{j=n}^{\infty} \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T) \]

\[ = \sum_{j=n}^{\infty} (j+1-n)B(K_j, K_{j+1}, K_{j+2}, T) \]

\[ = \sum_{j=n}^{\infty} (j+1-n)\Delta K \cdot \left(\frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K}\right) \]

\[ = \sum_{j=n}^{\infty} (K_{j+1} - K_n)w(K_{j+1}, T) \quad w(K_{j+1}, T) = \frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K} \]

\[ = \sum_{j=0}^{\infty} (K_{j+1} - K_n)^+ w(K_{j+1}, T) \]

The weights correspond to values of Butterfly spreads centered at each \( K_j \). In particular, they are positive.
From weights to probabilities

\( w(K,T) > 0 \)

\[
\sum_{j=1}^{\infty} w(K_j,T) = \frac{1}{\Delta K} \sum_{j=1}^{\infty} B(K_{j-1},K_j,K_{j+1},T) = \frac{1}{\Delta K} CS(0,\Delta K, T)
\]

\( = PV(\$1) = e^{-rT} \) (assuming that the stock can only take values > \( \Delta K \))

\( w(K_j,T) = e^{-rT} p(K_j,T) \); \( \sum_j p(K_j,T) = 1, \ p(K_j,T) > 0. \)

\[
Call(K,T) = e^{-rT} \sum_j \max(K_j - K,0)p(K_j,T)
\]

\( = e^{-rT} E^\rho \{ \max(S_T - K,0) \} \)
First moment of $p$ is the forward price

$$E^p\{S\} = e^{rT} Call(0,T)$$

$$= e^{rT} e^{-qT} S_0 = S_0 e^{(r-q)T}$$

$$= F_{0,T}$$

• A call with strike 0 is the option to buy the stock at zero at time $T$
  Its value is therefore the present value of the forward price (pay now, get stock later).

• It follows that the first moment of $p$ is the forward price.

• It also follows that put prices are given by a similar formula, namely

$$Put(K,T) = Call(K,T) + Ke^{-rT} - Se^{-qT}$$

$$= e^{-rT} E^p\{(S_T - K)^+\} + e^{-rT} K - e^{-rT} F$$

$$= e^{-rT} E^p\{(S_T - K)^+\} + e^{-rT} E^p\{K - S_T\}$$

$$= e^{-rT} E^p\{(K - S_T)^+\}$$
General Payoffs

- Any twice differentiable function \( f(S) \) can be expressed as a combination of put and call payoffs, using the formula

\[
f(S) = f(0) + f'(0)(S - F) + \int_0^F (S - Y)^+ f''(Y) dY + \int_F^\infty (Y - S)^+ f''(Y) dY\]

\[
= f(0) + f'(0)(S - F) + \int_0^F (Y - S)^+ f''(Y) dY + \int_F^\infty (S - Y)^+ f''(Y) dY
\]

- Thus, a European-style payoff can be viewed as a spread of puts and calls. By linearity of pricing,

Fair value of a claim with payoff \( f(S_T) = \)

\[
e^{-rT} \int_0^\infty e^{-rF} \left\{ f(S_T - F) + e^{-rT} \int_0^F (Y - S_T)^+ f''(Y) dY + \int_F^\infty (S_T - Y)^+ f''(Y) dY \right\} dF
\]

\[
e^{-rT} E^p \left\{ f(S_T) \right\}
\]
Fundamental theorem of pricing (one period model)

• An arbitrage opportunity is a portfolio of derivative securities and cash which has the following properties:
  
  - The payoff is non-negative in all future states of the market
  - The price of the portfolio is zero or negative (a credit)

Assume that each security has a unique price (i.e. assume bid-offer).

• If there are no arbitrage opportunities, then there exists a probability distribution of future states of the market such that, for any function \( f(S) \), the price of a security with payoff \( f(S_T) \) is

\[
P_f = e^{-rT} E^p \{ f(S_T) \}
\]

• Conversely, if such a probability exists there are no arbitrage opportunities
Practical Application to European Options

• A pricing measure is a probability of future prices of the underlying asset with the property that

\[ E^p \{ S_T \} = F_{0,T} \]

• If we determine a suitable pricing measure, then all European options with expiration date \( T \) should have value given by

\[
\begin{align*}
Call(K,T) &= e^{-rT} E \left\{ (S_T - K)^+ \right\}, \\
Put(K,T) &= e^{-rT} E \left\{ (K - S_T)^+ \right\}
\end{align*}
\]

• The main issue is then to determine a suitable pricing measure in the real, practical world.
What does a pricing measure achieve in the case of options?

The pricing measure gives a model to compute the option’s fair value as a function of the price of the underlying asset, the strike and the maturity.

Option fair value is a smoothed out version of the payoff.
The Black-Scholes Model

Assume that the pricing measure is log-normal, i.e. log-returns are normal

$$S_T = S_0 e^X, \quad X \sim N(\mu T, \sigma^2 T)$$

$$E\{S_T\} = \int_{-\infty}^{+\infty} S_0 e^y e^{-\frac{(y-\mu T)^2}{2\sigma^2 T}} dy = S_0 e^{\mu T + \frac{\sigma^2 T}{2}} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right) T}$$

$$\therefore \quad \mu + \frac{\sigma^2}{2} = r - q \quad \therefore \quad \mu = r - q - \frac{\sigma^2}{2}$$

$$X = Z \sigma \sqrt{T} - \frac{\sigma^2}{2} T + (r - q) T, \quad Z \sim N(0, 1)$$
Call pricing with the Black-Scholes model

\[
\text{Call}(S, K, T) = e^{-rT} \mathbb{E}\left[(S_T - K)^+ \right] = e^{-rT} \int_{-\infty}^{+\infty} \left( Se^{\sigma \sqrt{T} - \sigma^2 T/2 + (r-q)T} - K \right)^+ e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}
\]

\[
= e^{-rT} \int_{A}^{+\infty} Se^{\sigma \sqrt{T} - \sigma^2 T/2 + (r-q)T} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - e^{-rT} K \int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}
\]

\[
A = \frac{1}{\sigma \sqrt{T}} \left( \ln \left( \frac{K}{S} \right) + \frac{\sigma^2 T}{2} - (r-q)T \right)
\]

\[
= e^{-qT} S \left( \int_{A}^{+\infty} e^{\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left( \int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right)
\]

\[
= e^{-qT} S \left( \int_{A - \sigma \sqrt{T}}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left( \int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right)
\]

\[
= e^{-qT} S \left( \int_{-\infty}^{-A + \sigma \sqrt{T}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left( \int_{-\infty}^{-A} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right)
\]
Black-Scholes Formula

\[ BSCall(S, T, K, r, q, \sigma) = S e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \]

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left( \ln \left( \frac{F_{0,T}}{K} \right) + \frac{\sigma^2 T}{2} \right), \quad d_2 = \frac{1}{\sigma \sqrt{T}} \left( \ln \left( \frac{F_{0,T}}{K} \right) - \frac{\sigma^2 T}{2} \right), \quad F_{0,T} = S e^{(r-q)T} \]

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \]  

cumulative normal distribution
Black-Scholes Formula at work

S=$48, K=$50, r=6%, sigma=40%, q=0
Multi-period asset model

- Derivative securities may depend on multiple expiration/cash-flow dates. Furthermore, the 1-period model described above is rigid in the sense that it cannot price American-style options.

- We consider instead a more realistic approach to pricing based on the statistics of stock returns over short periods of time (e.g. 1 day).

- We assume that the underlying price has returns satisfying

\[
\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} \sim N(\mu\Delta t, \sigma^2\Delta t)
\]

- We also assume that successive returns are uncorrelated.
Model closing prices, for example. The % changes between closing prices are normal and uncorrelated. 

\( S_t = \text{closing price of period } (t-1,t) \)
Parameterization

\[ \mu \Delta t = E \left\{ \frac{\Delta S_t}{S_t} \right\}, \quad \sigma^2 \Delta t = E \left\{ \left( \frac{\Delta S_t}{S_t} \right)^2 \right\} - \left( E \left\{ \frac{\Delta S_t}{S_t} \right\} \right)^2 \]

\[ \mu = \text{annualized expected return} \]

\[ \sigma = \text{annualized standard deviation} \]

1% daily standard deviation => 15.9% annualized standard deviation

\[ \Delta t = \frac{1}{252}, \quad \sqrt{252} = 15.9 \]
Pricing Derivatives

- Let us model the value of a derivative security as a function of the underlying asset price and the time to expiration

\[ V_t = C(S_t, t) \quad 0 < t < T \]

Change in market value over one period:

\[ \Delta V_t = \Delta C(S_t, t) \]

\[
\frac{\partial C(S_t, t)}{\partial t} \Delta t + \frac{\partial C(S_t, t)}{\partial S} \Delta S + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial t^2} \Delta t^2 + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} (\Delta S_t)^2 + \ldots
\]

\[
= \frac{\partial C(S_t, t)}{\partial t} \Delta t + \frac{\partial C(S_t, t)}{\partial S} \Delta S + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \left( \frac{\Delta S_t}{S_t} \right)^2 + o(\Delta t)
\]

\[
= \left( \frac{\partial C(S_t, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2 \right) \Delta t + \frac{\partial C(S_t, t)}{\partial S} \Delta S_t + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \left( \frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right] + o(\Delta t)
\]

\[ = \alpha \Delta t + \beta \Delta S_t + \epsilon \]
The hedging argument

• Consider a portfolio which is long 1 derivative and short $\beta$ stocks.

• Assume derivative does not pay dividends

Profit and loss, including financing and dividends:

$$PNL = -V_t \cdot r\Delta t + \Delta V_t - \beta(\Delta S_t - S_t r\Delta t + S_t q\Delta t)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta \Delta S_t + \varepsilon_t - \beta(\Delta S_t - S_t r\Delta t + S_t q\Delta t)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta S_t (r - q)\Delta t + \varepsilon_t$$

$$= \left(-C(S_t,t)r + \frac{\partial C(S_t,t)}{\partial t} + \frac{\partial C(S_t,t)}{\partial S} S (r - q) + \frac{1}{2} \frac{\partial^2 C(S_t,t)}{\partial S^2} S_t^2 \sigma^2 \right) \Delta t + \varepsilon_t$$
Analyzing the residual term $\mathcal{E}_t$

$$\mathcal{E}_t = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} \left[ \left( \frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right] + o(\Delta t)$$

$$E\{\mathcal{E}_t \mid S_t\} = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} E\left\{ \left( \frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \middle| S_t \right\} + o(\Delta t)$$

$$= S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} \mu^2 \Delta t^2 + o(\Delta t)$$

$$= o(\Delta t)$$

- The residual term has essentially zero expected return (vanishing exp. return in the limit $Dt_\rightarrow 0$.)
The fair value of our derivative security is...

• The PNL for the long short portfolio of 1 derivative and –$	ext{beta}$ shares has expected value

\[
E\{PNL\} = \alpha \Delta t + o(\Delta t)
\]

\[
= \left(-C(S_t, t) r + \frac{\partial C(S_t, t)}{\partial t} + \frac{\partial C(S_t, t)}{\partial S} S (r - q) + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2 \right) \Delta t + o(\Delta t)
\]

• This portfolio has no exposure to the stock price changes. Therefore, if $C(S_t, t)$ represents the ``fair value’’ of the derivative, the portfolio should have zero rate of return (we already took into acct its financing). Thus:

\[
\frac{\partial C(S_t, t)}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} + (r - q) S \frac{\partial C(S_t, t)}{\partial S} - C(S_t, t) r = 0
\]

• This is the Black-Scholes partial differential equation (PDE).
American-style calls & puts

- Consider a call option on an underlying asset paying dividends continuously. Since the option can be exercised anytime, we have

\[ C(S,t) \geq \max(S - K, 0), \quad t < T. \tag{1} \]

- The terminal condition at \( t=T \) corresponds to the final payoff

\[ C(S,T) = \max(S - K, 0). \]

- Thus, the function \( C(S,t) \) should satisfy the Black-Scholes PDE in the region of the \((S,t)\)-plane for which strict inequality holds in (1), and it should be equal to \( \max(S-K,0) \) otherwise.

- The solution of this problem is done numerically and will be addressed in the next lecture.