1. Consider a linear system of algebraic equation $Ax = b$. Here the matrix $A$ has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.

(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.

(c) Formulate, if possible, necessary and sufficient conditions on $A$ and $b$ which guarantee that at least one solution exists.

(d) Formulate, if possible, necessary and sufficient conditions on $A$ which guarantee that at least one solution exists for any choice of $b$.

2. A real matrix $A$ is said to skew-symmetric if $A^T = -A$. Prove that such a matrix has a full set of eigenvectors and that all its eigenvalues are purely imaginary, i.e., are of the form $ai$ where $a$ is a real number and $i^2 = -1$.

3. Consider

\[ < f, g > = \int_{-1}^{+1} t^2 f(t) \overline{g(t)} dt. \]

Here $f(t)$ and $g(t)$ are continuous functions.

(a) Show that this defines an inner product space $V$ over the complex field.

(b) Let the span of $1, t,$ and $t^2$ define a subspace of $V$. Create an orthonormal basis of this subspace.

(c) Find the second order polynomial which is closest to the function $t^3$ in the norm defined by the inner product defined above.
4. In Gaussian elimination, three matrices $P$, $L$, and $U$ are computed for a given square matrix $A$. Here $P$ represents a permutation of the rows of $A$, $L$ is lower triangular with diagonal elements all $= 1$, and $U$ is upper triangular and $PA = LU$.

(a) What is partial pivoting?
(b) How can $P$, $L$, and $U$ be used to solve a linear system of equations $Ax = b$?
(c) Can this algorithm ever fail if we use exact arithmetic?
(d) What characterizes a matrix $A$ for which the solution of $Ax = b$ is very sensitive to small changes in $b$?
(e) How does the work grow as a function of $n$ where $A$ is $n$ by $n$.

5. Show that the rank of any matrix is unchanged if it is multiplied from the left or the right by a square, nonsingular matrix of the appropriate size.

6. Consider a square matrix of order $n$ defined by $I - 2vv^T$. Here $v$ is column vector with $n$ real components.

(a) Under what condition is this a matrix with orthonormal columns?
(b) Give the geometric context of this type of transformation.
(c) How are such matrices used to solve linear least squares problems?

7. Consider a square matrix with real entries which has the following structure:
   • the only non-zero elements are on the diagonal and in the first row and first column of the matrix;
   • all the entries in the first row and first column are strictly larger than 0;
   • all diagonal elements are nonzero.

(a) Prove that such a matrix has only real eigenvalues;
(b) Prove that the rank of such a matrix must equal $n$ or $n - 1$, where $n$ is the number of rows of the matrix;
(c) Assume that the matrix is positive definite and symmetric and that we use Cholesky’s method to factor it into products of triangular matrices. What is the cost in terms of multiplications as a function of $n$?
(d) Can you think of a reordering of the matrix so that it is still symmetric and which would lead to a substantial decrease in the cost of using the Cholesky algorithm?