1 Interest Based Instruments

e.g., Bonds, forward rate agreements (FRA), and swaps.

Note that the higher the credit risk, the higher the interest rate.

Zero Rates: \( n \) year zero rate (or simply \( n \)-year zero) is the rate of interest earned on an instrument starting today and lasting for \( n \) years without intermediate payments.

For derivatives, there are 3 particularly important rates:

1. The Treasury rate — the rate applicable to borrowing by a government in its own currency.
   
   Assumption: Treasury rate is risk-free.
   
   This is because a government is unlikely to default, e.g., U.S. or Japan.

2. LIBOR rates (London InterBank Offer Rate) —
   
   the rate at which (1) large international banks fund much of their activities and at which (2) a large international bank is willing to lend to another large bank.

   N.B. LIBOR zero rates are higher than Treasury zero rates because there are some risks, e.g., the bank which borrows money may default.

3. Repo rate
   
   Repo — repurchase agreement:
   
   Contract:

   \[
   A \text{ sells some securities to } B \text{ now at } P_1 \\
   \text{ buys back later at } P_2 \\
   P_2 > P_1 \\
   \text{i.e., } B \text{ provides loan} \\
   P_2 - P_1 \text{ is equivalent to some interest earned — the repo rate}
   \]

   If structured carefully, it has low risks (because even if \( A \) defaults, \( B \) keeps the securities).

   (a) Repo rate is slightly higher than Treasury rate;
   
   (b) Over-night repo is an agreement negotiated every day;
   
   (c) Term repo is a longer term agreement.
1.1 Bond Prices and Term Structures

1.1.1 Time-Value of Money

Time value of money — discount factor:

\[ B(t,T) = \text{the value at time } t \text{ of$1 received at time } T. \]

Note that

1. By its definition, \( B(t,T) \) is the price at time \( t \) of a zero-coupon bond ($1 at time \( T \));
2. If interest rate is stochastic, then \( B(t,T) \) is not known until time \( t \);
3. If we view \( B(t,T) \) as a function of its maturity \( T \) — the term structure of interest rates.

Assumption:

\[ \text{At } t = 0, \quad B(0,T) \text{ is observable for all } T. \]

Equivalent ways to represent the time-value of money:

1. Yield \( Y(t,T) \):

\[ B(t,T) = e^{-Y(t,T)(T-t)} \]

\[ \therefore Y = -\frac{1}{T-t} \ln B(t,T) \]

2. Term Rate \( R(t,T) \):

\[ B(t,T) = \frac{1}{1 + R(t,T)(T-t)} \]

Note that

\[ 1 + R(t,T)(T-t) = \frac{1}{B(t,T)} \]

\[ \therefore R(t,T) = \frac{1}{T-t} \left( \frac{1}{B(t,T)} - 1 \right) = \frac{1-B(t,T)}{B(t,T)(T-t)} \]

3. The instantaneous forward rate \( f(t,T) \):

\[ B(t,T) = e^{-\int_t^T f(t,\tau)d\tau} \]

4. For U.S. Treasury bills, often the discount rate \( I(t,T) \) is tabulated:

\[ B(t,T) = 1 - I(t,T)(T-t) \]
### 1.1.2 Coupon Payments and Final Payments

Note that: The value of the bond at \( t = 0 \) is the sum of the present value of all future payments.

- e.g., For a fixed rate bond, the coupon payments of amount \( c_j \) at time \( t_j \) and the final payment of amount \( F \) at \( t = T \) are fixed at \( t = 0 \).

By the argument of no arbitrage, the value of the bond at time \( t \) is

\[
\text{the cash price} = \sum c_j B(t, t_j) + FB(t, T)
\]

Note that the cash price is **not** quoted in the newspaper. Instead,

\[
\text{quoted price} = \text{cash price} - \text{accrued interest}
\]

### 1.1.3 Floating-rate bonds

This kind of bonds has its interest rate (coupon rate) reset at each coupon date, i.e., after each coupon payment, its value returns to its **face value**.

- e.g., a one-year floating-rate note with semi-annual payments and face value =$1, pegged to the LIBOR.

Suppose

\[
\begin{align*}
\text{At } t &= 0, \text{ the LIBOR 0.5yr term interest rate} = 5.25\% \text{ p.a.} \\
\text{At } t &= 0.5yr, \text{ the LIBOR 0.5yr term interest rate} = 5.6\% \text{ p.a.}
\end{align*}
\]

**Payments**:
- at \( t = 0.5yr \):
  - **Coupon**: \( 0.0525 \times 0.5 = 0.02625 \)
- at \( t = 1yr \):
  - **Coupon**: \( 0.056 \times 0.5 = 0.028 \)
  - Plus $1, i.e., the bond matures.

**N.B.**

1. Interest rate is set at the beginning of a period;

2. the interest is paid at the end of the period.
Question: How to value floating-rate bonds?

Continuing our example above, Just after the first coupon payment, the value of the floating-rate bond at \( t = 0.5 \text{yr} \) is

(Letting \( B(t, T) \) be the value at \( t \) of a LIBOR contract worth $1 at \( T \).)

\[
B(t, T) \times \left( \text{coupon payment + face value} \right) = \frac{1}{1 + R(t, T)(T - t)} \left( \frac{0.028}{\text{coupon payment}} + \frac{1}{\text{face value}} \right), \quad \text{for } t = 0.5 \text{yr}, \ T = 1 \text{yr}
\]

\[
= \frac{1}{1 + 0.056 \times 0.5} (0.028 + 1) = 1
\]

i.e., at 6 months, the bond is worth $1.

At time \( t = 0 \), the value of the bond is

\[
B(0, t) \times (\text{the first coupon + value at 6 months}) = \frac{1}{1 + 0.0526 \times 0.5} (0.02625 + 1) = 1
\]

Therefore, the value of a floating-rate bond is equal to its principal right after each reset.

### 1.2 Forward rates and Forward Rate Agreements (FRA)

Consider the agreement: We want to borrow (or lend) at later time \( t \) with maturity \( T \) and we do not want to have cash changing hand now. At which rate do we need to set in this agreement?

Recall:

1. If interest rate is deterministic, then

\[
B(0, t) B(t, T) = B(0, T)
\]

by a no-arbitrage argument.

2. If the interest rate is stochastic, then,

\[
\begin{align*}
B(0, t) \\
B(0, T)
\end{align*} \quad \text{is known at time } t = 0
\]

\[
B(t, T) \text{ is unknown at time } t = 0
\]
Consider the portfolio:

(1) : long a zero-coupon bond with $1 at time $T$

$\Rightarrow$ The present value $= B(0, T)$

(2) : short a zero-coupon bond worth $\frac{B(0, T)}{B(0, t)}$ at time $t$

$\Rightarrow$ The present value $= -B(0, T)$

The net (at time $t = 0$) = 0

at time $t$ : we will have to pay $\frac{B(0, T)}{B(0, t)}$

at time $T$ : we will receive $1$.

Therefore,

\[
F_0(t, T) \equiv \frac{B(0, T)}{B(0, t)}
\]

$F_0(t, T)$ is the payment at time $t$ for $1$ at time $T$.

i.e.,

$F_0(t, T)$ is the discount factor (known today)

for borrowing at time $t$ with maturity $T$.

Note that, there are three times which are involved: $0, t, T$

1.2.1 Term rates

Define $f_0(t, T)$ by

\[
F_0(t, T) = \frac{1}{1 + f_0(t, T)(T - t)}
\]

where $f_0(t, T)$ is the forward term rate for borrowing from $t$ to $T$.

The meaning of term rates

An agreement now to borrow (or lend) at later time $t$ with maturity $T$ has the present value 0, if it sets the term rate $= f_0(t, T)$.
1.3 Detour: Why $f$ in $B(t,T) = e^{-\int_t^T f(t,s)ds}$ is called an instantaneous forward rate?

Consider a forward contract:

Agreeing at time $t$
to make a payment at a later time $T_1$
to receive a payment in return at time $T_2$ ($T_2 > T_1$)

Question: What is the price for this forward?
It is essentially striking a forward on the $T_2$-bond.

Replicating portfolio:

At time $t$:

Buy $T_2$-bond: 1 unit
Sell $T_1$-bond: $k$ units
\[ \therefore k \text{ is the payment at } T_1 \]

Initial cost at time $t$: $B(t,T_2) - kB(t,T_1)$

At time $T_1$: we can pay $\$k$.
At time $T_2$: we receive $\$1$.

Now if the initial cost is zero, therefore,
\[ k = \frac{B(t,T_2)}{B(t,T_1)} \]
which is the forward price of purchasing the $T_2$-bond at time $T_1$.

The corresponding forward yield is
\[ -\frac{\log B(t,T_2) - \log B(t,T_1)}{T_2 - T_1} \]

Let $T_1 = T$, $T_2 = T + \Delta T$, as $\Delta T \to 0$, a forward rate for instantaneous borrowing
\[ f(t,T) = -\frac{\partial}{\partial T} \log B(t,T) \]

therefore,
\[ B(t,T) = e^{-\int_t^T f(t,s)ds} \]

Question: What about a contract to borrow (or lend) at a rate $R_k$ instead of $f_0(t,T)$? This leads to the pricing of FRA.
1.4 Forward Rate Agreement (FRA)

Question: How to price it?

Let \( L \) be the principal — amount borrowed at time \( t \)

Consider:

Borrow \( L \) at time \( t \),

the payment at time \( T \) is

\[
\frac{L}{B(t,T)} = L (1 + R(T - t))
\]

where \( R \) is the term rate. For \( R = R_k \), the payment at time \( T \) is

\[
\left(1 + R_k \Delta T\right) = (1 + f_0(t, T) \Delta T) L + \left(R_k - f_0(t, T)\right) \Delta TL
\]

Therefore,

the present value of the FRA = \( B(0, T) (R_k - f_0(t, T)) \Delta TL \)

1.5 Alternative characterization of FRAs — relation between FRA and Swap.

We observe:

N.B. a FRA is equivalent to the following agreement:

Lending party:

pay interest at the market rate \( R(t, T) \)

receive interest at the contract rate \( R_k \)

Why?

The argument is as follows: View the principal \( L \) borrowed by the lender at time \( t \) at the market rate

repaying \((1 + R \Delta T) L\) at \( t \) = \( T \)

But receiving \((1 + R_k \Delta T) L\) at \( t \) = \( T \)
For the above situation,

at time $t$ : the net cash flow is 0
at time $T$ : net $(R_k - R) \Delta TL$

Note that, $R(t,T)$ is not known at time $t = 0$, instead, we use

$$R = f_0(t,T)$$

which gives the correct value of the contract at time $t = 0$.

Therefore,

1. An FRA is equivalent to an agreement when a fixed rate $R_k$ is exchanged for a market interest rate $R$.
2. An FRA can be valued by assuming

$$R = f_0(t,T)$$

i.e., using the forward term rate for the market interest rate at time $t$.

### 1.6 Swaps

Question: How to value a swap?

Example:

A:

receives fixed payments at 7.15% p.a.
floating-payments determined by LIBOR
2 payments per year with maturity 2 years

and $N$ is the notional principal.

1. Consider the fixed side of the swap:

   For discount factor $B(0,T)$, we can use either LIBOR or T-bill discount rates.
   Assume, for the four payment dates, we have

   $$B(0,t_1 = 182\text{days}) = 0.9679$$
   $$B(0,t_2 = 365\text{days}) = 0.9362$$
   $$B(0,t_3 = 548\text{days}) = 0.9052$$
   $$B(0,t_4 = 730\text{days}) = 0.8749$$
therefore, the value of the fixed side \((r = 7.15\%)\) is

\[
V_{fix} = N \times r \left( \frac{t_1 - 0}{365} \right) B(0, t_1) \\
+ N \times r \left( \frac{t_2 - t_1}{365} \right) B(0, t_2) \\
+ N \times r \left( \frac{t_3 - t_2}{365} \right) B(0, t_3) \\
+ N \times r \left( \frac{t_4 - t_3}{365} \right) B(0, t_4) \\
= 0.1317N
\]

— only the coupon payments

no final payment of the principal

2. Value the floating side of the swap:

Note that we do not know \(B(t_i, t_{i+1})\) at time \(t = 0\).

The value of the floating bond at time \(t = 0\) is \(N\), i.e., the notional principal.

Since we did not include the notional principal \(N\) on the fixed side, we will exclude this here also.

Since the value of a floating-rate bond is equal to its principal just after each reset:

\[
:\therefore V_{float} = N - B(0, t_4) N \\
:\therefore V_{float} = 0.1251N
\]

Note that only \(B(0, t_4)\) is needed for \(V_{float}\)

3. To the party receiving the fixed rate, paying the floating rate, the value of the swap is

\[
V_{swap} = V_{fix} - V_{float} = 0.0066N
\]

the value to the other party is

\[-V_{swap} = -0.0066N\]

4. Par Swap Rate

At fixed rate \(r = 7.15\%\), the swap value does not vanish.

Question: What is the par swap rate, which is the fixed rate that makes the swap have zero value at time \(t = 0\)?
Let $x$ be the rate, obviously, it satisfies

\[
N \times x \left( \frac{t_1 - 0}{365} \right) B(0, t_1) + N \times x \left( \frac{t_2 - t_1}{365} \right) B(0, t_2) + N \times x \left( \frac{t_3 - t_2}{365} \right) B(0, t_3) + N \times x \left( \frac{t_4 - t_3}{365} \right) B(0, t_4) = N - B(0, t_4) N
\]

which yields

\[
1.8421x = 0.1251 \\
x = 0.0679
\]

i.e., the par swap rate is 6.79%.

**Foreign Currency Swaps:**
Exchanging a floating-rate income stream in a foreign currency for a fixed-rate income stream in dollars — to eliminate foreign currency risks.

In summary, by comparing two bonds, the value of a swap is

\[
\sum_{i=1}^{M} B(0, t_i) R_{fix} (t_i - t_{i-1}) N - (1 - B(0, t_M)) N
\]

Note that an alternative view is

\[
\text{swap} = \text{a collection of forward rate agreements}
\]

This is because the value at time $t = 0$ of FRA with payments at time $t = t_i$ is

\[
B(0, t_i) (R_{fix} - f_0 (t_{i-1}, t_i)) (t_i - t_{i-1}) \times N
\]

\[
\therefore f_0 (t_{i-1}, t_i) (t_i - t_{i-1}) = \frac{B(0, t_{i-1})}{B(0, t_i)} - 1
\]

for the payment dates, $0 < t_1 < t_2 < \cdots < t_M$, we have (Homework),

\[
\sum_{i=1}^{M} B(0, t_i) (R_{fix} - f_0 (t_{i-1}, t_i)) (t_i - t_{i-1}) \times N
\]

\[
= \sum_{i=1}^{M} B(0, t_i) R_{fix} (t_i - t_{i-1}) N - (1 - B(0, t_M)) N.
\]
1.7 Caps, Floors, and Swaptions

1.7.1 Swaption
Swaption — option on a swap, i.e., when it matures, its holder has the right to enter into a specified swap contract.

Note that

1. the holder will do so only if the swap contract has a positive value.
2. Swaps can be a liability.

Note that

1. Since a swap is equivalent to a pair of bonds, a swaption is an option on a pair of bonds;
2. Since a swap is a collection of FRAs, a swaption is an option on a collections of FRAs.

1.7.2 Caps
To insure against the worst scenario of a high interest rate, a cap pays the difference between the market interest rate and a specified cap rate at each coupon data if the difference is positive, i.e., the borrow never has to pay rates above the cap.

Note that, caps are call options on the market rate.

1.7.3 Floors
To insure against low interest rate. Note that, floors are put options on the market rate.

**Put-call parity:**

\[ \text{cap} - \text{floor} = \text{swap} \]

if the fixed rate is the same for all three.
2 Equivalent Martingale Measures

Recall: If \( f \) and \( g \) are some derivatives on the same, single process \( dW \),

\[
\text{No Arbitrage } \implies \frac{f}{g} \text{ is a martingale with respect to some measure}
\]

More specific,

\[
\begin{align*}
\text{df} &= \mu_f f dt + \sigma_f f dW \\
\text{dg} &= \mu_g g dt + \sigma_g g dW 
\end{align*}
\]

which are not necessarily geometric Brownian motions since \( \mu_f \) and \( \sigma_f \) can depend on \( f \), etc. An no arbitrage argument yields

\[
\frac{\mu_f - r}{\sigma_f} = \lambda = \frac{\mu_g - r}{\sigma_g}
\]

For a new market price of risk \( \lambda^* \),

\[
\lambda^* = \frac{\mu^* - r}{\sigma^*}
\]

we have

\[
\begin{align*}
\text{df} &= (r + \lambda^* \sigma_f) f dt + \sigma_f f dW \\
\text{dg} &= (r + \lambda^* \sigma_g) g dt + \sigma_g g dW 
\end{align*}
\]

(1)

(2)

Note that

1. Market price of risk determines the drift;
2. Volatility does not change;
3. Choosing a drift is equivalent to choosing a market price of risk \( \lambda \).

From the above argument, we conclude that:

\[
\text{No Arbitrage } \implies \frac{f}{g} \text{ is a martingale for some } \lambda
\]

The question is which \( \lambda \). It turns out that

\[
\text{if } \lambda = \sigma_g, \\
\text{then, } \frac{f}{g} \text{ is a martingale for all derivative } f.
\]

This can be demonstrated as follows:
Substituting $\lambda = \sigma_g$ into Eqs. (1) and (2) yields
\[
\begin{align*}
    df &= (r + \sigma_g \sigma_f) f dt + \sigma_f f dW \\
    dg &= (r + \sigma_g^2) g dt + \sigma_g g dW
\end{align*}
\]

Applying Ito’s lemma to $\ln f$ and $\ln g$:
\[
\begin{align*}
    d\ln f &= \left(r + \sigma_g \sigma_f - \frac{1}{2} \sigma_f^2\right) dt + \sigma_f dW \\
    d\ln g &= \left(r + \frac{1}{2} \sigma_g^2\right) dt + \sigma_g dW
\end{align*}
\]

therefore,
\[
\begin{align*}
    d\ln \frac{f}{g} &= d(\ln f - \ln g) \\
                   &= d\ln f - d\ln g \\
                   &= \frac{1}{2} (\sigma_f - \sigma_g)^2 dt + (\sigma_f - \sigma_g) dW
\end{align*}
\]

Now, we want to use this to compute $d\left(\frac{f}{g}\right)$ by the following method:

If we know
\[
\begin{align*}
    dX &= \mu_X dt + \sigma_X dW \\
    d\ln X &= \mu dt + \sigma dW
\end{align*}
\]

what is the relation between $(\mu_X, \sigma_X)$ and $(\mu, \sigma)$? Since
\[
\begin{align*}
    d\ln X &= \frac{1}{X} dX + \frac{1}{2} \sigma_X^2 \left(-\frac{1}{X^2}\right) dt \\
           &= \left(\frac{1}{X} \mu_X - \frac{1}{2} \sigma_X^2 \frac{1}{X^2}\right) dt + \frac{\sigma_X}{X} dW \\
\end{align*}
\]

\[
\begin{align*}
    \mu &= \frac{1}{X} \mu_X - \frac{1}{2} \sigma_X \frac{1}{X^2} \\
    \sigma &= \frac{\sigma_X}{X}
\end{align*}
\]

i.e.,
\[
\begin{align*}
    \sigma_X &= \sigma X \\
    \mu_X &= \left(\mu + \frac{1}{2} \sigma^2\right) X
\end{align*}
\]

therefore,
\[
\begin{align*}
    d\left(\frac{f}{g}\right) &= \left[ -\frac{1}{2} (\sigma_f - \sigma_g)^2 + \frac{1}{2} (\sigma_f - \sigma_g)^2 \right] dt + (\sigma_f - \sigma_g) \frac{f}{g} dW
\end{align*}
\]
i.e.,

\[ d \left( \frac{f}{g} \right) = (\sigma_f - \sigma_g) \frac{f}{g} dW \]

in which there is no drift term, thus, \( \frac{f}{g} \) is a martingale.

Note that

1. When the market price of risk = \( \sigma_g \), it is a world of forward risk-neutral with respect to \( g \).
2. Since \( \frac{f}{g} \) is a martingale,

\[
\therefore \quad \frac{f_0}{g_0} = \mathbb{E}_g \begin{bmatrix} f_T \\ g_T \end{bmatrix}
\]

\[
\Rightarrow
\]

\[
f_0 = g_0 \mathbb{E}_g \begin{bmatrix} f_T \\ g_T \end{bmatrix}
\]

(a) Money Market Account as the Numeraire:

Since

\[ dg = rg \, dt \]

where \( r \) can be stochastic, but the volatility of \( g = 0 \), i.e., the market price of risk is zero for the money market. Then

\[
f_0 = g_0 \mathbb{E}_{RN} \begin{bmatrix} f_T \\ g_T \end{bmatrix}
\]

Since

\[
g_0 = 1
\]

\[
g_T = e^{\int_0^T r(\tau) \, d\tau}
\]

\[
\therefore \quad f_0 = \mathbb{E}_{RN} \left[ e^{-\int_0^T r(\tau) \, d\tau} f_T \right]
\]

If \( r \) is a constant, then

\[
f_0 = e^{-rT} \mathbb{E}_{RN} \left[ f_T \right]
\]

Hence, the money market account numerarie is equivalent to traditional risk-neutral world.

(b) Zero-Coupon bond price as the Numeraire:

Definition: \( B(t, T) \) is the price at time \( t \) of a zero-coupon bond worth of \$1 at time \( T \).

Then,

\[
g_T = B(T, T) = 1
\]

\[
g_0 = B(0, T)
\]
\[ f_0 = B(0, T) \mathbb{E}_T [f_T] \]

where \( \mathbb{E}_T \) denotes the forward risk-neutral measure with respect to \( B(t, T) \). Note that it is nice to have \( B(0, T) \) outside \( \mathbb{E} \)-operator.

Recall the forward price of \( f \) maturing at \( T \) is

\[ F = \frac{f_0}{B(0, T)} \]

e.g., \( F = S_0 e^{rT} \). Since

\[ f_0 = B(0, T) \mathbb{E}_T [f_T] \]
\[ F = \frac{f_0}{B(0, T)} \]

\[ \therefore F = \mathbb{E}_T [f_T] \]

i.e., In a forward risk-neutral measure with respect to \( B(0, T) \), the forward price of \( f \) is equal to the expected future spot price. In contrast, in the traditional risk-neutral measure, futures price is equal to the expected future spot price.

Intuitive way (via binomial trees) of understanding change of numeraire:

Since

\[ f_{\text{now}} = e^{-r\delta t} (q f_{\text{up}} + (1-q) f_{\text{down}}) \]

where \( q \) is the risk-neutral probability depending on the underlying movement. For another tradeable \( g \), we have

\[ g_{\text{now}} = e^{-r\delta t} (q g_{\text{up}} + (1-q) g_{\text{down}}) \]

therefore

\[ \frac{f_{\text{now}}}{g_{\text{now}}} = \frac{e^{-r\delta t} (q f_{\text{up}} + (1-q) f_{\text{down}})}{e^{-r\delta t} (q g_{\text{up}} + (1-q) g_{\text{down}})} \]

\[ = \frac{q g_{\text{up}} f_{\text{up}}}{q g_{\text{up}} + (1-q) g_{\text{down}} g_{\text{up}}} + \frac{(1-q) g_{\text{down}}}{q g_{\text{up}} + (1-q) g_{\text{down}} g_{\text{down}}} f_{\text{down}} \]

If

\[ q^* \equiv \frac{q g_{\text{up}}}{q g_{\text{up}} + (1-q) g_{\text{down}}} \]

Since

\[ \frac{(1-q) g_{\text{down}}}{q g_{\text{up}} + (1-q) g_{\text{down}} + q^*} + q^* \]
\[ = \frac{(1-q) g_{\text{down}}}{q g_{\text{up}} + (1-q) g_{\text{down}}} + \frac{q g_{\text{up}}}{q g_{\text{up}} + (1-q) g_{\text{down}}} \]
\[ = \frac{(1-q) g_{\text{down}} + q g_{\text{up}}}{q g_{\text{up}} + (1-q) g_{\text{down}}} = 1 \]
i.e.,
\[ \frac{(1 - q) g_{\text{down}}}{q g_{\text{up}} + (1 - q) g_{\text{down}}} = 1 - q^* \]
therefore,
\[ \frac{f_{\text{now}}}{g_{\text{now}}} = q^* \left( \frac{f_{\text{up}}}{g_{\text{up}}} \right) + (1 - q) \left( \frac{f_{\text{down}}}{g_{\text{down}}} \right) \]
Of course, \( q^* \) will vary from subtree to subtree. Therefore,
\[ \frac{f_{\text{now}}}{g_{\text{now}}} = \mathbb{E}^* \left[ \frac{f_{\text{next}}}{g_{\text{next}}} \right] \]
where \( * \) denotes the expectation with respect to \( q^* \). Iterate through the tree, we have
\[ \frac{f(t)}{g(t)} = \mathbb{E}^* \left[ \frac{f(T)}{g(T)} \right] \]
therefore,
\[ \frac{f}{g} \] is a martingale with respect to \( q^* \)
i.e.,
\[ f(t) = g(t) \mathbb{E}^* \left[ \frac{f(T)}{g(T)} \right] \]

3 Black’s Model
— Application of equivalent martingale measures

3.1 Black-Scholes Result with stochastic interest rates
1. Consider a European call option on a non-dividend-paying stock with maturity \( T \),
\[ c = B(0, T) \mathbb{E}_T \left[ (S_T - K)_+ \right] \]
where \( \mathbb{E}_T \) denotes the forward risk-neutral measure with respect to a zero-coupon bond maturing at \( T \).
Define an effective rate \( R \) (zero rate) by
\[ B(0, T) = e^{-RT} \]
Assume
\[ S_T \sim \text{lognormal in the forward risk-neutral world} \]
with standard deviation of \( \ln S_T = \tilde{\sigma} \)
therefore,
\[ E_T [(S_T - K)_+] = E_T [S_T] N(d_1) - KN(d_2) \]
\[ d_{1,2} = \frac{1}{\bar{\sigma}} \left[ \ln \left( \frac{E_T [S_T]}{K} \right) \pm \frac{\bar{\sigma}^2}{2} \right] \]

Since
\[ E_T [S_T] \] is the forward price
\[ : \quad E_T [S_T] = S_0 e^{RT} \]

Redefine
\[ \bar{\sigma} = \sigma \sqrt{T} \]

then
\[ c = S_0 N(d_1) - Ke^{-RT} N(d_2) \]
\[ d_{1,2} = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S_0}{K} \right) + \left( R \pm \frac{\sigma^2}{2} \right) T \right] \]

What we just described is the Black’s model.

### 3.2 Black’s Model

#### 3.3

Let \( V \) be an interest-rated instrument:

Payoff function — \( f(V) \)

Black’s model: Assumptions:

1. Option value = \( B(0, T) E_T[f(V_T)] \)

2. \( V_T \sim \text{lognormal, i.e., } V_T = e^X, \quad X \sim \text{normal.} \)

Forward price of \( V = E_T[V_T] \)

Example: a European call option on \( T_B \)-discount bond maturing at \( T < T_B \),
\[ c = B(0, T) E_T [(B(T, T_B) - K)_+] \]
Example: Black’s model to caps:

\[
\{t_i\}
\]

\[R_i = R(t_i, t_{i+1}) \quad \text{the term rate for } [t_i, t_{i+1}]\]

\[R_K = \text{the cap rate}\]

\[L = \text{principal}\]

The \(i^{th}\) caplet pays at \(t_{i+1}\):

\[L \cdot (t_{i+1} - t_i) \cdot (R_i - R_K)_+\]

Using the Black’s model,

the present value = \(B(0, t_{i+1}) L \Delta t \left[ f_i \times N(d_{1i}) - R_K \times N(d_{2i}) \right] \)

where \(f_0 = f_0(t_i, t_{i+1})\) is the forward term rate, i.e.,

\[
\frac{1}{1 + f_i \Delta t} = \frac{B(0, t_{i+1})}{B(0, t_i)}
\]

which is known today.

\[d_{i,2i} = \frac{1}{\sigma_i \sqrt{t_i}} \left[ \ln \left( \frac{f_i}{R_K} \right) \pm \frac{\sigma_i^2 t_i}{2} \right]\]

where \(\sigma_i\) has to be specified for each \(i\) — inferred from the market.

Finally, summing up all caplets gives the price of the cap.

Example: Swaption

Consider the swap:

\[T = t_0 < t_1 < t_2 < \cdots < t_N\]

which are coupon dates of the swap.

Let \(R_{\text{swap}}\) be the par swap rate at time \(t = T\), therefore

\[
\sum_{i=1}^{N} B(T, t_i) R_{\text{swap}} (t_i - t_{i-1}) L = (1 - B(T, t_N)) L
\]

\[\text{fixed side}\]

\[\text{floating side}\]
Consider a swaption (an option on the swap above): the holder has the right to pay the fixed rate $R_K$ and receive the floating rate on date $T$.

If it were not a right, then, the value to its holder at time $t = T$ is

$$V_{\text{float}} - V_{\text{fix}} = (1 - B(T, t_N))L - \sum_{i=1}^{N} B(T, t_i) \Delta_{i-1} t R_K L$$

$$= \sum_{i=1}^{N} B(T, t_i) R_{\text{swap}} \Delta_{i-1} t L - \sum_{i=1}^{N} B(T, t_i) \Delta_{i-1} t R_K L$$

$$= (R_{\text{swap}} - R_K) \sum_{i=1}^{N} B(T, t_i) \Delta_{i-1} t L$$

As an option, it has cash flow at $t = t_i$:

$$L \Delta_{i-1} t (R_{\text{swap}} - R_K)_+$$

therefore, the corresponding value at $t = 0$ is

$$B(0, t_i) L \Delta_{i-1} t \left[ F_{\text{swap}} \times N(d_1) - R_K \times N(d_2) \right]_{\text{forward swap rate}}$$

with

$$d_{1,2} = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F_{\text{swap}}}{R_K} \right) \pm \frac{\sigma^2 T}{2} \right]$$

which is independent of $i$. and

$$F_{\text{swap}} = \mathbb{E}_T[R_{\text{swap}}]$$

The total value at $t = 0$ is

$$\sum_{i=1}^{N} B(0, t_i) L \Delta_{i-1} t [F_{\text{swap}} \times N(d_1) - R_K \times N(d_2)]$$

$$= LA \left[ F_{\text{swap}} \times N(d_1) - R_K \times N(d_2) \right]$$

with

$$A \equiv \sum_{i=1}^{N} B(0, t_i) L \Delta_{i-1} t$$

Question: How to compute the forward swap rate?
Note that a swap from the fixed side is a claim $X$

$$X = L (1 - B (T, t_N)) - \sum_{i=1}^{N} B (T, t_i) K \Delta_{i-1} t \times L$$

where $K$ is some rate. The present value is

$$V (t = 0) = L (B (0, T) - B (0, t_N)) - \sum_{i=1}^{N} B (0, t_i) K \Delta_{i-1} t \times L$$

then the forward swap rate (i.e., making $V (t = 0) = 0$) is determined by

$$L (B (0, T) - B (0, t_N)) - \sum_{i=1}^{N} B (0, t_i) K \Delta_{i-1} t \times L = 0$$

i.e.,

$$K = \frac{B (0, T) - B (0, t_N)}{\sum_{i=1}^{N} B (0, t_i) \Delta_{i-1} t} \equiv F_{\text{swap}}$$

which is the forward swap rate, or

$$K = \frac{1 - F (T, t_N)}{\sum_{i=1}^{N} F (T, t_i) \Delta_{i-1} t}$$

where $F (T, t_N)$ etc. are the forward price at $t = 0$ to buy a $t_N$-bond at time=$T$. This can be achieved by the following way, in the equation used to determine the par swap rate, i.e.,

$$\sum_{i=1}^{N} B (T, t_i) R_{\text{swap}} \Delta_{i-1} t \times L = (1 - B (T, t_N)) L$$

replacing $B (T, t_i)$ with its forward price, and $R_{\text{swap}}$ with its forward swap rate $F_{\text{swap}}$ yields

$$\sum_{i=1}^{N} F (T, t_i) F_{\text{swap}} \Delta_{i-1} t \times L = (1 - F (T, t_N)) L$$

which gives an equation for $F_{\text{swap}}$.

### 3.4 Pricing using PDEs in the continuous payment setting

Stochastic spot rate:

$$dr = m (r, t) dt + \Sigma (r, t) dW$$

**Swap:**

$A$ pays interest on an amount $N$ to $B$ at a fixed rate $r^*$

$B$ pays floating interest rate $r$. 

20
In a time interval $dt$, $A$ receives

$$N(r - r^*)dt \quad \text{— viewed as coupons on bond}$$

therefore, $NV(r, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} + (m - \lambda \Sigma) \frac{\partial V}{\partial r} - rV + (r - r^*) = 0$$

final data— $V(r, T) = 0$

where $\lambda$ is the market price of risk. For many interest models, e.g., Vasick, CIR, Hull-White, $\lambda$ does not show up in the evaluation.

**Caps & Floors**

Caps:
$NV(r, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} + (m - \lambda \Sigma) \frac{\partial V}{\partial r} - rV + \min(r - r^*) = 0$$

final data— $V(r, T) = 1$

**Swaptions**

Swap: expires at $T_s$, its value $V_s(r, t)$ for $t \leq T_s$
Option: e.g., call on this swap for strike $K$ at $T < T_s$
Swaption:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V}{\partial r^2} + (m - \lambda \Sigma) \frac{\partial V}{\partial r} - rV = 0$$

final data— $V(r, T) = (V_s(r, T) - K)_+$

Swap:

$$\frac{\partial V_s}{\partial t} + \frac{1}{2} \Sigma^2 \frac{\partial^2 V_s}{\partial r^2} + (m - \lambda \Sigma) \frac{\partial V_s}{\partial r} - rV_s + (r - r^*) = 0$$

final data— $V_s(r, T) = 0$

N.B. Solve swap $V_s$ first, then solve for swaption $V$. 

21