Homework Assignment

1. (Due Dec 30) **Improving efficiency of Rejection Method**: By rejection from a Gaussian,

\[ \rho_0(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{x^2}{2\sigma^2} \right], \]

we can sample from the density

\[ \rho(x) = \frac{1}{Z} e^{-x^4/4} \]

where \( Z = \int_{-\infty}^{+\infty} e^{-x^4/4} dx \). Find an optimal value of \( \sigma \) such that the rejection method is efficient.

2. (Due Dec 7) **Rigidity of Covariance Matrix**: In the following, you are asked to compare results between \( N = 5 \) and \( N = 15 \), or even a larger value of \( N \), for \( n = 10 \) and \( n = 1000 \) or even larger value of \( n \).

   (a) Generate a valid \( N \times N \) covariance matrix \( \Sigma_{N\times N} \) (You can use any method at your disposal to generate the matrix. A valid covariance matrix should be able to pass the Choleski test).

   (b) Generate \( n \) independent observations, i.e., \( n \) normal deviates, \( \{x_i, i = 1, 2, \ldots, n\} \), from \( \mathcal{N}(0, \Sigma_{N\times N}) \). Note that \( x \) is an \( N \)-dimensional vector.

   (c) Then compute an empirical covariance matrix \( \tilde{\Sigma}_{N\times N} \) using these \( n \) observations.

   (d) Examine whether this empirical matrix \( \tilde{\Sigma}_{N\times N} \) can pass the Choleski test.

   (e) Perturb some entries in the matrix \( \tilde{\Sigma}_{N\times N} \) by a few percent to see if the resulting matrix can still pass the Choleski test.

3. (Due Dec 7) **The Order of Strong Convergence**: Solving the stochastic differential equation

\[ dS = \mu S dt + \sigma S dW \]

with the initial value \( S = S_0 \) at \( t = 0 \), where \( \mu, \sigma \) are constant, using the following methods,

   (a) the exact solution advancement to obtain \( J \) trajectories, \( \{S^j(t_i), i = 1, 2, \ldots, n\}, \ j = 1, 2, \ldots, J, t_0 = 0, \ t_i = t_{i-1} + h, \ t_n = T \).

   (b) the Euler scheme to obtain \( J \) approximate trajectories \( \{\hat{S}^j_h(t_i), i = 1, 2, \ldots, n\}, \ j = 1, 2, \ldots, J \).

   (c) Then analyze the error in the sense of strong convergence using the estimate,

\[ \varepsilon(h) \equiv \frac{1}{J} \sum_{j=1}^{J} \left| \hat{S}^j_h(T) - S^j(T) \right| \]

What is the slope of \( \varepsilon(h) \) vs \( h \) using the log-log plot?

You can choose \( S_0 = 20, \mu = 0.05, \sigma = 0.25, T = 1, J = 100, \) and \( h = T/n, \ n = 100, 1000, 10000, 100000. \)