Math Finance Lecture 1

References: See syllabus

Homeworks

One Final

Office Hours

Form of Exams:

1. Central Doma: No Arbitrage
   \[ \text{Reduces price of derivatives} \]

   Underlyings: e.g.
   1. Stocks, Bonds
   2. Currencies
   3. Commodities e.g. oil, gold

NB: it does not constrain on the underlyings

Why? Risk of underlying \( \rightarrow \) Profit/Loss
\( \text{No Arbitrage} \rightarrow \text{No Risk Pricing} \)

NB: Time-value of money

2. Martingale approach vs. PDE
Derivatives securities: forwards, futures, options (Call/put)

Riggs

No. Swaps can be decomposed into forwards x options

1. Forward Contract (at maturity $T$ and delivery price $K$)

Buy a forward — holder is obligated to buy the underlying asset at price $K$ on date $T$.

Terminology: Buy a forward $\rightarrow$ hold a long forward

How to price? — central question: Scholes

Stocks $\sim$ Log-normal

$$\log S_T = \log S_0 + \mathcal{N}(\mu, \sigma^2)$$

i.e. $S = S_0 e^{\mathcal{N}(\mu, \sigma^2)}$

Forward: (i.e. short stock long forward)

value at expiration $T$: $S_T - K$

time value: $e^{-rT}(S_T - K)$
Traditional (Conventional):

The expected value: \( \mathbb{E}(e^{-t(S_t - k)}) \)

(This should be zero: no one profits)

\[ \mathbb{E}[e^{-t(S_t - k)}] = 0 \]

\[ k = \mathbb{E}(S_t) \]

By parts:

\[
K = \int_{-\infty}^{\infty} s \cdot \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(s-\mu)^2}{2\sigma^2}} dx
\]

\[
= \int_{-\infty}^{\infty} s \cdot \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(s-\mu)^2}{2\sigma^2}} dx
\]

\[
= S_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(s-\mu)^2}{2\sigma^2}} dx
\]

\[
- \frac{x^2}{2\sigma^2} + \frac{6\mu x}{2\sigma^2} - \frac{6\mu^2}{2\sigma^2} + x
\]

\[
K = S_0 \cdot e^{\mu + \frac{15}{2}\sigma^2}
\]

Is this the correct price?
Alternative Pricing Scheme:

Assumptions:
1. Underlying asset pays no dividends
   and has no carrying cost
   (e.g. a non-dividend paying stock)
2. Time value of money is computed using compound interest code $r$

i.e. $x_D$ at time $T$ (future time)
   is worth $x_D e^{-rT}$

   e.g. $x_D$ is worth $x_D$ at money $T$

   Buy a bond — hold a long bond
   i.e. lend $x_D e^{-rT}$ to be paid at time $T$

Consider portfolios:

Portfolio 1: one long forward maturity $T$
   & delivers price $K$

  payoff: $S_T - K$

  put one stock, pays $K$
Claim:
The delivery price should be $S_0 e^{rt}$ if no cash change hand at $t = 0$.

Why? If not priced at $S_0 e^{rt}$

Suppose the delivery price is $P$

2 possibilities:

1. $P > S_0 e^{rt}$
2. $P < S_0 e^{rt}$

Consider 1:

Constant portfolio:
- Short bond: $S_0$
- Stock $S_0$

$S_0 + S_0$ (present)
$= S_0 + S_0 = 0$
(no cash changing hands)

In the future:
- Cash: $S_0 \rightarrow k = S_0 e^{rt}$
- Stock: $S_0 \rightarrow S_f$

$p - S_0 e^{rt} > 0$
Sell forward, buy (*)
-$p$ to the bank
Gain $S_0 e^{rt}$
Payoff $S_0 e^{rt}$

Regardless of the future price of the stock,

- Profit at no risk.
Either way → Market forces (oversupply of sellers or buyers)

→ Price adjustment

Determining the delivery price

\[ K = S_0 e \]

The underlying principle is the no-arbitrage!

Examples of options:

1. European call option (251 maturity \( T \) and strike price \( K \))

   Buy a call ↔ hold a long call

   i.e. Holder has right to buy the underlying asset at price \( K \) on date \( T \).

2. European put option (251 maturity \( T \) and strike price \( K \))

   Buy a put ↔ hold a long put

   i.e. Holder has right to sell the underlying asset at price \( K \) on date \( T \).

1. \( \text{Forward} \) — obligation for the holder

1. \( \text{Call/put} \) — right

1. \( \text{An writer of an option} \) — potential obligation to sell if the holder chooses to pay.
Forward option — Contingent Claim

- The value at maturity — not known
- Depending on the value of the underlying on date $T$

Payoff:

**Forward:** $S_T - K$

**Call:** $(S_T - K)^+$

- If $S_T > K$ at $T$ → exercise $S_T - K$
- If $S_T < K$ → Log on stock $S_T$ → Payoff = 0

$$ (S_T - K)^+ $$

**Call PDO:**

- Notation: $x^+ = \begin{cases} S_x, & x \geq 0 \\ 0, & x < 0 \end{cases}$

**Put:** Payoff $(K - S_T)^+$

**Put PDO:**

- Notation: $x^-$

- If you think the stock is going up → Call

- If you expect the stock is going down → Buy a put
18: 10 A long position vs. short position
Buyer of a claim has a long position
⇒ seller has a short position

Payoff diagram of a short position
= negative of payoff diagram of long position

E.g. Short a call

\[
\text{payoff}
\]

\[
\begin{align*}
K & \quad 5_t \\
- (5_t - K) + &
\end{align*}
\]

18: The loss is unlimited.

2° European vs. American options
also cf. asian option

American:
Can exercise anytime \( t \leq T \)

E.g. American call on strike \( K \) at maturity \( T \)
Can be exercised at anytime \( t \in [0, T] \)
⇒ The holder has right to buy the underlying
for price \( K \) at \( 0 \leq t \leq T \).

Q: What is the optimal time for exercise?
Why Contingent Claims?
1. Hedge
2. Speculate

Q. What is hedge? You want certainty

E.g. A Chinese airline has a contract to buy an Airbus for a price fixed in £ payable one year from now.

How to hedge: go long on a forward contract for £ (payable in ¥)

⇒ Elimination of the foreign currency risk.

E.g. Holder of a forward

a Call:

Eliminates the downside risk
But you have to pay more for a call.

If you don’t want to pay that much, you can use Bull spread to protect the downside risk.
**Butterfly Spread**

payoff: 

\[ \text{bull} \rightarrow (S_t-K_1)^+ \rightarrow \text{give up some upside of a call} \rightarrow \text{should be cheaper than a call (K_1, T)} \]

\[ K_1 \quad K_2 \]

\[ \text{long a call at } K_1 \]

\[ \text{short a call at } K_2 \]

\[ K_1 \quad K_2 \quad S_t \]

\[ \text{iff: } \text{long a call (K_1, T) and short a call (K_2, T)} \]

\[ \quad = \text{Butterfly Spread (K_1, K_2, T)} \]

\[ \text{i.e. } (S_t-K_1)^+ - (S_t-K_2)^+ = \text{payoff of a butterfly spread} \]

\[ (K_1 < K_2) \]

**Option for Speculation**

**Reason:** The option is more sensitive to price change than the underlying.

i.e. Leveraged by option to speculate

**Consider:** A Euro call at strike K = 50.

- At time \( t \) near maturity, the value of the option

\[ \sim (S_t - K)^+ \]
$S_t = 60$ now

Q: 1) What happens if $S_t$ increases by 10% i.e. $S_t = 66$. The value of the option increases from about $60.50 = 10 \rightarrow 66\%$ increase to $66.50 = 16$

Q: 2) If $S_t$ decreases by 10% i.e. $S_t = 54$. The value of the option (from $60.50 = 10 \rightarrow 66\%$ loss to $54.50 = 4$

Sensitivity $\frac{\Delta C}{\Delta S} = \Delta$ (not necessarily near maturity)

the percentage change is $\frac{\Delta C}{\Delta S}$ vs $\frac{\Delta S}{S}$

$\Delta < 1$ in Black-Scholes but $\frac{\Delta S}{S}$ can be greater than 1

Q: What are principles in pricing?

1) If 2 portfolios have the same payoff, then their present values must be the same

2) If portfolio 1's payoff is always at least as good as portfolio 2's, then the present value of portfolio 1 $\geq$ the present value of portfolio 2

Q: Why? Violation $\rightarrow$ Arbitrage.
1. Price a Forward

Assume 1. Underlying asset pays no dividend and has no carrying cost (e.g., a non-dividend-paying stock)

2. Time value of money is compounded using compound interest (rate \( r \))
   i.e., \( \$D \) at time \( T \) (in the future) is worth now \( \$D e^{-rT} \)

In other words, we have a bond:

\[ \text{Bond} \quad \text{worth } \$D \text{ at } T \text{ (maturity)} \]

Buy a bond \( \rightarrow \) hold a long bond
\( \rightarrow \) lend \( \$D e^{-rT} \) to be repaid at time \( T \) at interest.

Consider 2 portfolios:

Portfolio 1: one long forward at maturity \( T \) and delivery price \( K \)

Payoff: \( S_T - K \)

Portfolio 2: long one unit of stock \( \{ \$S_T \text{ at maturity} \}
short one bond \( \{ \text{Present value } - K e^{-rT} \text{ value at maturity} - K \}

\rightarrow \text{payoff } S_T - K \text{ also same as portfolio 1.}

Our principle \( \rightarrow \) same present value

\[ \text{Present value of a forward} = S_0 - K e^{-rT} \]
In practice, forwards are written so that the present value is 0 (i.e., no payment changing hands) → a special strike (i.e., delivery price) a.k.a. forward price

$$\text{Forward price} = S_0 e^{rT}$$

\[ T \text{ the spot price.} \]

Q: Why violation of the principle → arbitrage?

Suppose the forward price is not priced at $S_0 - K e^{-rT}$ (from our principles)

But at $P$

→ 2 possibilities: (Homework)

1. $P > S_0 - K e^{-rT}$
   → Sell portfolio 1 at $P$
   Buy portfolio 2 at $S_0 - K e^{-rT}$

at $T$ P there is one stock in portfolio 2 which covers the stock needed in the fund position of portfolio 1

2. Use Cash $K$ received in portfolio 1 to cover $K$ needed in portfolio 2

Net position sells in the end but at initial profit $P - (S_0 - K e^{-rT})$

at no risk
Possibility 2 \( P < S_0 - Ke^{-rT} \)

\[
\begin{array}{c}
\text{LOW} \\
\text{HIGH}
\end{array}
\]

\[
\Rightarrow \text{instant profit at no cost}
\]

Either way \( \Rightarrow \) (Market forces (oversupply of sellers or buyers))

\[
\Rightarrow \text{price adjustment}
\]

Restoring the price to no-arbitrage value \( \approx \) (approximately).

**Put-Call Parity**

Another example of application of our principles.

\[ P(S_0, T, K) \text{ \# price of Every put when spot price is } S_0 \text{ \# strike } K \text{ at maturity } T \]

\[ C(S_0, T, K) \text{ \# price of Every call when spot price } S_0 \text{ \# strike } K, \text{ maturity } T \]

So far, we don't know how to price them yet but we know

\[ C(S_0, T, K) - P(S_0, T, K) = S_0 - Ke^{-rT} \text{ - put-call parity} \]

Why? 

\[ \frac{K}{(K-S_0)^+} \]

This is the forward rate \( K \) and \( K \)
More algebraically

Portfolio 1: A long call and a short put
payoff = $(S_T - k)_+ - (k - S_T)_+ = S_T - k$

Portfolio 2: A forward at delivery $k$ at $T$
payoff = $S_T - k$

Same payoff $\Rightarrow$ Same present value, i.e., $S_0 - Ke^{-rT}$

NB: 1° Put-Call parity holds, independent of models for underlying $S$

2° $C - p = S_0 - K e^{-rT}$ is true for any time

Some inequalities — application of our principles

For European options

1° $C \geq (S_0 - K e^{-rT})_+$

2° $p \geq (K e^{-rT} - S_0)_+$

Proof e.g. 1° Optionality $\Rightarrow C \geq 0$

\[ C \geq S_0 - K e^{-rT} \]

\[ \frac{S_0}{K} \geq \frac{K}{K} \]

\[ \therefore C \geq (S_0 - K e^{-rT})_+ \]
As: Again, the above inequalities hold independent of models for underlyings.

Some hypothesis involved:

1. No transaction cost, no bid-ask spread
2. No tax
3. Unlimited possibilities of long & short positions
   No restriction on borrowing

Institutions vs individuals

No Interest Rates

In general, instead of $e^{-rt}$

\[ B(t, T) = \text{Price at time } t \text{ of a risk-free bond worth $1 at time } T. \]

1. Constant interest rate
   \[ B(t, T) = e^{-r(T-t)} \]

2. Deterministic but nonconstant (i.e., known at present)

\[ \text{No risk} \rightarrow B(t_1, t_2)B(t_2, t_3) = B(t_1, t_3) \]

If stochastic, i.e., $B(t_2, t_3)$ not known at $t$,
   But $B(t_1, t_2)$ and $B(t_2, t_3)$ of course are known today $t$. 

\[ - B(t_1, t_2)B(t_2, t_3) = B(t_1, t_3) . \]
NB: In pricing forwards, put-call parity, only one periodorrowing is used.

Price of forwards → valid even if interest
rate is non-constant
and even stochastic!

\[ S_0 - KB(0,T) : \text{forward} \]

\[ C - p = S_0 - KB(0,T) \]

Forwards vs Futures

Future — the writer must sell the underlying to
its holder at maturity.

Differences plus forwards and futures

1° Futures — standardized and traded

2° Futures — mark-to-market

— issue of interest rate.

NB: Can prove

of interest rate is deterministic

\[ \text{MBS} \rightarrow \text{Future} = \text{Forward} \]

of stochastic, in general

\[ \text{Future} \neq \text{Forward} \]

But Future ≈ Forward
Covered Call

A reverse of a covered call

Protective Put

Bear Spread

Butterfly Spread

Butterfly spread can also be created with 4 puts.
Calendar spreads

Options at same strike price $K$ but different $T$.

Diagonal spreads: both $K$ and $T$ are different.

Combination: both call & put on the same stock.

A straddle

(Bottom straddle) 2 puts + 1 call

(a p r o c h) Fig jumps if the stock

A straddle

2 puts + 1 call

A top straddle

Mangle

A top vertical combination