Final Examination.

Due Dec 15.

1. We have a population of size $N$ which completely renews itself every generation. The total size is $N$ in every generation. The population consists of two types of individuals $A$ and $B$. If in a given generation the population consists of $x$ individuals of type $A$ and $N - x$ individuals of type $B$, then each member of the next generation will be of type $A$ with probability $\frac{x}{N}$ and type $B$ with probability $\frac{N - x}{N}$ independently of prior history. The types of different individuals are mutually independent. If $X_n$ is the number of type $A$ individuals in the $n$-th generation, then $X_n$ is a Markov process on the finite state space consisting of $\{0, 1, 2, \ldots, N\}$.

i. What is the transition probability
   \[
   \pi(x, y) = P[X_{n+1} = y | X_n = x]
   \]
   and what is special when $x = 0$ or $N$?

ii. Show that $X_n$ has a limit as $n \to \infty$ which is either 0 or $N$.

iii. If $\tau = \inf\{n : X_n = 0 \text{ or } N\}$ what is the value of
    \[
    P[X_\tau = 0 | X_0 = x]
    \]

iv. Is $E[\tau] = V_N(x) < \infty$?

v. If so can you get a bound for $\sup_x V_N(x)$?

vi. Show that the distribution of $\frac{X_{[Nt]}}{N}$ converges to a diffusion process whose generator is
    \[
    \frac{1}{2} x(1 - x) \frac{d^2}{dx^2}
    \]

vii. Carry out the analogs of ii, iii, iv in this case. Calculate $E[\tau|x(0) = x] = v(x)$.

viii. How is $V_N(x)$ related to $v(x)$?

2. $x(t)$ is the diffusion process with generator
    \[
    \frac{1}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx}
    \]
    The function $b(x)$ is smooth but perhaps unbounded for large $x$. Moreover $b(x) < 0$ for $x > 0$ and $b(x) > 0$ for $x < 0$. Show that the process never explodes. In other words if
    \[
    \tau_n = \inf\{t : |x(t)| \geq n\}
    \]
    then for any $T < \infty$,
    \[
    \lim_{n \to \infty} P[\tau_n \leq T] = 0
    \]