Assignment 6.

1. Assume a family $p_h(dy)$ of probability distributions on $\mathbb{R}^d$ satisfy

$$\lim_{h \to 0} \frac{1}{h} \int y_i p_h(dy) = b_i$$

$$\lim_{h \to 0} \frac{1}{h} \int y_i y_j p_h(dy) = a_{i,j}$$

and for some $\delta > 0$

$$\lim_{h \to 0} \frac{1}{h} \int \|y\|^{2+\delta} p_h(dy) = 0$$

Conclude that for any smooth function $f(y)$

$$\lim_{h \to 0} \frac{1}{h} \int [f(y) - f(0)] p_h(dy) = \frac{1}{2} \sum_{i,j} a_{i,j} (\frac{\partial^2 f}{\partial x_i \partial x_j})(0) + \sum_j b_j (\frac{\partial f}{\partial x_j})(0)$$

2. Show that the transition probability $p(s, x, t, dy)$ of the solution $x(t)$ to the SDE $dx = b dt + \sigma d\beta$ satisfies

$$\lim_{h \to 0} \frac{1}{h} \int [f(y) - f(x)] p(t, x, t+h, dy) = \frac{1}{2} \sum_{i,j} a_{i,j}(t, x) (\frac{\partial^2 f}{\partial x_i \partial x_j})(x) + \sum_j b_j(t, x) (\frac{\partial f}{\partial x_j})(x)$$