15 Solutions

\[ \frac{\partial \log p}{\partial \theta} = \frac{-2(x - \theta)}{1 + (x - \theta)^2} \]

\[ I(\theta) = \frac{1}{\pi} \int \left[ \frac{-2(x - \theta)}{1 + (x - \theta)^2} \right]^2 \frac{1}{1 + (x - \theta)^2} dx = \frac{1}{\pi} \int \frac{x^2}{(1 + x^2)^3} dx = \frac{1}{2} \]

\[ f(\theta, x_1, \ldots, x_n) = \theta^n \exp[-\theta \sum x_i ] \]

\[ \frac{\partial \log f(\theta, x_1, \ldots, x_n)}{\partial \theta} = \frac{n}{\theta} - \sum x_i \]

\[ \hat{\theta} = \frac{n}{\sum x_i} \]

\[ E[\hat{\theta}] = \int_0^\infty \frac{n}{t \Gamma(n)} \exp[-\theta t]t^{n-1} dt = \frac{n \theta}{n-1} \]

It is biased and \( \hat{\theta} = \frac{n-1}{n} \hat{\theta} \) is unbiased.

\[ \text{Var}(\hat{\theta}) = (n - 1)^2 \theta^2 \left[ \frac{1}{(n-1)(n-2)} - \frac{1}{(n-1)^2} \right] = \frac{\theta^2}{n-2} \geq \frac{\theta^2}{n} = \frac{1}{nI(\theta)} \]

It is asymptotically efficient.

- MLE is the Median. By symmetry it is unbiased.

\[ \frac{\partial \log f}{\partial \theta} = -\text{sign}(x - \theta) \]

\[ I(\theta) = 1 \]

Cramer-Rao lower bound is \( \frac{1}{n} \). Asymptotic variance of the MLE is

\[ \frac{4}{n[f(\theta)]^2} = \frac{1}{n} \]
A statistic is $T(0), \ldots, T(n)$. Unbiased means

$$\sum_j \binom{n}{j} \left[ \frac{3}{4} \right]^j \left[ \frac{1}{4} \right]^{n-j} T(j) = \frac{3}{4}$$

and

$$\sum_j \binom{n}{j} \left[ \frac{1}{4} \right]^j \left[ \frac{3}{4} \right]^{n-j} T(j) = \frac{1}{4}$$

Two equations. $n + 1$ unknowns. Lots of solutions. Easy to construct unbiased estimators with variance that is very small. For example if $n$ is odd $T(x) = a$ if $x < \frac{n}{2}$ and $T(x) = b$ if $x > \frac{n}{2}$ can be made unbiased by proper choice of $a$ and $b$. If we denote by

$$p_n = \sum_{j < \frac{n}{2}} \binom{n}{j} \left[ \frac{3}{4} \right]^j \left[ \frac{1}{4} \right]^{n-j} = \sum_{j > \frac{n}{2}} \binom{n}{j} \left[ \frac{1}{4} \right]^j \left[ \frac{3}{4} \right]^{n-j}$$

we need

$$a p_n + b(1 - p_n) = \frac{1}{4}$$

and

$$a(1 - p_n) + b p_n = \frac{3}{4}$$

giving us $a = \frac{4p_n - 3}{8p_n - 4}$ and $b = \frac{4p_n - 1}{8p_n - 4}$. The variance is given by

$$\sigma^2 = p_n(a - \frac{1}{4})^2 + (1 - p_n)(b - \frac{1}{4})^2$$

and is seen to be very very small for large $n$.

The log-likelihood is

$$-\frac{n}{2} \log \theta - \frac{1}{2\theta} \sum (x_i - \theta)^2 = -\frac{n}{2} \log \theta - \frac{1}{2\theta} \sum x_i^2 + \sum x_i - \frac{n}{2} \theta$$

Clearly $U_n = \frac{1}{n} \sum x_i^2$ is sufficient and the likelihood equation is

$$-\frac{1}{\theta} + \frac{U_n}{\theta^2} - 1 = 0$$
or
\[ \theta^2 + \theta = U_n \]
This gives
\[ \theta_n = -\frac{1}{2} + \sqrt{\frac{1}{4} + U_n} \]
which is consistent because
\[ -\frac{1}{2} + \sqrt{\frac{1}{4} + \theta^2 + \theta} = \theta \]
Has an asymptotic variance
\[ \frac{1}{n} \var(x^2)[f'(\theta^2 + \theta)]^2 \]
with
\[ f(y) = -\frac{1}{2} + \sqrt{\frac{1}{4} + y} \]
and
\[ f'(\theta^2 + \theta) = \frac{1}{2\theta + 1} \]
On simplification this reduces to \( \frac{1}{n} \frac{2\theta}{(1+2\theta)^2} \). The Cramer-Rao lower bound is exactly the same. The efficiency of the mean with variance \( \frac{\theta}{n} \) is given by \( \frac{3\theta}{(1+2\theta)^2} \).

- The log-likelihood function is
\[ -n \log \Gamma(p) - \sum x_i + (p - 1) \sum \log x_i \]
Tha likelihood equation is
\[ \frac{\Gamma'(p)}{\Gamma(p)} = G(p) = \frac{1}{n} \sum \log x_i \]
and the MLE is
\[ \hat{\theta}_n = G^{-1}(\frac{1}{n} \sum \log x_i) \]
It is consistent because
\[ \hat{\theta}_n \to G^{-1}(m) \]
where
\[ m = \int \frac{1}{\Gamma(p)} e^{-x} x^{p-1} \log x \, dx = G(p) \]
and
\[ G^{-1}(G(p)) = p \]
\( \hat{\theta}_n \) is asymptotically normal with variance \( \frac{\text{var} \, (\log x_i)}{n[G'(p)]^2} \). The quantity \( I(p) \) is calculated easily as
\[
I(p) = E[(\log x - G(p))^2] = \text{var} (\log x) = G''(p)
\]

- To test \( f_0(x) = 2x \) against \( f_1(x) = 2(1-x) \) the critical region is \( \frac{1-x}{x} > c \) or \( x < c \). Size is \( \int_0^c 2x \, dx = c^2 = \alpha \) or \( c = \sqrt{\alpha} \). Power is calculated to be
\[
\int_0^{\sqrt{\alpha}} 2(1-x) \, dx = 2\sqrt{\alpha} - \alpha
\]

- The critical region can be any subset of \([0, \frac{1}{2}]\) if \( \alpha < \frac{1}{2} \) or the entire \([0, \frac{1}{2}]\) along with a subset of \([\frac{1}{2}, 1]\) if \( \alpha > \frac{1}{2} \). The power is \( 2\alpha \) if \( \alpha < \frac{1}{2} \) and 1 if \( \alpha > \frac{1}{2} \).

- The critical region is of the form \( \sqrt{n} |\bar{x}_n| > c \) and \( c = 1.96 \) from the tables. Power at \( \mu = 1 \) is \( P[|z - \sqrt{n}| > 1.96] \) is essentially \( P[z < \sqrt{n} - 1.96] \) and this is .95 if \( \sqrt{n} > 1.96 + 1.64 = 3.61 \) or \( n > (3.61)^2 \).