The space of functions of Bounded Mean Oscillation (BMO) plays an important role in Harmonic Analysis.

A function $f$, in $L_1(loc)$ in $\mathbb{R}^d$ is said to be a BMO function if
\begin{equation}
\sup_x \inf_a \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} |f(y) - a| dy = \|u\|_{BMO} < \infty \tag{8.1}
\end{equation}
where $B_{x,r}$ is the ball of radius $r$ centered at $x$, and $|B_{x,r}|$ is its volume.

**Remark.** The infimum over $a$ can be replaced by the choice of $a = \bar{a} = \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} f(y) dy$ giving us an equivalent definition. We note that for any $a$,
\[|a - \bar{a}| \leq \frac{1}{|B_{x,r}|} \int_{y \in B_{x,r}} |f(y) - a| dy\]
and therefore if $a^*$ is the optimal $a$,
\[|\bar{a} - a^*| \leq \|f\|_{BMO}\]

**Remark.** Any bounded function is in the class BMO and $\|f\|_{BMO} \leq \|f\|_{\infty}$.

**Theorem 8.1 (John-Nirenberg).** Let $f$ be a BMO function on a cube $Q$ of volume $|Q| = 1$ satisfying $\int_Q f(x) dx = 0$ and $\|f\|_{BMO} \leq 1$. Then there are finite positive constants $c_1, c_2$, independent of $f$, such that, for any $\ell > 0$
\begin{equation}
|\{x : |f(x)| \geq \ell\}| \leq c_1 \exp\left[-\frac{\ell}{c_2}\right] \tag{8.2}
\end{equation}

**Proof.** Let us define
\[F(\ell) = \sup_f |\{x : |f(x)| \geq \ell\}|\]
where the supremum is taken over all functions with $\|f\|_{BMO} \leq 1$ and $\int_Q f(x) dx = 0$. Since $\int_Q f(x) dx = 0$ implies that $\|f\|_1 \leq \|f\|_{BMO} \leq 1$, $F(\ell) \leq \frac{1}{\ell}$. Let us subdivide the cube into $2^d$ subcubes with sides one half the original cube. We pick a number $a > 1$ and keep the cubes $Q_i$ with $\frac{1}{|Q_i|} \int_{Q_i} |f(x)| dx \geq a$. We subdivide again those with $\frac{1}{|Q_i|} \int_{Q_i} |f(x)| dx < a$ and keep going. In this manner we get an atmost countable collection of disjoint cubes that we enumerate as $\{Q_j\}$, that have the following properties:
1. \( \frac{1}{|Q_j|} \int_{Q_j} |f(x)| \, dx \geq a. \)

2. Each \( Q_j \) is contained in a bigger cube \( Q'_j \) with sides double the size of the sides of \( Q_j \) and \( \frac{1}{|Q_j|} \int_{Q_j} |f(x)| \, dx < a. \)

3. By the Lebesgue theorem \( |f(x)| \leq a \) on \( Q \cap (\cup_j Q_j)^c \).

If we denote by \( a_j = \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \), we have

\[
|a_j| \leq \frac{1}{|Q_j|} \int_{Q_j} |f(x)| \, dx \leq \frac{2^d}{|Q'_j|} \int_{Q'_j} |f(x)| \, dx \leq 2^d a
\]

by property 2). On the other hand \( f - a_j \) has mean 0 on \( Q_j \) and BMO norm at most 1. Therefore (scaling up the cube to standard size)

\[
|Q_j \cap \{ x : |f(x)| \geq 2^d a + \ell \}| \leq |Q_j \cap \{ x : |f(x) - a_j| \geq \ell \}|
\]

\[
\leq |Q_j| F(\ell)
\]

Summing over \( j \), because of property 3),

\[
|\{ x : |f(x)| \geq 2^d a + \ell \}| \leq F(\ell) \sum_j |Q_j|
\]

On the other hand property 1) implies that \( \sum_j |Q_j| \leq \frac{1}{a} \) giving us

\[
F(2^d a + \ell) \leq \frac{1}{a} F(\ell)
\]

which is enough to prove the theorem. \( \square \)

**Corollary 8.1.** For any \( p > 1 \) there is a constant \( C_{d,p} \) depending only on the dimension \( d \) and \( p \) such that

\[
\sup_Q \frac{1}{|Q|} \int_Q |f(x)| - \frac{1}{|Q|} \int_Q f(x) \, dx \leq C_{d,p} \|f\|_{\text{BMO}}^p
\]

The importance of BMO, lies partly in the fact that it is dual to \( \mathcal{H}_1 \).

**Theorem 8.2.** There are constants \( 0 < c \leq C < \infty \) such that

\[
c \|f\|_{\text{BMO}} \leq \sup_{g : \|g\|_{\mathcal{H}_1} \leq 1} |\int f(x) g(x) \, dx| \leq C \|f\|_{\text{BMO}}
\]

and every bounded linear functional on \( \mathcal{H}_1 \) is of the above type.
The proof of the theorem depends on some lemmas.

**Lemma 8.1.** The Riesz transforms $R_i$ map $L_\infty \to \text{BMO}$ boundedly. In fact convolution by any kernel of the form $K(x) = \frac{\Omega(x)}{|x|^d}$ where $\Omega(x)$ is homogeneous of degree zero, has mean 0 on $S^{d-1}$ and satisfies a Hölder condition on $S^{d-1}$ maps $L_\infty \to \text{BMO}$ boundedly.

**Proof.** Let us suppose that $Q$ is the unit cube centered around the origin and denote by $2Q$ the doubled cube. We write $f = f_1 + f_2$ where $f_1 = f 1_{2Q}$ and $f_2 = f - f_1 = f 1_{(2Q)^c}$.

$$g(x) = g_1(x) + g_2(x)$$

where

$$g_i(x) = \int_{R^d} K(x-y) f_i(y) dy$$

$$\int_Q |g_1(x)| dx \leq \|g_1\|_2 \leq \sup_\xi |\widehat{K}(\xi)| \|f_1\|_2 \leq 2^d \sup_\xi |\widehat{K}(\xi)| \|f\|_\infty$$

On the other hand with $a_Q = \int_Q K(-y) f_2(y) dy$

$$\int_Q |g_2(x) - a_Q| dx$$

$$\leq \int_Q dx \int_{R^d} |K(x-y) - K(-y)| f_2(y) dy$$

$$\leq \|f\|_\infty \int \int_{x \in Q \atop y \in 2Q} |K(x-y) - K(-y)| dx dy$$

$$\leq \|f\|_\infty \sup_x \int_{|y| \geq 2|x|} |K(x-y) - K(-y)| dy$$

$$\leq B \|f\|_\infty$$

The proof for arbitrary cube is just a matter of translation and scaling. The Hölder continuity is used to prove the boundedness of $\hat{K}(\xi)$. 

**Lemma 8.2.** Any bounded linear function $\Lambda$ on $\mathcal{H}_1$ is given by

$$\Lambda(f) = \sum_{i=0}^d \int (R_i f)(x) g_i(x) dx = -\int f(x) \sum_{i=0}^d (R_i g_i)(x) dx$$

where $R_0 = I$ and $R_i$ for $1 \leq i \leq d$ are the Riesz transforms.
Proof. The space \( \mathcal{H}_1 \) is a closed subspace of the direct sum \( \oplus L_1(R^d) \) of \( d+1 \) copies of \( L_1(R^d) \). Hahn-Banach theorem allows us to extend \( A \) boundedly to \( \oplus L_1(R^d) \) and the Riesz representation theorem gives us \( \{g_i\} \). Finally \( g_0 + \sum_{i=1}^d R_i g_i \) is in BMO.

Lemma 8.3. If \( g \in \text{BMO} \) then

\[
\int_{\mathbb{R}^d} \frac{|g(y)|}{1 + |y|^{d+1}} dy < \infty \tag{8.4}
\]

and

\[
G(t, x) = \int g(y)p(t, x - y)dy
\]

exists where \( p(\cdot, \cdot) \) is the Poisson kernel for the half space \( t > 0 \). Moreover \( g(t, x) \) satisfies

\[
\sup_x \int_{|y-x| < h, \atop 0 < t < h} t|\nabla G(t, y)|^2 \, dt \, dy \leq A\|g\|^2_{\text{BMO}} t^d \tag{8.5}
\]

for some constant independent of \( g \). Here \( \nabla G \) is the full gradient in \( t \) and \( x \).

Proof. First let us estimate \( \int_{\mathbb{R}^d} \frac{|g(x)|}{1 + |x|^{d+1}} \, dx \). If we denote by \( Q_n \) the cube of side \( 2^n \) around the origin

\[
\int_{\mathbb{R}^d} \frac{|g(x)|}{1 + |x|^{d+1}} \, dx \leq \int_{Q_0} \frac{|g(x)|}{1 + |x|^{d+1}} \, dx + \sum_n \int_{Q_{n+1} \cap Q_n} \frac{|g(x)|}{1 + |x|^{d+1}} \, dx
\]

\[
\leq \int_{Q_0} |g(x)| \, dx + \sum_n \frac{1}{2^{n(d+1)}} \int_{Q_{n+1}} |g(x)| \, dx
\]

\[
\leq \int_{Q_0} |g(x)| \, dx + \sum_n \frac{1}{2^{n(d+1)}} \int_{Q_{n+1}} |g(x) - a_{n+1}| \, dx
\]

\[
+ \sum_n \frac{|a_{n+1}|}{2^n}
\]

\[
\leq \int_{Q_0} |g(x)| \, dx + \|g\|_{\text{BMO}} \sum_n \frac{2^{n+1}d}{2^{n(d+1)}} + \sum_n \frac{|a_{n+1}|}{2^n}
\]
where

\[ a_{n+1} \leq \frac{1}{2^{(n+1)d}} \int_{Q_{n+1}} g(x) dx \]

Moreover

\[ |a_{2Q} - a_Q| = \frac{1}{|Q|} \int_Q |g(x) - a_{2Q}| dx \leq 2^d \|g\|_{BMO} \]

and this provides a bound of the form

\[ |a_{Q_n}| \leq Cn \|g\|_{BMO} + \int_{Q_0} |g(x)| dx \]

establishing (8.4). We now turn to proving (8.5). Again because of the homogeneity under translations and rescaling, we can assume that \( x = 0 \) and \( h = 1 \). So we only need to control

\[ \int_{|y| < 1} \int_{0 < t < 1} t |\nabla G(t, y)|^2 dtdy \leq A \|g\|^2_{BMO} \]

We denote by \( Q_4 \) the cube \(|x| \leq 2\) and write \( g \) as

\[ g = a_{Q_4} + (g_1 - a_{Q_4}) + g_2 \]

where \( g_1 = g 1_{Q_4}, g_2 = g - g_1 = g 1_{Q_5} \). Since constants do not contribute to (8.5), we can assume that \( a_{Q_4} = 0 \), and therefore the integral \( \int_{Q_4} |g(x)| dx \) can be estimated in terms of \( \|g\|_{BMO} \). An easy calculation, writing \( G = G_1 + G_2 \) yields

\[ |\nabla G_2(t, y)| \leq \int_{Q_4} \frac{|g(x)|}{1 + |x|^{d+1}} dx \leq A \|g\|_{BMO} \]

As for the \( G_1 \) contribution in terms of the Fourier transform we can control it by

\[ \int_0^\infty \int_{R^d} t |\nabla G|^2 dtdy = \int_0^\infty \int_{R^d} t |\xi|^2 e^{-2t|x|} |\widehat{g_1}(\xi)|^2 d\xi dt = \int_{R^d} |\widehat{g_1}(\xi)|^2 d\xi \]

which is controlled by \( \|g\|_{BMO} \) because of the John-Nirenberg theorem. \( \square \)
Lemma 8.4. Any function \( g \) whose Poisson integral \( G \) satisfies (8.5) defines a bounded linear functional on \( \mathcal{H}_1 \).

Proof. The idea of the proof is to write

\[
2 \int_0^\infty \int_{\mathbb{R}^d} t \nabla G(t, x) \nabla F(t, x) dt dx = 4 \int_0^\infty \int_{\mathbb{R}^d} te^{-2|\xi|} |\xi|^2 \hat{f}(\xi) \hat{g}(\xi) d\xi dt
\]

\[
= \int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) d\xi
\]

\[
= \int_{\mathbb{R}^d} f(x) g(x) dx
\]

and concentrate on

\[
\int_0^\infty \int_{\mathbb{R}^d} t |\nabla_x G(t, x)| |\nabla_x F(t, x)| dt dx
\]

We need the auxiliary function

\[
(S_h u)(x) = \left( \int \int_{|x-y|<t<h} t^{1-d} |\nabla u|^2 dy dt \right)^{\frac{1}{2}}
\]

Clearly \((S_h u)(x)\) is increasing in \( h \) and we show in the next lemma that

\[
\|S_\infty F\|_1 \leq C \|f\|_{\mathcal{H}_1}
\]

Let us assume it and complete the proof. Define

\[
h(x) = \sup \{ h : (S_h F)(x) \leq MC \}
\]

then

\[
(S_{h(x)} F)(x) \leq MC
\]

In addition it follows from (8.5) that

\[
\sup_{y,h} \int_{|y-x|\leq h} |(S_h F)(x)|^2 dx \leq Ch^d
\]

Now \( h(x) < h \) means \((S_h F)(x) > MC \) and therefore

\[
\left| \{ x : |x-y| < h, h(x) < h \} \right| \leq \frac{Ch^d}{M^2}
\]
By the proper choice of $M$, we can be sure that
\[ \{|x : |x - y| < h, h(x) \geq h\}| \geq ch^d \]
Now we complete the proof.

\[
\int_0^\infty \int_{R^d} t |\nabla_x G(t, x)||\nabla_x F(t, x)|dt\,dx \\
\leq C \int_0^\infty \int_{R^d} \int_{|y-x| < t \leq h(y)} t^{1-d} |\nabla_x G(t, x)||\nabla_x F(t, x)|dt\,dx\,dy \\
\leq \int_{R^d} dy \left( \int_0^\infty \int_{|y-x| < t \leq h(y)} t^{1-d} |\nabla_x G(t, x)|^2 dt \right)^\frac{1}{2} \\
\quad \times \left( \int_0^\infty \int_{|y-x| < t \leq h(y)} t^{1-d} |\nabla_x F(t, x)|^2 dt \right)^\frac{1}{2} \\
\leq M \int_{R^d} (S F)(y) dy \leq M \|f\|_{H_1}
\]

Lemma 8.5. If $f \in H_1$ then $|(S F)(x)| \leq C \|f\|_{H_1}$.

Proof. This is done in two steps.

**Step 1.** We control the nontangential maximal function

\[ U^*(x) = \sup_{y,t: |x-y| \leq kt} |U(t, y)| \]

by

\[ \|U^*\|_1 \leq C_k \|u\|_{H_1} \]

If $U_0(x) \in H_1$ then $U_0$ and its $n$ Riesz transforms $U_1, \ldots, U_n$ can be recognized as the full gradient of a Harmonic function $W$ on $R_{n+1}^n$. Then $V = (U_0^2 + \cdots + U_n^2)^{\frac{p}{2}}$ can be verified to be subharmonic provided $p > \frac{n-1}{n-2}$.

This depends on the calculation

\[
\Delta V = \frac{p}{2} \left( \frac{p-2}{2} V^{\frac{p-2}{2}} \right) \|\nabla V\|^2 + \frac{p}{2} V^{\frac{p-4}{2}} \Delta V \\
= pV^{\frac{p-2}{2}} \left( (p-2) \sum U_j \nabla U_j \|^2 + V \sum_j \|\nabla U_j\|^2 \right) \\
\geq 0
\]
provided either \( p \geq 2 \), or if \( 0 < p < 2 \),
\[
\|H\xi\|^2 \leq \frac{1}{2-p} \mathrm{Tr} (H^*H)\|\xi\|^2 
\]  
(8.6)
where \( H \) is the Hessian of \( W \) with trace 0 and \( \xi = (U_0, \ldots, U_n) \). Then if \( \{\lambda_j\} \) are the \( n+1 \) eigenvalues of \( H \), and \( \lambda_0 \) is the one with largest modulus, the remaining ones have an average of \( -\frac{\lambda_0}{n} \) and therefore
\[
\mathrm{Tr} (H^*H) = \sum \lambda_j^2 \geq (1 + \frac{1}{n})\lambda_0^2
\]
This means that for equation (8.6) to hold we only need \( n+1 \frac{n}{n+1} \leq \frac{1}{2-p} \) or \( p \geq \frac{n-1}{n+1} \). In any case there is a choice of \( p = p_n < 1 \) that is allowed.

Now consider the subharmonic function \( V \). If we denote by \( h(t, x) \) the Poisson integral of the boundary values of \( h(0, x) = V(0, x) \),
\[
V(t, x) \leq h(t, x)
\]
and we have
\[
U^*(x) = \sup_{(y,t):|x-y|\leq kt} U(t, y) \leq \sup_{(y,t):|x-y|\leq kt} V[(t, y)]^{\frac{1}{p}} \leq \sup_{(y,t):|x-y|\leq kt} h[(t, y)]^{\frac{1}{p}}
\]
By maximal inequality, valid because \( \frac{1}{p} > 1 \),
\[
\|U^*\|_1 \leq \|h^*\|_p^p \leq C_{k,p}\|h(0, x)\|_p^p = C_{k,p}\|V(0, x)\|_p^p \leq C_k\|U\|_{\mathcal{H}_1}
\]

**Step 2.** It is now left to control \( \|(S_\infty U)(x)\|_1 \leq C\|U^*\|_1 \). We use the room between the regions \( |x-y| \leq t \) in the definition of \( S \) and the larger regions \( |x-y| \leq kt \) used in the definition of \( U^* \) to control \( S \) through \( U \). Let us pick \( k = 4 \). Let \( \alpha > 0 \) be a number. Consider the set \( E = \{ x : |U^*(x)| \leq \alpha \} \) and \( B = E^c = \{ x : |U^*(x)| > \alpha \} \). We denote by \( G \) the union \( G = \cup_{x \in E}\{(t, y) : |x-y| \leq t\} \). We want to estimate
\[
\int_E |S_\infty U|^2(x)dx = \int \int \int_{|x-y|\leq t} t^{1-d}d|\nabla U|^2(t, y)dxdtdy
\]
\[
\leq C \int_G t|\nabla U|^2(t, y)dtdy
\]
\[
\leq C \int_G t(DU^2)(t, y)dtdy
\]
\[
\leq C \int_{\partial G} [t|\frac{\partial U^2}{\partial n}(t, y)| + |U^2(t, y)|\frac{\partial t}{\partial n}(t, y)]d\sigma
\]
by Greens's theorem. We have cheated a bit. We have assumed some smoothness on $\partial G$. We have assumed decay at $\infty$ so there are no contributions from $\infty$. We can assume that we have initially $U(0, x) \in L_2$ so the decay is valid. We can approximate $G$ from inside by regions $G_*$ with smooth boundary. The boundary consists of two parts. $B_1 = \{ t = 0, x \in E \}$ and $B_2 = \{ x \in E^c, t = \phi(x) \}$. Moreover $|\nabla \phi| \leq 1$. We will show below that $t|\nabla U(t, y)| \leq C\alpha$ in $G$. On $B_1$ one can show that $t|U||\nabla U| \to 0$ and $U^2 \frac{\partial U}{\partial n} \to U^2$. Moreover $d\sigma \approx dx$. The contribution from $B_1$ is therefore bounded by $\int_E |U(0, x)|^2 dx \leq \int_E |U^*(0, x)|^2 dx$. On the other hand on $B_2$ since it is still true that $d\sigma = dx$, using the bound $t|\nabla U| \leq C\alpha$, $|\frac{\partial U}{\partial n}| \leq 1$, we see that the contribution is bounded by $C\alpha^2 |E^c|$. Putting the pieces together we get

$$\int_E |S_\infty U|^2(x) dx \leq C\alpha^2 T_{U^*}(\alpha) + C \int_E |U^*|^2(x) dx$$

$$\leq C\alpha^2 T_{U^*}(\alpha) + C \int_{0}^{\alpha} zT_{U^*}(z)dz$$

where $T_{U^*}(z) = \text{mes}\{ x : |U^*(x)| > z \}$. Finally since $\text{mes}(E^c) = T_{U^*}(\alpha)$

$$\text{mes}\{ x : |S_\infty U(x)| > \alpha \} \leq C T_{U^*}(\alpha) + \frac{C}{\alpha^2} \int_{0}^{\alpha} zT_{U^*}(z)dz$$

Integrating with respect to $\alpha$ we obtain

$$\|S_\infty U\|_1 \leq C\|U^*\|_1$$

**Step 3.** To get the bound $t|\nabla U| \leq C\alpha$ in $G$, we note that any $(t, x) \in G$ has a ball around it of radius $t$ contained in the set $\cup_{x \in E}\{ y : |x - y| \leq 4t \}$ where $|U| \leq \alpha$ and by standard estimates, if a Harmonic function is bounded by $\alpha$ in a ball of radius $t$ then its gradient at the center is bounded by $\frac{C\alpha}{t}$. $\square$