Hormander’s Theorem
Let \( L \) denote the operator
\[
L = \frac{1}{2} \sum_{i=1}^{k} X_i^2 + Y
\]
where \( X_1, \ldots, X_k, Y \) are \( C^\infty \) vector fields in \( R^n \). Assume that the Lie Algebra generated by \( X_1, \ldots, X_k, Y \) span \( R^n \) at every \( x \). Then, if \( u \) is a distribution such that
\[
Lu = f
\]
and \( f \) is \( C^\infty \) in an open set \( G \), it is always true that \( u \) is \( C^\infty \) in \( G \).

Proof is in three steps.

**Step 1.** Let \( L \) have a fundamental solution \( p(t, x, y) \) with the following properties:
\[
\sup_{0 < t \leq 1} \sup_{|x - y| \geq \epsilon} |D_t^r D_y^s p(t, x, y)| \leq C_{r, s}(\epsilon) < \infty
\]
\[
\int p(t, x, y) f(y) dy \rightarrow f(x)
\]
in every \( C^r \) if \( f \in C^r \). And the same is true of the adjoint.
\[
\int p(t, x, y) f(x) dx \rightarrow f(y)
\]
in every \( C^r \) if \( f \in C^r \). In particular for \( f \in C^r \),
\[
\sup_{0 \leq t \leq 1} \|P_t f\|_{(r)} \leq C_r \|f\|_{(r)}
\]
and
\[
\sup_{0 \leq t \leq 1} \|P_t^* f\|_{(r)} \leq C_r \|f\|_{(r)}
\]

Then \( L \) is hypoelliptic.

**Proof:** Let \( x_0 \in G \). Let us find \( \epsilon \) such that
\[
\overline{B(x_0, 3\epsilon)} \subset G
\]
and a \( C^\infty \) function \( g \) which is 1 in \( B(x_0, 2\epsilon) \) and 0 outside \( B(x_0, 3\epsilon) \).
\[
L(g u) = g L u + u L g + < a \nabla g, \nabla u > = f + h
\]
where \( h \) is a distribution supported in \( B(x_0, 2\epsilon)^c \)
\[
P_t (g u) - g u = \int_0^1 P_t [L(g u)] dt
\]
\[
= \int_0^1 P_t f dt + \int_0^1 P_t h dt
\]
It is done.

**Step 2.** Hormander condition does not give smooth fundamental solution for the parabolic equation. We do the following trick. Introduce an extra variable $x_{n+1}$ and a function

$$\rho(x_{n+1}) = 2 + \sin x_{n+1}$$

Define new vector fields

$$\hat{X}_i = \sqrt{\rho(x_{n+1})}X_i$$

for $1 \leq i \leq k$,

$$\hat{X}_{k+1} = \frac{\partial}{\partial x_{n+1}}$$

and

$$\hat{Y} = \rho(x_{n+1})Y$$

so that

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{k} \hat{X}_i^2 + \hat{Y} = \rho(x_{n+1})\mathcal{L} + \frac{1}{2} \frac{\partial^2}{\partial x_{n+1}^2}$$

One checks that $\mathcal{L}$ satisfies the assumptions so that the conditions are fulfilled. So $\mathcal{L}$ is Hypoelliptic in the Hormander sense.

**Step 3.** If $Lu = f$ in $\mathbb{R}^n$ then $\hat{L}u = \rho(x_{n+1})f$ in $\mathbb{R}^{n+1}$ and if $f$ is regular in an open set $G \subset \mathbb{R}^n$, then $\rho(x_{n+1})f$ is regular in $\mathbb{R} \times G \subset \mathbb{R}^{n+1}$. Therefore $u$ which does not depend on $x_{n+1}$ is regular in $\mathbb{R} \times G \subset \mathbb{R}^{n+1}$ and therefore in $G$. 