More generally if $\mathcal{L}$ is the generator of a not necessarily self adjoint operator that generates a Markov semigroup, we can ask the following question. Can we estimate

$$ P_x \left[ \frac{1}{t} \int_0^t V(x(s)) ds \geq \ell \right] $$

The starting point for such estimates is the Feynman-Kac formula that says

$$ u(t, x) = E_x \left[ \exp \left[ \int_0^t V(x(s)) ds \right] f(x(t)) \right] $$

is the solution of

$$ u_t = \mathcal{L} u + V u; \ u(0, x) = f(x) $$

In particular, if $u(t, x) \equiv u(x)$ and $0 < c \leq u(x) \leq C < \infty$

$$ 0 = u_t = \mathcal{L} u - \frac{\mathcal{L} u}{u} u $$

and

$$ E_x \left[ \exp \left[ \int_0^t -\frac{\mathcal{L} u}{u} (x(s)) ds \right] \right] \leq \frac{u(x)}{c} $$

If we denote by $m(t, A)$, the empirical measure

$$ m(t, A) = \frac{1}{t} \int_0^t 1_A(x(s)) ds $$

and $Q_{t, x}$ the distribution of the empirical measure on the space $\mathcal{M}$ of all probability measures on our state space $X$, then the bound we have is

$$ E^{Q_{t, x}} \left[ \exp \left[ -t \langle \mathcal{L} u \rangle, m \rangle \right] \right] \leq \frac{u(x)}{c} $$

By Tchebechev’s inequality we can estimate

$$ Q_{t, x} [E] \leq \frac{u(x)}{c} \exp \left[ t \sup_{m \in \mathcal{E}} \langle \mathcal{L} u \rangle, m \rangle \right] $$

Therefore

$$ \limsup_{t \to \infty} \frac{1}{t} \log Q_{t, x} [E] \leq \sup_{m \in \mathcal{E}} \langle \mathcal{L} u \rangle, m \rangle $$

Since $u \in \mathcal{D}^+$ is arbitrary

$$ \limsup_{t \to \infty} \frac{1}{t} \log Q_{t, x} [E] \leq \inf_{u \in \mathcal{D}^+} \sup_{m \in \mathcal{E}} \langle \mathcal{L} u \rangle, m \rangle $$
If $N$ is small neighborhood of $m$, then

$$\lim_{N \downarrow m} \limsup_{t \to \infty} \frac{1}{t} \log Q_{t,x}[N] \leq \inf_{u \in \mathcal{D}^+} \langle \mathcal{L}u, m \rangle$$

The rate function for large deviation is the function

$$I(m) = - \inf_{u \in \mathcal{D}^+} \langle \mathcal{L}u, m \rangle$$

With this rate function we have upper bound for small neighborhoods. Since the sum of a finite number of exponentials decays like the worst, this yields an upper bound immediately for compact sets $K$, which can be covered by a finite number of arbitrary small neighborhoods.

$$\limsup_{t \to \infty} \frac{1}{t} \log Q_{t,x}[K] \leq - \inf_{m \in K} I(m)$$

If $X$ is not compact some additional control is needed to prove exponential tightness, i.e.

$$\limsup_{K \uparrow X} \limsup_{t \to \infty} \frac{1}{t} \log Q_{t,x}[K^c] \leq -\infty$$

Then we can estimate for any closed set $C$,

$$Q_{t,x}[C] \leq Q_{t,x}[C \cap K] + Q_{t,x}[K^c]$$

Since the second term can be made to decay with a large exponential decay rate by the choice of $K$, our estimate for compact sets can now be extended to closed sets.

To prove exponential tightness, when $X$ is not compact, for instance $\mathbb{R}^d$, it is enough to get an estimate of the form

$$E^{P_x} \left[ \exp \left[ \int_0^t V(x(s))ds \right] \right] \leq c(x)e^{at}$$

with $c(x) < \infty$ and $a < \infty$ for some $V(x) \geq 0$, $V(x) \to +\infty$ as $|x| \to \infty$. This would give us with

$$K_\ell = \{ m : \int V(x)m(dx) \leq \ell \} \subset \mathcal{M}$$

$$Q_{t,x}[K_\ell^c] \leq Q_{t,x} \left[ \{ m : \int V(x)m(dx) \geq \ell \} \right] \leq e^{-t\ell} E^{P_x} \left[ \exp \int_0^t V(x(s))ds \right] \leq c(x)e^{(a-\ell)t}$$

we can pick $\ell$ to be large and we will have our exponential tightness. For instance if

$$\mathcal{L} = \frac{1}{2} \Delta - \langle x, \nabla \rangle$$

is the OU process, with $u(x) = eV(1+x^2)$, it is not hard to see that $V(x) = -\frac{\mathcal{L}u}{u} \to +\infty$ as $|x| \to \infty$. We can add a constant to make it non negative.
Proving lower bounds involves changing the generator from $\mathcal{L}$ to $\widehat{\mathcal{L}}$, such that, $\mu$ is an (ergodic?) invariant measure for $\widehat{\mathcal{L}}$ and and the relative entropy of $P_x$ to $\widehat{P}_x$ in time $t$ is bounded by $Ht$. $\widehat{\mathcal{L}}$ may not be unique, but the optimal choice, i.e the smallest possible $H$ will equal $I(\mu)$, providing the lower bound.

We will illustrate this in the context of diffusions on a $d$-torus.

$$\mathcal{L} = \frac{1}{2} \Delta + < b(x), \nabla >$$

and

$$\widehat{\mathcal{L}} = \frac{1}{2} \Delta + < c(x), \nabla >$$

d$\mu = \phi(x)dx$

c(x) should be such that

$$\frac{1}{2} \Delta \phi = \nabla \cdot c\phi$$

and

$$H(c) = \frac{1}{2} \int ||c - b||^2 \phi dx$$

What needs to be proven is the identity

$$- \inf_{u \in \mathcal{D}^+} \int \frac{\mathcal{L}u}{u} \phi dx = I(\phi) = \inf_{c: \frac{1}{2} \Delta \phi = \nabla \cdot c\phi} H(c)$$

Replacing $u$ by $e^{-v}$, the left hand side can be written as

$$\sup_v \int [\mathcal{L}v - \frac{1}{2} ||\nabla v||^2] \phi dx$$

The right hand side is rewritten as

$$\inf_c \sup_u \int [\frac{1}{2} ||c - b||^2 + \widehat{\mathcal{L}}u] \phi dx$$

Note that $\sup_u$ is $+\infty$ unless $\frac{1}{2} \Delta \phi = \nabla \cdot c\phi$, in which case it is 0. We now calculate

$$RHS = \inf_c \sup_u \int [\frac{1}{2} ||c - b||^2 + \frac{1}{2} \Delta u + c(x) \cdot \nabla u] \phi dx$$

$$= \sup_u \inf_c \int [\frac{1}{2} ||c - b||^2 + \frac{1}{2} \Delta u + c(x) \cdot \nabla u] \phi dx$$

$$= \sup_u \int [\mathcal{L}u - \frac{1}{2} ||\nabla u||^2] \phi dx$$

$$= LHS$$

because the $\inf_c$ can be done pointwise and

$$\inf_c \frac{1}{2} [||b - c||^2 + c \cdot p] = b \cdot c - \frac{1}{2} ||p||^2$$
Interesting counter example:

\[ \mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + a \frac{d}{dx} \]

If \( \phi(x) \) satisfies

\[ \frac{1}{2} \phi_{xx} = (c \phi)_x \]

then

\[ \frac{1}{2} \phi_x = c(x) \phi(x) + k \]

\( k = 0 \), because \( (c(x) - a)^2 \phi, \phi \in L_1(R) \). This forces \( \int c(x)\phi(x)dx = 0 \) and

\[ \int (c(x) - a)^2 \phi(x) dx \geq a^2 \]

In particular \( I(\mu) \geq \frac{a^2}{2} \). There is a locally uniform exponential rate. But the total probability is 1.