Let us return to our model. We have a solution of Kolmogorov’s forward equation

\[
\frac{\partial f_N(t, \xi_1, \ldots, \xi_N)}{\partial t} = N^2 A_N f_N
\]

where \( f_N \) is the density with respect to the invariant measure given by \( \exp[-\sum \phi(\xi_i)] \). We saw that the Dirichlet form satisfies.

\[
\int \sum |(D_i - D_{i+1}) f_N|^2 e^{-\sum \phi(\xi_i)} d\xi \leq \frac{C}{N}
\]

If we denote the marginal in a block of length \( \ell \) around \( x \) by \( f_{N,x,\ell}(\xi_1, \ldots, \xi_\ell) \) and by \( \bar{f}_{N,\ell} \) the average

\[
\bar{f}_{N,\ell} = \frac{1}{N} \sum_x f_{N,x,\ell}
\]

then the Dirichlet form

\[
\int \sum_{1 \leq i < \ell-1} |(D_{i+1} - D_i) \bar{f}_{N,\ell}|^2 e^{-\sum \phi(\xi_i)} d\xi \leq \frac{C \ell}{N^2}
\]

With a log Sobolev constant of \( \ell^2 \), we can apply log Sobolev inequality on each hyperplane \( \sum_i x_i = \ell a \). If we write \( \bar{f}_{N,\ell} e^{-\sum \phi(\xi_i)} d\xi = G_{N,\ell}(da) \bar{\nu}_{N,\ell}(a, d\xi) \) and similarly \( e^{-\sum_i \phi(\xi_i)} = \phi_{N,\ell}(a) \bar{\mu}_{N,\ell}(a, d\xi) \)

\[
\int H(\bar{\nu}_{N,\ell}(a, \cdot) | \bar{\mu}_{N,\ell}(a, \cdot)) G_{N,\ell}(da) \leq \frac{c \ell^3}{N^2}
\]

If we can prove a conditional version of Cramér’s Large deviation result, i.e. with \( \lambda = h'(a) \), and

\[
F(a) = \frac{1}{M(\lambda)} \int e^{\lambda \xi - \phi(\xi)} F(\xi) d\xi
\]

\[
P[\frac{1}{n} \sum_i F(\xi_i) - \bar{F}(a) \geq \gamma] \leq \frac{1}{n} \sum \xi_i = a \leq \exp[-nC(\delta)]
\]

where \( C(\delta) > 0 \), then it is not hard to see that with \( \ell = N\epsilon \),

\[
\int \frac{1}{N} \sum_i F(\xi_i) \bar{f}_{N,\ell}(\xi) \leq \frac{c (N\epsilon)^3}{N^3 \epsilon} = c \epsilon^2
\]

This will allow replacing averages like

\[
\frac{1}{N} \sum J(\frac{x}{N}) F(\xi)
\]

with

\[
\frac{1}{N} \sum J(\frac{x}{N}) \bar{F}(\frac{1}{2N\epsilon} \sum_{y: |y-x| \leq N\epsilon} \xi_y)
\]

with a small \( \epsilon \). The conditional version of Cramér’s theorem is not hard to prove.
Let us start with i.i.d.r.v’s with density $g(\xi)$. Let $g_n(\xi)$ be the density of $\frac{1}{n} \sum \xi_i$. Then

$$g_n(a) = \exp[-nh(a) + o(n)]$$

is the density version of the Large deviation result. Let us take this for granted. We want to compute

$$\lim_{n \to \infty} \frac{1}{n} \log E[e^{\rho \sum_i F(\xi_i)} | \sum_i \xi_i = na] = \psi(\rho, a)$$

Suppose we define

$$g_{\rho}(\xi) = \frac{1}{T(\rho)} e^{\rho F(\xi)} g(\xi)$$

normalized to be a probability distribution. Then the distribution of $\frac{1}{n} \sum \xi$ under the new distribution is given by a density $g_{n,\rho}(a)$ that will satisfy

$$g_{n,\rho}(a) = \exp[-n\psi(\rho, a) + o(n)]$$

It is not hard to see that

$$\psi(\rho, a) = \log T(\rho) - h(\rho, a) + h(a)$$

and

$$h(\rho, a) = \sup_{\theta} [\theta a - \log M(\rho, \theta)]$$

with

$$M(\rho, \theta) = \frac{1}{T(\theta)} \int e^{\rho F(\xi)} + \theta \xi g(\xi) d\xi$$

Getting exponential error estimate is just the differentiability of $\psi(\rho, a)$ at $\rho = 0$ which is obvious. Moreover

$$\psi_{\rho}(0, a) = \frac{T'(0)}{T(0)} - h_{\rho}(0, a)$$

and

$$h_{\rho}(0, a) = -\frac{M_{\rho}(0, \theta)}{M(0, \theta)} \frac{T'(0)}{T(0)} - \frac{1}{M(0, \theta)} \int e^{\theta \xi} F(\xi) g(\xi) d\xi$$

with $\theta = h'(a)$. Finally

$$\psi_{\rho}(0, a) = \frac{1}{M(0, \theta)} \int e^{\theta \xi} F(\xi) g(\xi) d\xi$$

with $\theta = h'(a)$. Finally let us prove the density version of Cramér’s theorem.

We write with $g_{\theta}(\xi) = \frac{1}{M(\theta)} e^{\theta \xi} g(\xi)$ and $\theta = h'(\xi)$, so that

$$\int \xi g_{\theta}(\xi) d\xi = a$$
\[
\int_{\xi=a} g(\xi_1) \cdots g(\xi_n) d\xi = M(\theta)^n e^{-na} \int_{\xi=a} g_{\theta}(\xi_1) \cdots g_{\theta}(\xi_n) d\xi
\]

\[
= M(\theta)^n e^{-na} g_{n,\theta}(a)
\]

Since now \(a\) is the mean, the density version of CLT yields

\[
g_{n,\theta}(a) \simeq \frac{1}{\sqrt{2\pi nC(a)}}
\]

Density version of CLT. It is sufficient to prove with mean 0 and variance 1

\[
\lim_{n \to \infty} \frac{1}{2\pi} \int \hat{g}(\frac{t}{\sqrt{n}})^n e^{-itx} dt \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\]

It is not hard to get a bound of the form

\[
|\hat{g}(t)| \leq (1 + dt^2)^{-c}
\]

and for \(n >> 1\),

\[
(1 + c\frac{t^2}{n})^{-nc} \leq (1 + c't^2)^{-1}
\]