Lecture 12.

Log-Sobolev Inequality. Let us consider on $\mathbb{R}$ the generator

$$\mathcal{L} = \frac{1}{2} D_x^2 + \frac{1}{2} b(x) D_x$$

where $b(x) = \frac{\phi'(x)}{\phi(x)}$. Clearly $\mathcal{L}$ is self adjoint with respect to the weight $\phi(x)$ and

$$\mathcal{L}u = \frac{1}{2} \phi D_x \phi(x) D_x u$$

or

$$\langle \mathcal{L}u, v \rangle = \int (\mathcal{L}u)(x) v(x) \phi(x) dx = -\frac{1}{2} \int u_x v_x \phi(x) dx$$

We will take $\phi(x) = \exp[-\psi(x)]$ with a uniformly convex $\psi$, i.e. $\psi_{xx} \geq c > 0$. $b = -\psi'(x)$ and $b_x \leq -c < 0$. We let $f(t, x) \geq 0 \in L_1(\phi dx)$ evolve according to the equation

$$f_t(t, x) = (\mathcal{L}f)(t, x)$$

Since $\mathcal{L}$ is self adjoint $L^* = L$ with respect to the weight $\phi$. We denote by

$$H(t) = \int f(t, x) \log f(t, x) \phi(x) dx$$

Then

$$\frac{dH}{dt} = -\frac{1}{2} \int \frac{[f_x(t, x)]^2}{f(t, x)} \phi(x) dx$$

We are interested in calculating

$$\frac{d^2H(t)}{dt^2} = -\frac{d}{dt} \frac{1}{2} \int \frac{[f_x(t, x)]^2}{f(t, x)} \phi(x) dx$$

which equals

$$\frac{1}{2} \int \frac{[f_x(t, x)]^2}{[f(t, x)]^2} (\mathcal{L}f)(t, x) \phi(x) dx - \int \frac{f_x(t, x)}{f(t, x)} (\mathcal{L}f)_x(t, x) \phi(x) dx$$

We note that

$$(\mathcal{L}f)_x = \mathcal{L}f_x + \frac{1}{2} b(x) f_x$$

Moreover

$$\frac{1}{2} \int \frac{[f_x(t, x)]^2}{[f(t, x)]^2} (\mathcal{L}f)(t, x) \phi(x) dx$$

$$= \frac{1}{2} \langle \frac{f_x^2}{f^2}, \mathcal{L}f \rangle = \frac{1}{2} \langle \mathcal{L}(\frac{f_x^2}{f^2}), f \rangle$$

$$\geq \langle \frac{f_x}{f} \mathcal{L}(\frac{f_x}{f}), f \rangle = \langle \mathcal{L}(\frac{f_x}{f}), f_x \rangle = \langle \frac{f_x}{f}, \mathcal{L}f_x \rangle$$

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Therefore
\[ \frac{d^2H(t)}{dt^2} \geq -\frac{1}{2} \int \frac{b_x f_x^2}{f} \phi(x) dx \geq \frac{c}{2} \int \frac{[f_x(t,x)]^2}{f(t,x)} \phi(x) dx = -c \frac{dH(t)}{dt} \]

If we denote by \( I(t) = -\frac{dH(t)}{dt} \), then \( \frac{dI(t)}{dt} \leq -cI(t) \), providing \( \int_0^\infty I(s) ds \leq \frac{1}{c} I(0) \). But
\[ H(0) = \int_0^\infty I(t) dt \leq \frac{1}{c} I(0) \]

We have assumed \( H(\infty) = 0 \). True for a dense set.

Suppose we are in \( \mathbb{R}^d \) and we have a generator of the type
\[ \mathcal{L}u = \frac{1}{2\phi} \nabla \cdot \phi \nabla u \]
with a positive definite symmetric \( C \), (independent of \( x \)) which is self adjoint with respect to the weight \( \phi(x) = e^{-\psi(x)} \). Then
\[ H(t) = \int f(t,x) \log f(t,x) \phi(x) dx \]
\[ I(t) = \frac{1}{2} \int \frac{\langle C \nabla f, \nabla f \rangle}{f} \phi dx \]

The crucial step is to estimate
\[ \frac{dI(t)}{dt} = -\frac{1}{2} \langle \mathcal{L} f, \frac{\langle C \nabla f, \nabla f \rangle}{f^2} \rangle + \int \frac{\langle C \nabla \mathcal{L} f, \nabla f \rangle}{f^2} \phi dx \]
and note that \( \nabla \mathcal{L} f = \nabla \mathcal{L} f + (Db) \nabla f \). We need to bound
\[ \langle C \nabla f, (Db) \nabla f \rangle \leq -c \langle C \nabla f, \nabla f \rangle \]

In our particular context we are looking at \( \mathbb{R}^{N-1} \) represented as the hyperplane \( \sum_{i=1}^N x_i = Nm \). \( C \) is the quadratic form
\[ \langle Cu, u \rangle = \sum_{i=2}^N (u_i - u_{i-1})^2 \]
\( \hat{\psi} \) is the restriction of \( \sum_i \psi(x_i) \) to the hyperplane. \( -Db \) is the Hessian of \( \psi \). The matrix representing \( C \) is
\[
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\end{pmatrix}
\]
It is not hard to see that \( c = c_N = aN^{-2} \) where \( a \) is a lower bound on \( \psi''(x) \).

Our density \( f(x) \) on \( \mathbb{R}^N \) is written as the superposition of densities \( \mu_m = f_m(x)\lambda_m(dx) \) on the hyperplane relative to the conditionals \( \lambda_m \), which are just \( \pi\phi(x_i) \) normalized on the hyperplane.

\[
f(x)dx = \int \mu_m(x)d\nu(m)
\]

The entropy under control is

\[
\int H(\mu_m|\lambda_m)d\nu(m) \leq cN^2I(f)
\]

which is obtained on each hyperplane and integrated with respect to \( m \).

**Large deviation estimates.** With respect to \( \lambda_m \) the probability

\[
\lambda_m\left[ \left| \frac{1}{N} \sum g(x_i) - \hat{g}(m) \right| \geq \delta \right] = \exp[-c(\delta)N]
\]

where

\[
\hat{g}(m) = \frac{1}{M(\lambda)} \int g(x)e^{\lambda x}\phi(x)dx
\]

with \( \lambda = \lambda(m) \). Note that \( N = N\epsilon, I(f) = \frac{\epsilon}{N} \). Therefore

\[
\int H(\mu_m|\lambda_m)d\nu(m) \leq cN\epsilon^3
\]

Since \( m = \bar{x} \), with respect to \( \lambda_m \) and \( \mu_m \), we have

\[
\mu_m\left[ \left| \frac{1}{N} \sum g(x_i) - \hat{g}(m) \right| \geq \delta \right] \leq c(\delta)\epsilon^2
\]

This estimate is valid uniformly over bounded set of values of \( m \). Integrate w.r.t \( \nu \). Entropy controls integrability. For instance, since \( \psi(x) \geq cx^2 \), with \( w(x) = \sqrt{1 + x^2} \),

\[
\int \exp\left[ \sum_{i=1}^{N} w(x_i) \right] e^{-\sum_{i=1}^{N} \psi(x_i)} dx \leq e^{CN}
\]

If \( f(t, x) \) is the density at time \( t \), then

\[
H(t) = \int f(t, x)\log f(t, x)\Pi\phi(x_i)dx \leq H(0) \leq CN
\]

The entropy inequality states

\[
\int Fd\lambda \leq H(\lambda|\mu) + \log \left[ \int e^F d\mu \right]
\]
Proof:

\[
\sup_x \{xy - x \log x + x\} = ye^y - ye^y + e^y = e^y
\]

Therefore

\[
\int F f d\mu \leq \int e^F d\mu + \int [f \log f - f] d\mu
\]

Since \( \int f d\mu = 1 \), we can replace \( F \) by \( F + c \) to get

\[
\int F f d\mu \leq \int e^{F+c} d\mu - c - 1 + \int f \log f d\mu = e^c \int e^F d\mu - c - 1 + H(\lambda|\mu)
\]

Minimize with respect to \( c \). \( c = -\log \int e^F d\mu \). Provides control. In particular taking \( F = k 1_A(x) \),

\[
k\lambda(A) \leq H(\lambda|\mu) + \log[e^k \mu(A) + (1 - \mu(A))]
\]

with \( k = \log \frac{1}{\mu(A)} \), we get

\[
\lambda(A) \leq \frac{2 + H(\lambda|\mu)}{\log \frac{1}{\mu(A)}}
\]

Convexity of \( Q(f) = D(\sqrt{f}) \) is an immediate consequence of the variational formula

\[
Q(f) = -\inf_{u > 0} \int \frac{(Lu)(x)}{u(x)} f(x) d\mu
\]