Chapter 5

Stochastic Integrals and Itô’s formula.

We will call an Itô process a progressively measurable almost surely continuous process $x(t, \omega)$ with values in $\mathbb{R}^d$, defined on some $(\Omega, \mathcal{F}_t, P)$ that is related to progressively measurable bounded functions $[a(s, \omega), b(s, \omega)]$ in the following manner.

$$\exp[(\theta, x(t, \omega) - x(0, \omega) - \int_0^t b(s, \omega)ds) - \frac{1}{2} \int_0^t \langle \theta, a(s, \omega)\theta \rangle ds]$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ for all $\theta \in \mathbb{R}^d$. A canonical example is Brownian motion that corresponds to $b(s, \omega) \equiv 0$ and $a(s, \omega) \equiv 1$ or $a(s, \omega) \equiv I$ in higher dimensions. We will abbreviate it by $x(\cdot) \in I(a, b)$. Such processes are not of bounded variation unless $a \equiv 0$. In fact they have nontrivial quadratic variation.

Lemma 5.1. If $x(\cdot)$ is a one dimensional process and $x(\cdot) \in I(a, b)$ then

$$\lim_{n \to \infty} \sum_{j=1}^n |x(jT/n) - x((j-1)T/n)|^2 = \int_0^T a(s, \omega)ds$$

in probability and in $L_1(P)$.

Proof. If $y(t) = x(t) - x(0) - \int_0^t b(s, \omega)ds$, then $y(\cdot) \in I(a, 0)$ and the difference between $x(\cdot)$ and $y(\cdot)$ is a continuous function of bounded variation. It is therefore sufficient to show that

$$\lim_{n \to \infty} \sum_{j=1}^n |y(jT/n) - y((j-1)T/n)|^2 = \int_0^T a(s, \omega)ds$$

If we denote by

$$Z_j = |y(jT/n) - y((j-1)T/n)|^2 - \int_{(j-1)T/n}^{jT/n} a(s, \omega)ds$$

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then \( E[Z_j] = 0 \) and for \( i \neq j \), \( E[Z_i Z_j] = 0 \). It is therefore sufficient to show
\[
E[|Z_j|^2] \leq \frac{C(T)}{n^2}.
\]
This follows easily from the Gaussian bound
\[
E[e^\lambda(y(t_2) - y(t_1))] \leq e^{\frac{C\lambda^2(t_2 - t_1)}{2}}
\]
provided \( a(s, \omega) \leq C \). We see that \( E[(y(t_2) - y(t_1))^4] \leq C(t_2 - t_1)^2 \).

This means that integrals of the form \( \int_0^t e(s, \omega) dx(s, \omega) \) have to be carefully defined. Since the difference between \( x(\cdot) \) and \( y(\cdot) \) is of bounded variation it suffices to concentrate on \( \int_0^t e(s, \omega) dx(s, \omega) \). We develop these integrals in several steps, each one formulated as a lemma.

**Lemma 5.2.** Let \( S \) be the space of functions \( e(s, \omega) \) that are uniformly bounded piecewise constant progressively measurable functions of \( s \). In other words there are intervals \([t_{j-1}, t_j)\) in which \( e(s, \omega) \) is equal to \( e(t_{j-1}, \omega) \) which is \( F_{t_{j-1}} \)-measurable. We define for \( t_{k-1} \leq t \leq t_k \)
\[
\xi(t) = \int_0^t e(s, \omega) dy(s) = \sum_{j=1}^{k-1} e(t_{j-1}, \omega)[y(t_j) - y(t_{j-1})] + e(t_{k-1}, \omega)[y(t) - y(t_{k-1})]
\]
The following facts are easy to check.

1. \( \xi(t) \) is almost surely continuous, progressively measurable. Moreover \( \xi(\cdot) \in \mathcal{I}(e^2(s, \omega)a(s, \omega), 0) \).
2. The space \( S \) is linear and the map \( e \to \xi \) is a linear map.
3. \[
E\left[ \sup_{0 \leq s \leq t} |\xi(s, \omega)|^2 \right] \leq 4E\left[ \int_0^t |e(s, \omega)|^2 a(s, \omega) ds \right]
\]
4. In particular if \( e_1, e_2 \in S \), and for \( i = 1, 2 \)
\[
\xi_i(t) = \int_0^t e_i(s, \omega) dy(s)
\]
then
\[
E\left[ \sup_{0 \leq s \leq t} |\xi_1(s, \omega) - \xi_2(s, \omega)|^2 \right] \leq 4E\left[ \int_0^t |e_1(s, \omega) - e_2(s, \omega)|^2 a(s, \omega) ds \right]
\]

**Proof.** It is easy to see that, because for \( \lambda \in \mathbb{R} \),
\[
E[\exp\{\lambda y(t) - y(s)\}] = e^{\frac{\lambda^2}{2} \int_s^t a(u, \omega) du} \mathbb{P}(s) = 1
\]
it follows that if \( \lambda \) is replaced by \( \lambda(\omega) \) that is bounded and \( \mathcal{F}_t \) measurable then

\[
E[\exp[\lambda(s, \omega)(y(t) - y(s))] - \frac{\lambda(s, \omega)^2}{2} \int_s^t a(u, \omega)du | \mathcal{F}_s] = 1
\]

We can take \( \lambda(s, \omega) = \lambda e^t(s, \omega) \). This proves 1. 2 is obvious and 3 is just Doob’s inequality. 4 is a restatement of 3 for the difference.

**Lemma 5.3.** Given a bounded progressively measurable function \( e(s, \omega) \) it can be approximated by a sequence \( e_n \in S \), such that \( \{e_n\} \) are uniformly bounded and

\[
\lim_{n \to \infty} E[\int_0^T |e_n(s, \omega) - e(s, \omega)|^2ds] = 0
\]

As a consequence the sequence \( \xi_n(t) = \int_0^t e_n(s, \omega)dy(s) \) has a limit \( \xi(t, \omega) \) in the sense

\[
\lim_{n \to \infty} E[\sup_{0 \leq s \leq t} |\xi_n(s) - \xi(s)|^2] = 0
\]

It follows that \( \xi(t, \omega) \) is almost surely continuous and \( \xi(\cdot) \in \mathcal{I}(e^2(s, \omega)a(s, \omega)) \).

**Proof.** It is enough to prove the approximation property. Since

\[
Y_\lambda(t) = \exp[\lambda \xi_n(t)] - \frac{\lambda^2}{2} \int_0^s e_n^2(s, \omega)a(s, \omega)ds
\]

are martingales and \( e_n^2a \) has uniform bound \( C \), it follows that

\[
\sup_{0 \leq t \leq T} \sup_{n} E[\exp[\lambda \xi_n(t)]] \leq \exp[\frac{C\lambda^2T}{2}]
\]

providing uniform integrability. We note that

\[
\lim_{n,m \to \infty} E[\sup_{0 \leq s \leq t} |\xi_n(s) - \xi_m(s)|^2] = 0
\]

Now it is easy to show that \( \xi_n(\cdot) \) has a uniform limit in probability and pass to the limit. To prove the approximation property we approximate first \( e(s, \omega) \) by

\[
e_h(s, \omega) = \frac{1}{h} \int_{(s-h)v0}^s e(u, \omega)du
\]

It is a standard result in real variables that \( \|e_h(\cdot) - e_h(\cdot)\|_2 \to 0 \) as \( h \to 0 \) and \( e_h \) is continuous in \( s \). Note that we only look back and not ahead, thus preserving progressive measurability. We can now approximate \( e_h(s, \omega) \) by \( e_h(\frac{\text{int}}{h}, \omega) \) that are again progressively measurable, but simple as well, so they are in \( S \).

**Lemma 5.4.** If \( e(s, \omega) \) is progressively measurable and satisfies

\[
E[\int_0^T e^2(s, \omega)a(s, \omega)ds] < \infty
\]
we can define on \([0, T]\),
\[
\xi(t) = \int_0^t e(s, \omega)dy(s)
\]
as a square integrable martingale and
\[
\xi(t)^2 - \int_0^t e^2(s, \omega)a(s, \omega)ds
\]
will be a martingale.

**Proof.** The proof is elementary. Just approximate \(e\) by truncated functions
\[
e_n(s, \omega) = e(s, \omega)1_{|e(s, \omega)| \leq n}(\omega)
\]
and pass to the limit. Again
\[
\lim_{n,m \to \infty} E[ \sup_{0 \leq s \leq t} |\xi_n(s) - \xi_m(s)|^2 ] = 0
\]

**Remark 5.1.** If \(x(\cdot) \in I(a, b)\) we can let \(y(t) = x(t) - \int_0^t b(s, \omega)ds\) and define
\[
\xi(t) = \int_0^t e(s, \omega)dx(s) = \int_0^t e(s, \omega)dy(s) + \int_0^t e(s, \omega)b(s, \omega)ds
\]
If
\[
E[\int_0^t b^2(s, \omega)e^2(s, \omega)ds] < \infty
\]
then we can check \(\xi\) is well defined. In fact we can define for bounded progressively measurable \(e, c,\)
\[
\xi(t) = \int e(s, \omega)dx(s) + \int c(s, \omega)ds
\]
It is easy to check that
\[
\xi(\cdot) \in I(e^2(s, \omega)a(s, \omega), e(s, \omega)b(s, \omega) + c(s, \omega))
\]
Recall that if \(X \sim N[\mu, \sigma^2]\) and \(Y = aX + b\) then \(Y \sim N[a\mu + b, a^2\sigma^2]\).

**Remark 5.2.** We can have \(x(t) \in \mathbb{R}^d\) and \(x(\cdot) \in I(a, b)\), where \(a = a(s, \omega)\)
is a symmetric positive semidefinite matrix valued bounded progressively measurable function and \(b = b(s, \omega)\) is an \(\mathbb{R}^d\) valued, bounded and progressively measurable. We can the define
\[
\xi(t) = \int_0^t e(s, \omega) \cdot dx(s) + \int c(s, \omega)ds
\]
where \( e(s, \omega) \) is a progressively measurable bounded \( k \times d \) matrix and \( c \) is \( \mathbb{R}^k \) valued, bounded and progressively measurable. The integral is defined by each component. For \( 1 \leq i \leq k \),

\[
\xi_i(t) = \sum_j \int_0^t e_{i,j}(s, \omega) \cdot dx_j(s) + \int c_i(s, \omega)ds
\]

The one verifies easily that

\[
\xi(\cdot) \in \mathcal{I}(ca^*, eb + c)
\]

**Theorem 5.5. Itô’s formula.** Consider a smooth function \( f(t, x) \) on \([0, T] \times \mathbb{R}^d\). Let \( x(t) \) with values in \( \mathbb{R}^d \) belong to \( \mathcal{I}(a, b) \). Then almost surely

\[
f(t, x(t)) = f(0, x(0)) + \int_0^t f_s(s, x(s))ds + \int_0^t (\nabla_x f)(s, x(s)) \cdot dx(s) + \frac{1}{2} \int_0^t \sum a_{i,j}(s, \omega)(D_{x_i,x_j}f)(s, x(s))ds
\]

**Proof.** Consider the \( d + 1 \) dimensional process \( Z(t) = (f(t, x(t)), x(t)) \). If \( \sigma \in \mathbb{R} \) and \( \lambda \in \mathbb{R}^d \), then if we consider \( g(t, x) = \sigma f(t, x) + \langle \lambda, x \rangle \) we know that

\[
\exp[\sigma g(t, x(t)) - g(0, x(0)) - \int_0^t e^{-g}[\partial_s e^g + L_{s, \omega} e^g](s, x(s))ds]
\]

is a martingale. A computation yields

\[
e^{-g}[\partial_s e^g + L_{s, \omega} e^g] = \partial_s g + L_{s, \omega} g + \frac{1}{2} \langle \nabla g, a \nabla g \rangle
\]

\[
= \sigma \partial_s f + \sigma L_{s, \omega} f + \langle \lambda, b(s, \omega) \rangle + \frac{1}{2} \langle (\sigma \nabla f + \lambda), a(s, \omega)(\sigma \nabla f + \lambda) \rangle
\]

Implies that \( Z(t) \in \mathcal{I}(\tilde{a}, \tilde{b}) \), where

\[
\tilde{a} = \begin{pmatrix}
\langle \nabla f, a \nabla f \rangle & (a \nabla f)^{tr} \\
(a \nabla f) & a
\end{pmatrix}
\]

and

\[
\tilde{b} = (\partial_s f + L_{s, \omega} f, b)
\]

Now we can compute that \( w(\cdot) \in \mathcal{I}(A, B) \) where

\[
w(t) = \int_0^t \cdot df(s, x(s)) - \int_0^t (\partial_s f)(s, x(s))ds - \int_0^t (\nabla_x f)(s, x(s)) \cdot dx(s)
\]

\[- \frac{1}{2} \int_0^t \sum_{i,j} a_{i,j}(s, \omega)(D_{x_i,x_j}f)(s, x(s))ds
\]
If we can calculate and show that $A = 0$ and $B = 0$, this would imply that $w(t) \equiv 0$ and that proves the theorem.

$$
A = (1, -\nabla f) \left( \begin{pmatrix} \langle \nabla f, a \nabla f \rangle \\ a \end{pmatrix} - \nabla f \right) = 0
$$

$$
B = \partial_s f + L_{s, \omega} f - b \cdot \nabla f - \frac{1}{2} \sum_{i,j} a_{i,j}(s, \omega)(D_{x_i x_j} f) = 0
$$

Remark 5.3. If $x(\cdot) \in \mathcal{I}(a, b)$ and $y(t) = \int_0^t \sigma(s, \omega) \cdot dx(s) + \int_0^t c(s, \omega) ds$ we saw that

$$
y(\cdot) \in \mathcal{I}(\tilde{a}, \tilde{b})
$$

where

$$
\tilde{a} = \sigma a \sigma^*, \tilde{b} = \sigma b + c
$$

This is like linear change of variables of a Gaussian vector. $dx \simeq N[adtdt, bdtdt]$ and $\sigma dx + c \simeq N[\sigma a \sigma^* dt, (\sigma b + c) dt]$. We can develop stochastic integrals of $y(\cdot)$ and if $dz = \sigma' dy + c' dt$ then $dz = \sigma' \sigma dx + (\sigma' c + c') dt$. If $\sigma$ is a invertible then $dy = \sigma dx + c dt$ can be inverted as $dx = \sigma^{-1} dy - \sigma^{-1} c dt$.

Finally one can remember Itô’s formula by the rules

$$
df(t, x(t)) = f_t dt + \sum_i f_{x_i} dx_i + \frac{1}{2} \sum_{i,j} f_{x_i x_j} dx_i dx_j
$$

If $x(\cdot) \in \mathcal{I}(a, b)$ then $dx_i dx_j = a_{i,j} dt$. $(dt)^2 = dt dx_i = 0$. Because the typical paths have half a derivative (more or less) $dx \simeq \sqrt{dt}$. $dx_i dx_j$ is of the order of $dt$ and $dt dx_i, (dt)^2$ are negligible.