Assignment 6.

Due Oct 30.

1. We have a multinomial distribution with 4 possibilities 1, 2, 3, 4 with probabilities \( \{p_i(\theta)\}, i = 1, 2, 3, 4 \) given by

\[
p_1(\theta) = \frac{\theta}{3}, \quad p_2(\theta) = \frac{2\theta}{3}, \quad p_3(\theta) = \frac{2(1 - \theta)}{3}, \quad p_4 = \frac{1 - \theta}{3}
\]

that depend on a parameter \( \theta \) between 0 and 1. We have \( n \) independent observations \( \{X_i\} \) each one being 1, 2, 3 or 4. What is the MLE? What is \( I(\theta) \) and the Cramér-Rao bound? Can you verify directly the asymptotic normality of the MLE with variance achieving the Cramér-Rao bound? Use the fact that the frequencies \( f_1, f_2, \ldots, f_k \) of an i.i.d sample from a multinomial with probabilities \( p_1, p_2, \ldots, p_k \) satisfy a multivariate CLT with \( E[f_i] = np_i \) and

\[
E\left[ \left( \frac{f_i - np_i}{\sqrt{n}} \right)^2 \right] = p_i(1 - p_i)
\]

and for \( i \neq j \)

\[
E\left[ \left( \frac{f_i - np_i}{\sqrt{n}} \right) \left( \frac{f_j - np_j}{\sqrt{n}} \right) \right] = -p_i p_j
\]

2. A sample of size 65 from a normal population with unknown mean \( \mu \) and unknown variance \( \theta \) had a sample mean of 2.1 with a sample variance of 0.64. Construct a 99% confidence interval for \( \mu \).