Homework 7

1. Find all integer solutions of $y^2 = x^3 + 1$.

2. Show that
$$\log(\zeta(s)) - \sum_p \frac{1}{p^s}$$
remains bounded for $s \to 1 + 0$.

3. Let χ be a nontrivial character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $0 < a < b < p$. Show that
$$|\sum_{a<n\leq b} \chi(n)| < \sqrt{p}\log(p)$$

4. Prove that
$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \quad n \geq 1.$$

5. Show that for $B \to \infty$, the number of ordered quadruples (a, b, c, d) of integers in the interval $[1, B]$ such that $\gcd(a, b) = \gcd(c, d)$ is asymptotic to $\frac{2}{5}B^4$.