Lecture 9

Plan

- Transcendence

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- Adeles

Approximations

Reminder: For all $\alpha \in \mathbb{R}$ and all $t \in \mathbb{N}$ there exist $a, q \in \mathbb{Z}$ such that

$$|\alpha - \frac{a}{q}| < \frac{1}{qt} < \frac{1}{q^2}$$

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α is irrational iff

$$|\alpha - \frac{a}{q}| < \frac{1}{q^2}$$

has infinitely many solutions.

Examples

$$\alpha := \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k!}}$$

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Put

$$\frac{a}{q} := \sum_{k=0}^{m} \frac{(-1)^k}{2^{k!}}.$$

Then

$$|\alpha - \frac{a}{q}| < \frac{1}{2^{(m+1)!}} = \frac{1}{q^{m+1}}, \quad q := 2^{m!}$$

and

$$|\alpha - \frac{a}{q}| < \frac{1}{q^m}$$

infinitely often.

Examples

Theorem (Khinchin)

Let $\psi(x)$ be a decreasing function on $\mathbb{N}$, taking values in $(0, 1/2)$.

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Let $\psi(x)$ *be a* *decreasing* *function on* $\mathbb{N}$*, taking values in* $(0, 1/2)$*. Consider the inequality*

$$|\alpha - \frac{a}{q}| < \frac{\psi(q)}{q} \quad (*).$$

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$$|\alpha - \frac{a}{q}| < \frac{\psi(q)}{q} \quad (*).$$

Consider

$$\sum_{q \geq 1} \psi(q) \quad (**)$$

1. *If (**) diverges then, for almost all α (in the sense of Lebesgue measure), (*) has infinitely many solutions in the rationals.*

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Consider

$$\sum_{q \geq 1} \psi(q) \quad (**)$$

1. *If (**) diverges then, for almost all α (in the sense of Lebesgue measure), (*) has infinitely many solutions in the rationals.*
2. *Otherwise, for almost all α, (*) has finitely many solutions.*

Algebraic tools

Recall the basic theory of $\mathbb{Q}[x]$: division with remainder, Euclidean algorithm, etc.

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If $f \in \mathbb{Q}[x]$ is such that $f(\alpha) = 0$ then $\phi \mid f$.

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Proof: Division with remainder.

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Let $f \in \mathbb{Q}[x]$ be irreducible, and suppose that $g \in \mathbb{Q}[x]$ has common roots with f. Then $f \mid g$, and all roots of f are roots of g.

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Corollary: If $f \in \mathbb{Q}[x]$ is irreducible, then it has no multiple roots, otherwise $f'(\alpha) = 0$, but $\deg(f') < \deg(f)$, contradiction.

Algebraic numbers

Algebraic numbers form a field:

$$\alpha, \beta \in \bar{\mathbb{Q}} \Rightarrow \alpha \pm \beta, \alpha \cdot \beta, \alpha/\beta \in \bar{\mathbb{Q}}.$$

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Theorem (Cantor 1874)

$\bar{\mathbb{Q}}$ is countable, but $\mathbb{R}$ is uncountable.

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Theorem (Cantor 1874)

$\bar{\mathbb{Q}}$ *is countable, but* $\mathbb{R}$ *is uncountable.*

Thus there are infinitely many transcendental numbers. But it is hard to produce an explicit transcendental number!

Algebraic numbers

Let $\alpha \in \mathbb{R} \cap \bar{\mathbb{Q}}$ be of degree $n \geq 2$.

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Let $\alpha \in \mathbb{R} \cap \bar{\mathbb{Q}}$ be of degree $n \geq 2$. Then there exists a constant $c = c(\alpha) \in (0, 1)$ such that for all $a, q \in \mathbb{Z}$ one has

$$|\alpha - \frac{a}{q}| > \frac{c}{q^n}$$

Algebraic numbers

Proof: Let

$$f(x) = a_n x^n + \cdots a_1 x + a_0, \quad a_i \in \mathbb{Z}, a_n > 0$$

with root $\alpha = \alpha_1$ and other roots $\alpha_2, \ldots, \alpha_n$.

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If $|\frac{p}{q} - \alpha| \geq 1$ then there is nothing to prove. Assume < 1 and estimate

$$|\alpha_i - \frac{p}{q}| \leq |\alpha_i - \alpha| + |\alpha - \frac{p}{q}| \leq \underbrace{2\max(|\alpha_i|) + 1}_{R}$$

Algebraic numbers

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$$\frac{1}{q^b} \cdot \underbrace{c}_{1/a_n R^{n-1}} \leq |\alpha - \frac{p}{q}|$$

Algebraic numbers

A reformulation: Let $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$, of order d. Then there exist at most finitely many approximations to

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Theorem (Roth)

In fact, there are only finitely many approximations to

$$|\alpha - \frac{p}{q}| \leq \frac{1}{q^{2+\epsilon}}, \quad \epsilon > 0$$

Applications

Theorem

There are only finitely many solutions in $x, y \in \mathbb{Z}$ to

$$x^3 - 2y^3 = a, \quad a \in \mathbb{Z}.$$

Applications

Proof: As before, write

$$x^3 - 2y^3 = (x - \sqrt[3]{2}y)(x - \zeta\sqrt[3]{2}y)(x - \zeta^2\sqrt[3]{2}y)$$

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&\geq |\frac{x}{y} - \sqrt[3]{2}| \cdot |\Im(\zeta\sqrt[3]{2})| \cdot |\Im(\zeta^2\sqrt[3]{2})| \\
&= |\frac{x}{y} - \sqrt[3]{2}| \cdot \frac{3}{2^{4/3}}.
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&= |\frac{x}{y} - \sqrt[3]{2}| \cdot \frac{3}{2^{4/3}}.
\end{aligned}$$

Every solution gives an approximation to $\sqrt[3]{2}$ of order $\frac{1}{y^3}$, thus there are finitely many such solutions.

Special functions

Among all nonelementary functions encountered in solving the simplest and most important equations are functions that appear repeatedly, and therefore have been well studied and given various names. Such functions are customarily called special functions.

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given by

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- Dirichlet series $\sum \frac{a_n}{n^s}$

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Examples:

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Main interest: Study objects where arithmetic and transcendental nature are closely combined, e.g.,

$$e^{i\pi} = -1, \quad \bar{\mathbb{Q}}^{ab} = \cup_n \mathbb{Q}(e^{\frac{2\pi i}{n}})$$

e

$$e = \sum_{n \geq 0} \frac{1}{n!}$$

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- Fourier 1815:

$$0 < n! \cdot e - A_n = O(\frac{1}{n})$$

with

$$A_n := n! \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Z}.$$

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If $e \in \mathbb{Q}$ then we get an infinite sequence of integers with limit 0.

e

- Liouville 1840:

$$n! \cdot e^{-1} - C_n = O(\frac{1}{n}), \quad C_n := n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

which implies that $e^2 \notin \mathbb{Q}$ and $ae^2 + be + c = 0$ is not solvable, with $a, b, c \in \mathbb{Z}$.

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Indeed,

$$n!(ae + b + ce^{-1}) - \underbrace{(aA_n + bn! + cC_n)}_{d_n} = O(\frac{1}{n})$$

Thus $d_n = O(\frac{1}{n})$ and $d_n = 0$, $\forall n > n_0$.

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Thus $d_n = O(\frac{1}{n})$ and $d_n = 0$, $\forall n > n_0$. On the other hand,

$$d_{n+2} - (n+1)(d_{n+1} + d_n) = 2a \quad \Rightarrow \quad \exists d_n \neq 0.$$

e

$$e^r \notin \mathbb{Q}, \quad \forall r \in \mathbb{Q} \setminus 0$$

e

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I.e., the graph of $y = e^x$ avoids all rational points, except $(0, 1)$.

Proof: Consider

$$f(x) := \frac{x^n(1-x)^n}{n!} = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i, \quad c_i \in \mathbb{Z}.$$

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- $$f(x) = f(1-x) \quad \Rightarrow f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$$
  $$f^{(k)}(x) = \frac{k!}{n!}c_k + \cdots$$

e

- Let

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- $$(e^{sx} F(x))' = se^{sx} F(x) + e^{sx} F'(x) = s^{2n+1} e^{sx} f(x)$$

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Assume that

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Consider

$$N := b\int_0^1 s^{2n+1} e^{sx} f(x)dx = b[e^{sx}F(x)]_0^1$$

$$aF(1) - bF(0) \in \mathbb{Z}$$

since $F(1), F(0) \in \mathbb{Z}$.

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$$0 < \underbrace{N}_{\in\mathbb{Z}} < b \cdot \frac{s^{2n+1}e^s}{n!}.$$

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$$0 < \underbrace{N}_{\in \mathbb{Z}} < b \cdot \frac{s^{2n+1} e^s}{n!}.$$

It remains to choose n so that the right side is < 1. □

π

Same argument works for π^2.

Assume that $\pi^2 = \frac{a}{b}, a, b \in \mathbb{Z}_{>0}$. Put

$$F(x) = b^n(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \cdots \pm \cdots)$$

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As before, we find

$$F''(x) = -\pi^s F(x) + b^n \pi^{2n+2} f(x), \quad F(0), F(1) \in \mathbb{Z}.$$

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Have:

$$\begin{aligned}(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x))' &= (F''(x) + \pi^2 F(x))\sin(\pi x) \\ &= b^n \pi^{2n+2} f(x)\sin(\pi x) \\ &= \pi^2 a^n f(x)\sin(\pi x)\end{aligned}$$

π

$$
\begin{aligned}
0 < N &:= \pi \int_0^1 a^n f(x) \sin(\pi x) dx \\
&= [\frac{1}{\pi} F'(x) \sin(\pi x) - F(x) \cos(\pi x)]_0^1 \\
&= F(0) + F(1) \in \mathbb{Z}
\end{aligned}
$$

but

$$
< \pi \frac{a^n}{n!} \quad n \gg 0
$$

which is a contradiction.

Theorem (Lambert 1766)

$$\pi \notin \mathbb{Q}$$

Proof (Hermite):

1. $$\pi^{2n+1}\int_0^1 t^n(1-t)^n\sin(\pi t)\,dt = n!\,Q(\pi), \quad Q\in\mathbb{Z}[x], \quad \forall n\in\mathbb{N}$$

   This is proved by induction, using integration by parts.

Proof (Hermite):

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2. $Q(\pi)\neq 0$ – look at the integral

Further results

- Lindemann 1882: $\alpha \in \bar{\mathbb{Q}} \setminus 0 \Rightarrow e^{\alpha} \notin \bar{\mathbb{Q}}$

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- Lindemann-Weierstrass 1885: Let $\alpha_0, \ldots, \alpha_m \in \bar{\mathbb{Q}}$ be distinct. Then

$$e^{\alpha_0}, \ldots, e^{\alpha_m}$$

  are linearly independent over $\bar{\mathbb{Q}}$

Rationality

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Periods

A period is a number $s \in \mathbb{C}$ such that

$$\Im(s), \Re(s)$$

are absolute convergent integrals of rational functions $f \in \mathbb{Q}[x_1, \ldots, x_n]$ over domains in $\mathbb{R}^n$, given by polynomial inequalities with $\mathbb{Q}$-coefficients.

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$$\mathcal{P} := \{s\}_{\text{periods}}, \quad \hat{\mathcal{P}} := \cup_{n \geq 0} \frac{1}{(2\pi i)^n} \mathcal{P}$$

Periods

Fact: $\mathcal{P}$ is countable,

$$\bar{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}.$$

Period relations: examples

$$\log(4) = \int_1^4 \frac{dx}{x} = \int_1^2 \frac{dx}{x} + \int_2^4 \frac{dx}{x} = 2\int_1^2 \frac{dx}{x} = 2\log(2)$$

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Substitute

$$x := u^2 \frac{1+v^2}{1+u^2}, \quad y := v^2 \frac{1+u^2}{1+v^2},$$

this will lead to

$$I := 2\int_0^\infty \frac{du}{1+u^2} \cdot \int_0^\infty \frac{dv}{1+v^2} = \frac{\pi^2}{2}$$

Period relations

Standard rules:

- Additivity

$$\int_a^b (f(x)+g(x))dx = \cdots, \quad \int_a^b f(x)dx = \int_a^c \cdots + \int_c^b \cdots$$

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- Stokes (Newton–Leibniz)

$$\int_a^b f'(x)dx = F(b) - F(a)$$

Periods

Conjecture

- If $s \in \mathcal{P}$ has two integral representations, then one can pass between them using the above rules.

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- If $s \in \mathcal{P}$ has two integral representations, then one can pass between them using the above rules.
- Any polynomial relation between periods is obtained through manipulations of the defining integrals using the above rules.

Periods: Γ-version

For $a \in \mathbb{Q}$ we have

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1. $\Gamma(a+1) = a\Gamma(a)$
2. $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$
3. $$\prod_{k=0}^{n-1} \Gamma(a + \frac{k}{n}) = (2\pi)^{\frac{n-1}{2}} n^{-na+\frac{1}{2}} \Gamma(na)$$

Periods: Γ-version

Proof:

$$\frac{\Gamma(a)\Gamma(1)}{\Gamma(a+1)} = \int_0^1 x^{a-1}(1-x)^{1-1}dx = \frac{1}{a}$$

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$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= \int_0^1 x^{a-1}(1-x)^{-a}dx \\ &= \int_0^1 \left(\frac{x}{1-x}\right)^a \frac{dx}{x} \\ &= \int_0^\infty u^a \frac{du}{1+u}\end{aligned}$$

Periods: Γ-version

Conjecture (Rohrlich)

Every multiplicative relation of the form

$$\prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \pi^{\mathbb{Z}/2}\bar{\mathbb{Q}}, \quad m_a \text{ or its square } \in \mathbb{Z}$$

is generated by the relations (1), (2), (3).

Restricted products

Let $\{\mathcal{T}_i\}_{i \in I}$ be collection of topological spaces.

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$$\prod_{i\in I}{}' \mathcal{T}_i =: \mathcal{T}$$

as the set of

$$\{(x_i)_{i\in I} \mid \text{ for almost all } i \text{ we have } x_i \in \mathcal{U}_i\},$$

i.e., vectors, where all but finitely many x_i are in the corresponding $\mathcal{U}_i$.

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i.e., vectors, where all but finitely many x_i are in the corresponding $\mathcal{U}_i$. The topology is given by a basis of open subsets:

$$\prod_{i \in I_0} \mathcal{V}_i \times \prod_{i \in I \setminus I_0} \mathcal{U}_i,$$

where I_0 is finite and $\mathcal{V}_i \subseteq \mathcal{T}_i$ are open.

Restricted product

A basic example is

$$\mathbb{A}_{\mathbb{Q}} := \prod_p{}' \, \mathbb{Q}_p \times \mathbb{R}$$

where $\mathcal{U}_p = \mathbb{Z}_p \subset \mathbb{Q}_p$.

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The image is discrete and cocompact.

Proof

We have

$$\mathbb{A}_{\mathbb{Q}}/(\mathbb{Q} \times \prod_p \underbrace{\mathbb{Z}_p}_{\text{compact}}) \simeq \mathbb{R}/\mathbb{Z}.$$

Dualities

An additive character is a continuous homomorphism

$$\psi_p : \mathbb{Q}_p \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

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where

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Note that the left side is in $\mathbb{Q}$.

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Fix Haar measures μ_p, for all primes p, normalized by

$$\mu_p(\mathbb{Z}_p) = 1,$$

we have

$$\mu_p(p^n\mathbb{Z}_p) = \frac{1}{p^n}, \quad \mu_p(\mathbb{Z}_p^\times) = 1 - \frac{1}{p}.$$

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We also put

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the standard Lebesgue measure on $\mathbb{R}$. This gives a Haar measure on $\mathbb{A}_\mathbb{Q}$ and we can integrate functions on the adeles.

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We also have a character

$$\psi_a : \mathbb{A}_\mathbb{Q} \to \mathbb{S}^1,$$

where $a \in \mathbb{A}_\mathbb{Q}$, defined as

$$\psi_a = \prod_p \psi_{a_p} \times \psi_{a_\infty},$$

a product of local characters. This is indeed a continuous homomorphism.

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is self-dual, $(\mathbb{A}/\mathbb{Q})^{\perp} = \mathbb{Q}$. The Poisson summation formula takes the form

$$\sum_{x \in \mathbb{Q}} f(x) = \sum_{a \in \mathbb{Q}} \hat{f}(a),$$

where

$$\hat{f}(a) = \int_{\mathbb{A}_{\mathbb{Q}}} f(x)\psi_a(x)\mu(x),$$

provided we have convergence.

Application

Recall that

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Let

$$N(B) := \#\{(x_0, x_1) \mid H(x_0, x_1) \leq B\}$$

be the counting function.

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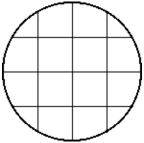
This is nothing but the Gauss circle problem, except that we are looking at coprime coordinates.



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This allows to rewrite the problem as follows.

Application

Consider the following height function

$$H = \prod_p H_p \times H_\infty : \mathbb{A}_\mathbb{Q} \to \mathbb{R},$$

with local factors given by

$$H_p(x) := \max(1, |x|_p), \quad H_\infty(x) := (1 + x^2)^{1/2}.$$

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$$N(B) := \{x \in \mathbb{Q} \mid H(x) \leq B\}, \quad B \to \infty.$$

Tauberian theorem

A convenient version of the Tauberian theorem is:

Consider
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Then

$$N(B) := \sum_{n \leq B} a_n \sim \frac{c}{a\Gamma(b)} \cdot B^a \log(B)^{b-1}, \quad B \to \infty.$$

Height zeta function

Therefore, we introduce and study the function

$$Z(s) := \sum_{x \in \mathbb{Q}} H(x)^{-s}.$$

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However, because H_p is invariant under $\mathbb{Z}_p$, only characters which are trivial on $\mathbb{Z}_p$, for all p contribute. So a must be in $\mathbb{Z}$.

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- put $U(0) := \{x \mid |x|_p \leq 1\}$ and $U(j) := \{x \mid |x|_p = p^j\}$, note that

$$\mathrm{vol}(U(j)) = p^j(1 - \frac{1}{p}).$$

Local integrals

We have

$$\int_{\mathbb{Q}_p} H_p(x_p)^{-s}\mu_p \quad = \quad \int_{U(0)} H_p(x_p)^{-s}\mu_p + \sum_{j\geq 1}\int_{U(j)} H_p(x_p)^{-s}\mu_p$$

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Characters, once again

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- unramified: trivial on $\mathbb{Z}_p$, i.e., $a_p \in \mathbb{Z}_p$
- $\psi_\infty : x \mapsto e^{2\pi i a \cdot x}$, $a \in \mathbb{R}$

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- unramified: trivial on $\mathbb{Z}_p$, i.e., $a_p \in \mathbb{Z}_p$
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Characters, once again

For $a \in \mathbb{Z} \setminus 0$ and $p \nmid a$, we compute

$$\hat{H}_p(s, \psi_a) = 1 + \sum_{j \geq 1} p^{-sj} \int_{|x|_p = p^j} \psi_a(x_p)\mu_p = 1 - p^{-s}.$$

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For $i \geq 1$ and $p \nmid a$, we integrate a nontrivial character over a compact group, thus get 0. For $i = 0$, we get 1. Since $U(i) = V(i) \setminus V(i-1)$, we have

$$\int_{U(i)} \psi_a(x)\mu_p = \begin{cases} 0 & i \geq 2 \\ -1 & i = 1 \end{cases},$$

which implies the claim.

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Thus

$$\prod_{p|a} |\hat{H}_p(s, \psi_a)| \ll (1 + |a|)^{\delta}$$

for some (small) $\delta > 0$.

Characters, once again

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For $\Re(s) > 2 - \delta$, one has:

- $|\prod_{p|a} \hat{H}_p(s,a)| \ll |\prod_{p|a} \int_{\mathbb{Q}_p} H_p(x_p)^{-s} \mu_p| \ll (1+|a|)^{\delta}$
- $|\int_{\mathbb{R}} (1+x^2)^{-s/2} \cdot e^{2\pi i a x} dx| \ll_N \frac{1}{(1+|a|)^N}$, for any $N \in \mathbb{N}$, (integration by parts)

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This gives a meromorphic continuation of $Z(s)$ and its pole at $s = 2$.