Lecture 8

Plan

- Complex analytic tools (entire functions with prescribed zeroes)

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- More about $\Gamma(s)$

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- More about $\Gamma(s)$
- Zero-free region of $\zeta(s)$
- Tauberian theorems
- Prime Number Theorem

Analytic tools

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It is called entire if $\forall \Omega \subset \mathbb{C}$ bounded,

$$g|_{\Omega} \quad \text{is analytic.}$$

Lemma

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Our goal will be to write down entire functions with prescribed zeros.

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- $\lim_{n\to\infty} 1/|a_n| = 0$

$\Rightarrow \exists$ an entire function

$$g : \mathbb{C} \to \mathbb{C}$$

such that $\{a_n\}$ are its zeroes, with multiplicities.

Proof

$$u_n(s) := \left(1 - \frac{s}{a_n}\right) \cdot e^{\sum_{j=1}^{n-1} \frac{1}{j}\left(\frac{s}{a_n}\right)^j}$$

$$u(s) := \prod_n u_n(s)$$

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We have

$$\ln(u_r(s)) = \ln(1 - \frac{s}{a_r}) + \sum_{j=1}^{r-1} \frac{1}{j}(\frac{s}{a_r})^j$$

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$$\Omega_n := \{s \in \mathbb{C} \mid \ |s| \leq (1-\epsilon)|a_n|\}$$

We have

$$\ln(u_r(s)) = \ln(1 - \frac{s}{a_r}) + \sum_{j=1}^{r-1} \frac{1}{j}(\frac{s}{a_r})^j = -\sum_{j=r}^{\infty} \frac{1}{j}(\frac{s}{a_r})^j$$

Proof

We consider $r \geq n$. We claim absolute convergence:

$$\sum_{r=n}^{\infty} \sum_{j=r}^{\infty} \frac{1}{j} \left| \left(\frac{s}{a_r} \right)^j \right| < \infty.$$

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$$\sum_{r=n}^{\infty} \sum_{j=r}^{\infty} \frac{1}{j} \left((1-\epsilon) \frac{|a_n|}{|a_r|} \right)^j$$

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$$\leq \sum_{r} \frac{1}{\epsilon r} (1-\epsilon)^r < \infty$$

It follows that $u(s)$ is holomorphic in Ω_n. Since

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Remark

If

$$\sum_{n\geq 1} \frac{1}{|a_n|^{\rho+1}} < \infty$$

then

$$g(s) = \prod_{n=1}^{\infty} \left(1 - \frac{s}{a_n}\right) e^{\sum_{j=1}^{\rho} \frac{1}{j}\left(\frac{s}{a_n}\right)^j}$$

is holomorphic with prescribed zeroes.

Theorem

Every entire function has the form

$$g(s) = e^{h(s)} \cdot s^m \cdot \prod_{n=1}^{\infty} \left(1 - \frac{s}{a_n}\right) e^{\sum \cdots}$$

with $h(s)$ entire.

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Proof:

$$\frac{g(s)}{s^m} \cdot \prod \cdots$$

entire, no zeroes $\Rightarrow$ $h(s)$ also entire.

Definition

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- $$\beta = \beta(g) := \inf\{b \mid \sum_{n\geq 1} \frac{1}{|a_n|^b} < \infty\} \quad \text{or } \infty$$

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- $\beta \leq \alpha$
- $h \in \mathbb{C}[s], \ \deg(h) \leq \alpha$
- $\alpha = \max(\beta, \deg(h))$
- *If* $\exists \quad r_1, \ldots, r_n, \ldots \to \infty$ *with*

$$m_g(r_j) \geq e^{c \cdot r_j^\alpha}$$

then

$$\alpha = \beta \quad \textit{and} \quad \sum \frac{1}{|a_n|^\beta} = \infty.$$

The sin function

Previously, we have established

$$\sin(\pi s) = s e^{h(s)} \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right), \quad h(s) = as + b$$

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Taking the logarithmic derivative, $\log(\cdot)'$, we find

$$\pi \cdot \frac{\cos(\pi s)}{\sin(\pi s)} = \frac{1}{s} + h'(s) - \sum_{n=1}^{\infty} \frac{2s}{n^2 - s^2}$$

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Taking the limit $s \to 0$, we conclude that $a = 0$, so that

$$\frac{\sin(\pi s)}{s} = c \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right),$$

taking the limit $s \to 0$, we find that $c = \pi$.

The sin function

It follows that

$$\sin(\pi s) = \pi s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$

Gamma function

Definition

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

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$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

This is an entire function.

Properties of the Gamma function

(1)

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1} \quad \text{(Euler)}$$

with

$$\gamma = \lim_{m\to\infty}\left(\sum_{j=1}^{m}\frac{1}{j} - \ln(m)\right).$$

Properties of the Gamma function

Proof: From the definition, we have

$$\frac{1}{\Gamma(s)} = s \cdot \lim_{m\to\infty} m^{-s} \prod_{n=1}^{m} \left(1 + \frac{s}{n}\right)$$

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$$= s \cdot \lim_{m\to\infty} \left(\prod_{n=1}^{m} \underbrace{\left(1 + \frac{1}{n}\right)^{-s} \cdot \left(1 + \frac{s}{n}\right)}_{\text{as claimed}} \right) \underbrace{\left(1 + \frac{1}{m}\right)^{s}}_{\to 0}$$

Properties of the Gamma function

(2)

$$\Gamma(s) = \lim_{n\to\infty} \frac{1\cdot 2\cdots(n-1)n^s}{s(s+1)\cdots(s+n-1)}$$

Properties of the Gamma function

(3)

$$\Gamma(s+1) = s \cdot \Gamma(s) \quad \text{e.g., } \Gamma(n+1) = n!$$

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$$\Gamma(s) \cdot \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

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(4) For $s \in \mathbb{C} \setminus \mathbb{Z}$

$$\Gamma(s) \cdot \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Proof: Use the definition to obtain

$$\Gamma(s) \cdot \Gamma(-s) = -\frac{1}{s^2} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right)^{-1} = -\frac{\pi}{s \cdot \sin(\pi s)}$$

Now use

$$\Gamma(1-s) = -s \cdot \Gamma(-s).$$

Properties of the Gamma function

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(7) $\arg(s) \in [-\pi + \delta, \pi - \delta]$, $\delta > 0 \Rightarrow$

$$\log(\Gamma(s)) = (s - \frac{1}{2}) \log(s) - s + \log(\sqrt{2\pi}) + \mathcal{O}(\frac{1}{|s|}) \quad \text{(Stirling)}$$

Properties of the Gamma function

It follows that, for $\sigma_0 \leq \sigma \leq \sigma_1$,

- $$\Gamma(\sigma + it) = t^{\sigma + it - \frac{1}{2}} \cdot e^{-\frac{\pi t}{2} - it + i\frac{\pi}{2}(\sigma - \frac{1}{2})}\sqrt{2\pi}\left(1 + \mathcal{O}(\frac{1}{t})\right)$$

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- $$\frac{\Gamma'(s)}{\Gamma(s)} = \log(s) + \mathcal{O}(\frac{1}{|s|}), \quad \text{for } |\arg(s)| < \pi.$$

Conclusion: $\Gamma(s)^{-1}$ is entire, with $\alpha = 1$,

Properties of the Gamma function

Conclusion: $\Gamma(s)^{-1}$ is entire, with $\alpha = 1$, and with the following estimate

$$\Gamma(\sigma + it) = t^{\sigma + it - 1/2} e^{-\pi t/2 - it + i(\sigma - 1)\pi/2} \sqrt{2\pi} \{1 + \mathcal{O}\left(\frac{1}{|t|}\right)\},$$

for $\sigma_0 \leq \sigma \leq \sigma_1$.

Applications to $\zeta(s)$

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- $\xi(0) = \xi(1) \neq 0$
- trivial zeroes of $\zeta(s)$ at $s = -2, -4, \ldots$ coming from poles of $\Gamma(\frac{s}{2})$ at $s \in 2\mathbb{N}$.

Riemann zeta function

Theorem

- *Every nontrivial zero of* $\zeta(s)$ *has real part in* $[0, 1]$.

Riemann zeta function

Theorem

- *Every nontrivial zero of* $\zeta(s)$ *has real part in* $[0,1]$.
- $$\sum \frac{1}{|\rho_n|} = \infty, \quad \sum \frac{1}{|\rho_n|^{1+\epsilon}} < \infty, \quad \forall \epsilon > 0$$

Proof

- $\alpha(\xi)$? Using Abel summation

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\int \frac{\rho(u)}{u^{s+1}}du$$

It follows that

$$|\zeta(s)| = \mathcal{O}(|s|).$$

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$$|\zeta(s)| = \mathcal{O}(|s|).$$

- $|\Gamma(s)| \le e^{c\cdot|s|\cdot\ln(|s|)} \quad \Rightarrow \alpha(\xi) \le 1.$

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- $|\Gamma(s)| \leq e^{c \cdot |s| \cdot \ln(|s|)} \quad \Rightarrow \alpha(\xi) \leq 1.$
- For $s \to \infty$, $\ln(\Gamma(s)) \sim s \cdot \ln(s) \quad \Rightarrow \alpha(\xi) = 1.$

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It follows that

$$|\zeta(s)| = \mathcal{O}(|s|).$$

- $|\Gamma(s)| \leq e^{c\cdot|s|\cdot\ln(|s|)} \quad \Rightarrow \alpha(\xi) \leq 1.$
- For $s \to \infty$, $\ln(\Gamma(s)) \sim s \cdot \ln(s) \quad \Rightarrow \alpha(\xi) = 1$. This implies that

$$\sum \frac{1}{|\rho_n|} = \infty,$$

i.e., ζ has infinitely many nontrivial zeroes. Furthermore,

$$\sum \frac{1}{|\rho_n|^{1+\epsilon}} < \infty, \quad \forall \epsilon > 0$$

Basic identity

Thus we have

$$\xi(s) = e^{a+bs} \cdot \prod_{n \geq 1} \left(1 - \frac{s}{\rho_n}\right) \cdot e^{\frac{s}{\rho_n}}$$

and also symmetry with respect to $\Re(s) = \frac{1}{2}$ and $\Im(s) = 0$.

Basic identity

Thus we have

$$\xi(s) = e^{a+bs} \cdot \prod_{n \geq 1} \left(1 - \frac{s}{\rho_n}\right) \cdot e^{\frac{s}{\rho_n}}$$

and also symmetry with respect to $\Re(s) = \frac{1}{2}$ and $\Im(s) = 0$.

$$\Rightarrow \quad \frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s} - \frac{1}{s-1} + \underbrace{\sum \left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right)}_{\text{from } \zeta(s)}$$

$$+ \underbrace{\sum \left(\frac{1}{s+2n} - \frac{1}{2n}\right)}_{\text{from } \Gamma(\frac{s}{2})} + \text{const.}$$

Lemma

Put $\rho_n := \beta_n + i\gamma_n$, $T \geq 2$. We have

$$\sum_{n=1}^{\infty} \frac{1}{1 + (T - \gamma_n)^2} \leq c \log(T).$$

Lemma

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$$\sum_{n=1}^{\infty} \frac{1}{1 + (T - \gamma_n)^2} \leq c \log(T).$$

Before we start the proof, some recollections regarding the von Mangoldt function.

von Mangoldt function

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Properties:

- $$\sum_{d|n} \Lambda(d) = \log(n)$$

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Properties:

- $$\sum_{d|n} \Lambda(d) = \log(n)$$

- $$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) \quad \text{Moebius inversion}$$

von Mangoldt function

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Properties:

- $$\sum_{d|n} \Lambda(d) = \log(n)$$

- $$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d) \qquad \text{Moebius inversion}$$

- $$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}, \qquad \Re(s) > 1.$$

Proof of the Lemma

Put $s := 2 + iT$.

- $$|\sum_{n\geq 1} \frac{1}{s+2n} - \frac{1}{2n}| \leq c_0 \log(T)$$

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Put $s := 2 + iT$.

- $$|\sum_{n\geq 1} \frac{1}{s+2n} - \frac{1}{2n}| \leq c_0 \log(T)$$

- $$\begin{aligned} -\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) &= \Re\left(\frac{1}{s} - B_0 - \sum_{n\geq 1}\left(\frac{1}{s+2n} - \frac{1}{2n}\right)\right) \\ &\quad - \Re\left(\sum_{n\geq 1}\left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right)\right) \\ &\leq c_1 \log(T) - \Re\left(\sum_{n\geq 1}\left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right)\right) \end{aligned}$$

Proof

With our choice the LHS

$$\sum \frac{\Lambda(n)}{n^{2+iT}} < c_2.$$

Proof

With our choice the LHS

$$\sum \frac{\Lambda(n)}{n^{2+iT}} < c_2.$$

It follows that

$$\Re\left(\sum_{n\geq 1}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right)\right) < c_3 \log(T).$$

Proof

With our choice the LHS

$$\sum \frac{\Lambda(n)}{n^{2+iT}} < c_2.$$

It follows that

$$\Re\left(\sum_{n\geq 1}\left(\frac{1}{s-\rho_n}+\frac{1}{\rho_n}\right)\right) < c_3 \log(T).$$

Now observe that

$$|\frac{1}{s-\rho_n}| \geq \frac{1}{1+(T-\gamma_n)^2}, \qquad |\frac{1}{\rho_n}| \geq \frac{\beta_n}{\beta_n^2+\gamma_n^2} \geq 0.$$

This concludes the proof.

Corollary

$$\#\{n \quad | \quad T \leq |\Im(\rho_n)| \leq T+1\} \leq \log(T).$$

Zero-free region

Theorem (Vallee-Poussin)

Let $s = \sigma + it$. There exists a constant $c > 0$ such that

$$\zeta(s) \neq 0,$$

for all s with

$$\Re(s) \geq 1 - \frac{c}{\log(|t|) + 2}.$$

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for all s with

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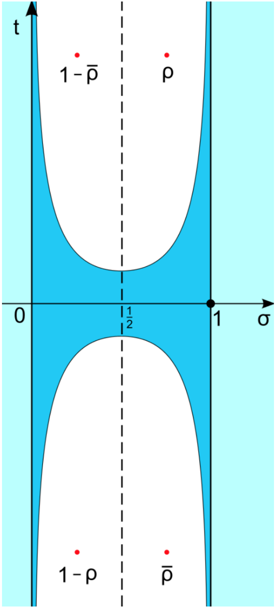
We cannot hope to get $\Re(s) > 1 - \epsilon$, but the given shape suffices for the Prime Number Theorem.

Zero-free region



Proof

$\rho_n = \beta_n + i\gamma_n$, $\gamma_n \geq \gamma_0$, $\Re(s) = \sigma > 1$

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= \sum \frac{\Lambda(n)}{n^s} \\
&= \sum \frac{\Lambda(n)}{n^\sigma} \cdot e^{it\log(n)}
\end{aligned}
$$

Proof

$$-\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum \frac{\Lambda(n)}{n^\sigma} \cdot \cos(t \log(n))$$

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Basic identity

$$3 + 4\cos(\phi) + \cos(2\phi) = 2(1+\cos(\phi))^2 \geq 0$$

Proof

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Basic identity

$$3 + 4\cos(\phi) + \cos(2\phi) = 2(1+\cos(\phi))^2 \geq 0$$

It follows:

$$3\left(-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) + 4\left(-\Re\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right)\right) + \left(-\Re\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right)\right) \geq 0$$

Proof: Assume that $1 < \sigma \leq 2$.

(1)

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + c_1$$

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$$-\Re\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) \leq c_2 \log(|t|+2) - \sum_{k\geq 1} \Re\left(\frac{1}{s-\rho_k} + \frac{1}{\rho_k}\right)$$

with $0 \leq \beta_k \leq 1$;

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with $0 \leq \beta_k \leq 1$; note that

$$\Re(\frac{1}{s-\rho_k}) = \frac{\sigma - \beta_k}{(\sigma-\beta_k)^2 + (t-\gamma_k)^2} > 0,$$

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with $0 \leq \beta_k \leq 1$; note that

$$\Re(\frac{1}{s-\rho_k}) = \frac{\sigma - \beta_k}{(\sigma-\beta_k)^2 + (t-\gamma_k)^2} > 0, \quad \Re(\frac{1}{\rho_k}) = \frac{\beta_k}{\beta_k^2 + \gamma_k^2} \geq 0.$$

Proof

It follows that

$$-\Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) \leq c_2 \log(|t|+2) - \frac{\sigma-\beta_n}{(\sigma-\beta_n)^2+(t-\gamma_n)^2}.$$

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It follows that

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(3)

$$-\Re\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) \leq c_2 \log(|t|+2).$$

Proof

Substituting everything into the trigonometric identity

$$3 + 4\cos(\phi) + \cos(2\phi) \geq 0$$

we find

$$\frac{3}{\sigma - 1} - 4\frac{\sigma - \beta_n}{(\sigma - \beta_n)^2 + (t - \gamma_n)^2} + c_3 \log(|t| + 2) \geq 0,$$

for all $t, |t| \geq \gamma_0$, and all $\sigma \in (1, 2]$.

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$$\frac{4}{\sigma - \beta_n} \leq \frac{3}{\sigma - 1} + c_3 \log(|\gamma_n| + 2).$$

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$$\frac{4}{\sigma - \beta_n} \leq \frac{3}{\sigma - 1} + c_3 \log(|\gamma_n| + 2).$$

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$$\frac{4}{\sigma - \beta_n} \leq \frac{3}{\sigma - 1} + c_3 \log(|\gamma_n| + 2).$$

This already implies that $\beta_n \neq 1$: there are no zeroes on the line $\Re(s) = 1$. Write $\beta_n = 1 - \delta_n$. Put

$$\sigma := 1 + c\delta_n, \quad \delta_n \in (0, 1]$$

Proof

We find that

$$\frac{4}{1 + c\delta_n - 1 + \delta_n} \leq \frac{3}{1 + c\delta_n - 1} + c_3 \log(|\gamma_n| + 2).$$

Proof

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$$\frac{4}{(c+1)\delta_n} \leq \frac{3}{c\delta_n} + c_3 \log(|\gamma_n| + 2).$$

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We find that

$$\frac{4}{1 + c\delta_n - 1 + \delta_n} \leq \frac{3}{1 + c\delta_n - 1} + c_3 \log(|\gamma_n| + 2).$$

$$\frac{4}{(c+1)\delta_n} \leq \frac{3}{c\delta_n} + c_3 \log(|\gamma_n| + 2).$$

$$\frac{1}{\delta_n}\left(\frac{4}{c+1} - \frac{3}{c}\right) \leq c_3 \log(|\gamma_n| + 2)$$

For $c = 4$, we find that

$$1 - \beta_n = \delta_n \geq \frac{1}{20c_3} \cdot \frac{1}{\log(|\gamma_n| + 2)},$$

thus

$$\beta_n \leq 1 - c_4 \cdot \frac{1}{\log(|\gamma_n| + 2)}.$$

Tauberian theorems

Idea:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}ds = \delta(x) := \begin{cases} 0 & 0 < x < 1 \\ 1/2 & x = 1 \\ 1 & x > 1 \end{cases}$$

Tauberian theorems

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$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}ds=\delta(x):=\begin{cases}0 & 0<x<1\\ 1/2 & x=1\\ 1 & x>1\end{cases}$$

This is a kind of δ-function.

To prove this identity we shift contour of integration.

Tauberian theorems

We will use a slightly different result

$$I(T) := \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s}ds = \delta(x) + \mathcal{O}\left(x^{\sigma}\cdot\min(1,\frac{1}{T\log(T)})\right)$$

Tauberian theorems

How is this done? Pick different contours of integration Γ and $\tilde{\Gamma}$, which are rectangles given by $\Im(s) = \pm T$, $\Re(s) = \sigma$, and where the fourth side is given by $\Re(s) = \sigma_- < 0$, for Γ, respectively, $\Re(s) = \sigma_+ \to +\infty$, for $\tilde{\Gamma}$.

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For $x > 1$, we integrate

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For $x > 1$, we integrate

$$\frac{1}{2\pi i} \int_\Gamma \frac{x^s}{s}\, ds = 1,$$

(the residue at $s = 0$). For $x \in (0, 1)$ we integrate

$$\frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{x^s}{s}\, ds = 0,$$

(there are no poles to the right of $\Re(s) = \sigma$).

Tauberian theorems

It remains to estimate contributions over the other 3 sides of the rectangle.

Tauberian theorems

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$$\left|\int_{\Gamma_1}\right|, \quad \left|\int_{\Gamma_3}\right|, \quad \left|\int_{\tilde{\Gamma}_1}\right|, \quad \left|\int_{\tilde{\Gamma}_3}\right| < \frac{1}{2\pi}\int \frac{x^{\gamma}}{\sqrt{T^2+\gamma^2}}\, d\gamma \leq \frac{x^{\sigma}}{T\log(\sigma)},$$

where these are horizontal segments of the respective contours.

Tauberian theorems

For the horizontal segments, we have:

$$|\int_{\Gamma_2}| \leq \frac{1}{2\pi}\int_{-T}^{T}\frac{x^{-u}}{\sqrt{u^2+t^2}}\,dt = \mathcal{O}(x^{-u}),$$

which goes to zero for $x > 1$ and $u \to \infty$.

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which goes to zero for $x > 1$ and $u \to \infty$. Similarly,

$$|\int_{\tilde{\Gamma}_2}| \leq \frac{1}{2\pi}\int_{-T}^{T}\frac{x^{u}}{\sqrt{u^2+t^2}}\,dt = \mathcal{O}(x^{u}),$$

which goes to zero for $x \in (0,1)$ and $u \to \infty$.

Applications

Proposition

Let

$$f(s) := \sum_n \frac{a_n}{n^s}$$

satisfy

- absolutely convergent $\Re(s) > 1$,
- $|a_n| \leq A(n), A(n+1) \geq A(n)$
- $$\sum \frac{|a_n|}{n^\sigma} = \mathcal{O}\left(\frac{1}{(\sigma-1)^\alpha}\right),$$

  for $\sigma \to 1$ and some $\alpha > 0$.

continued ...

Applications

Then:

For all $\sigma \in (1, \sigma_0]$ and $x = N + 1/2$,

$$\mathcal{N}(x) := \sum_{n \le x} a_n$$
$$= \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} f(s) \cdot \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^\sigma}{T(\sigma - 1)^\alpha} \right)$$
$$+ \mathcal{O}\left(x \cdot A(2x) \cdot \frac{\log(x)}{T} \right)$$

Proof

We have

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s}f(s)\,ds = \sum_{n\geq 1} a_n\left(\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s}\right)$$

Proof

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$$= \sum_{n\leq x} a_n + \mathcal{R},$$

remainder term.

Proof

We have

$$\mathcal{R} = \mathcal{O}\left(\sum_{n=1}^{\infty} |a_n| \left(\frac{x}{n}\right)^{\sigma} T^{-1} \log(\frac{x}{n})^{-1}\right)$$

Proof

We have

$$\mathcal{R} = \mathcal{O}\left(\sum_{n=1}^{\infty} |a_n| \left(\frac{x}{n}\right)^{\sigma} T^{-1} \log(\frac{x}{n})^{-1}\right)$$

To evaluate this, put

$$\Sigma_1 := \sum_{\frac{x}{n} \le \frac{1}{2}} \cdots \quad + \sum_{\frac{x}{n} \ge 2} \cdots .$$

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In this domain, we have

$$|\log(\frac{x}{n})| \ge \log(2)$$

and

$$\Sigma_1 = \mathcal{O}\left(\frac{x^{\sigma}}{T(\sigma-1)^{\alpha}}\right),$$

the first $\mathcal{O}$ in the statement of the theorem.

Proof

On the other hand,

$$\Sigma_2 := \sum_{\frac{x}{2}<n<2x} |a_n| \left(\frac{x}{n}\right)^{\sigma} T^{-1} \log(\frac{x}{n})^{-1}$$

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$$\leq T^{-1}A(2x)2^{\sigma} \sum_{\frac{x}{2}<n<2x} |\log\left(\frac{N+\frac{1}{2}}{n}\right)|^{-1}$$

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$$\int_{x/2}^{N-1} \log\left(\frac{N+\frac{1}{2}}{n}\right)^{-1} du + \int_{N+1}^{2x} \log\left(\frac{u}{N+\frac{1}{2}}\right)$$

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which is

$$\mathcal{O}(x\log(x)).$$

Application to the Prime Number Theorem

Theorem

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = x + \mathcal{O}\left(x \cdot e^{-c\sqrt{\log(x)}}\right)$$

Proof

Using the Tauberian theorem, we have

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds + \mathcal{O}\left(\frac{x \log^2(x)}{T} \right)$$

Proof

Using the Tauberian theorem, we have

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds + \mathcal{O}\left(\frac{x \log^2(x)}{T} \right)$$

Put

$$x := N + 1/2 \geq 100, \quad A(n) := \log(n),$$

Proof

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Put

$$x := N + 1/2 \geq 100, \quad A(n) := \log(n),$$

$$T := e^{\sqrt{\log(x)}}, \quad \sigma := 1 + \frac{1}{\log(x)}$$

Proof

Choose a contour

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_2 \cup \Gamma_4,$$

with horizontal Γ_1 and Γ_3, completely contained in the zero-free region of $\zeta(s)$.

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Proof

Choose a contour

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_2 \cup \Gamma_4,$$

with horizontal Γ_1 and Γ_3, completely contained in the zero-free region of $\zeta(s)$. When the imaginary part of s gets larger, the rectangle gets narrower and narrower.

We will use the fact that

$$-\frac{\zeta'(s)}{\zeta(s)}$$

has only one pole inside Γ, at $s = 1$; this is precisely were we need the zero-free region.

Proof

As before, the estimates over horizontal pieces are easy

$$|\Gamma_1|, |\Gamma_3| \leq | \int_{\sigma_1+iT}^{\sigma+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds$$

$$\leq | \int_{\sigma_1}^{\sigma} \underbrace{\frac{|\zeta'(u+iT)|}{|\zeta(u+iT)|} \frac{x^u}{T}}_{\mathcal{O}(x\frac{\log^2(T)}{T})}, du$$

Proof

As before, the estimates over horizontal pieces are easy

$$|\Gamma_1|, |\Gamma_3| \le |\int_{\sigma_1+iT}^{\sigma+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s}\, ds$$

$$\le |\int_{\sigma_1}^{\sigma} \underbrace{\frac{|\zeta'(u+iT)|}{|\zeta(u+iT)|} \frac{x^u}{T}}_{\mathcal{O}(x\frac{\log^2(T)}{T})}, du$$

Recall the basic estimate

$$|\frac{\zeta'(s)}{\zeta(s)}| = \mathcal{O}(\log(|t|)^2).$$

Proof

We have

$$\left|\int_{\Gamma_2}\right| \le \frac{1}{2\pi}\int_{-T}^{T}\left|\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}\right|\cdot\left|\frac{x^{\sigma_1+it}}{\sigma_1+it}\right|\,dt$$

Proof

We have

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$$= \mathcal{O}\left(x^{\sigma_1}\log^2(T)\left(\int_0^1\frac{dt}{\sigma_1}+\int_1^T\frac{dt}{t}\right)\right)$$

Proof

We have

$$
\begin{aligned}
|\int_{\Gamma_2}| &\le \frac{1}{2\pi}\int_{-T}^{T} |\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}| \cdot |\frac{x^{\sigma_1+it}}{\sigma_1+it}|\,dt \\
&= \mathcal{O}\left(x^{\sigma_1}\log^2(T)\left(\int_0^1 \frac{dt}{\sigma_1} + \int_1^T \frac{dt}{t}\right)\right) \\
&= \mathcal{O}\left(x^{\sigma_1}\log^3(T)\right)
\end{aligned}
$$

Proof

We have

$$\left|\int_{\Gamma_2}\right| \leq \frac{1}{2\pi}\int_{-T}^{T}\left|\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}\right| \cdot \left|\frac{x^{\sigma_1+it}}{\sigma_1+it}\right| dt$$

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$$= \mathcal{O}\left(x^{\sigma_1}\log^3(T)\right)$$

The next step is to extract primes, i.e., $\pi(x)$, from $\psi(x)$.

Proof

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The next step is to extract primes, i.e., $\pi(x)$, from $\psi(x)$. This is done with Abel's summation.

Abel's summation

Let

$$f \in \mathrm{C}^1([a,b]), \quad S(x) := \sum_{a \le n \le x} c_n.$$

Abel's summation

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Then

$$\sum_{a < n \le b} c_n f(n) = -\int_a^b S(x) f'(x)\, dx + S(b) f(b).$$

Abel's summation

Apply:

$$c_n = \Lambda(n), \qquad f(x) := \frac{1}{\log(x)}$$

$$S(x) := \sum_{n \le x} \frac{\Lambda(n)}{\log(n)}$$

$$= \underbrace{\pi(x)}_{\#\{p \le x\}} + \sum_{n = p^k, k \ge 2} \frac{\Lambda(n)}{\log(n)}$$

Abel's summation

To understand the error term, note that $k \leq \log(x)$ and the number of summands is $\leq \sqrt{x}$. We get for the error term:

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It follows that

$$S(x) = \int_2^x \frac{\psi(u)}{u\log^2(u)}\,du + \frac{\psi(x)}{\log(x)}$$

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$$O(\sqrt{x}\log(x)).$$

It follows that

$$S(x) = \int_2^x \frac{\psi(u)}{u\log^2(u)}\,du + \frac{\psi(x)}{\log(x)}$$

$$S(x) = \int_2^x \frac{du}{\log^2(u)} + \frac{x}{\log(x)} + \mathcal{R}$$

where $\mathcal{R}$ is the remainder term.

Abel's summation

Using

$$\psi(x) = x + \mathcal{O}(xe^{-c\sqrt{\log(x)}})$$

we find that

$$\mathcal{R} = \mathcal{O}\left(\int_x^2 e^{-c\sqrt{\log(x)}} \frac{du}{\log^2(u)}\right)$$

Prime number theorem

Going back to

$$S = \int_2^x \frac{du}{\log^2(u)} + \frac{x}{\log(x)} + \cdots$$

we find

$$S = -\frac{u}{\log(u)}\Big|_2^x + \int_2^x \frac{du}{\log(u)} + \frac{x}{\log(x)} + \ldots$$

$$S = \int_2^x \frac{du}{\log(u)} + \frac{2}{\log(x)} + \ldots$$

Prime number theorem

Going back to

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$$S = \int_2^x \frac{du}{\log(u)} + \frac{2}{\log(x)} + \ldots$$

This is the Prime Number Theorem.

Prime number theorem

$$\#\{p \leq x\} = \pi(x) \sim \frac{x}{\log(x)} + \text{ Error term}$$