Lecture 6

Plan

- Analytic properties of $\zeta(s)$

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- Special values

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- Special values
- Dirichlet L-functions

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- Special values
- Dirichlet L-functions
- Primes in arithmetic progressions

Riemann zeta function

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- For $\Re(s) > 0$, we have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\underbrace{\int_N^\infty \frac{\rho(u)}{u^{s+1}}\,du}_{\text{hol. for } \Re(s)>0}$$

where $\rho(u) = \frac{1}{2} - \{u\}$.

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where $\rho(u) = \frac{1}{2} - \{u\}$. I.e., we have an isolated pole at $s = 1$, with residue 1.

Theta-function

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Functional equation (Poisson summation formula):

$$\theta(y) = \frac{1}{\sqrt{y}} \cdot \theta(\frac{1}{y}).$$

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We have

$$\omega(s) = s^{-1/2} \cdot \omega(\frac{1}{s}) - \frac{1}{2} + \frac{s^{-1/2}}{2}.$$

Fundamental computation

Compute the Mellin transform:

$$M(\omega)(s) = \int_0^\infty \omega(t) t^{s-1}\,dt = \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t} t^{s-1}\,dt, \quad \Re(s) > 1/2.$$

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$$\int_0^\infty e^{-\pi n^2 t} t^s \frac{dt}{t} = \frac{1}{(\pi n^2)^s} \int_0^\infty e^{-t} t^s \frac{dt}{t} = \frac{\Gamma(s)}{\pi^s n^{2s}}$$

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$$\Rightarrow \quad M(\omega)(s) = \pi^{-s} \cdot \Gamma(s) \cdot \zeta(2s), \quad \Re(s) > 1/2.$$

Fundamental computation

$$
\begin{aligned}
M(\omega)(s) &= \int_0^1 \omega(t) t^{s-1}\, dt + \int_1^\infty \omega(t) t^{s-1}\, dt \\
&= \int_0^1 t^{s-1} \underbrace{\left(t^{-1/2}\omega(\frac{1}{t}) - \frac{1}{2} + \frac{t^{-1/2}}{2} \right)}_{\omega(t)} dt \\
&+ \int_1^\infty \omega(t) t^{s-1}\, dt
\end{aligned}
$$

Mellin transform

We change variables, putting $u = 1/t$. This turns the first summand into

$$\int_1^\infty u^{-1-s}\left(u^{1/2}\omega(u) - \frac{1}{2} + \frac{u^{1/2}}{2}\right) du$$
$$= \int_1^\infty u^{-1/2-s}\omega(u)\,du - \frac{1}{2s} - \frac{1}{1-2s}$$

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Since

$$\omega(t) < \sum_{n\geq 1} e^{-\pi n t} < 2e^{-\pi t}$$

the integral converges for all s.

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$$M(\omega)(s) = M(\omega)(\frac{1}{2} - s)$$

It has a simple pole at $s = 0, 1/2$.

Riemann zeta function

It follows that

$$\pi^{-s} \cdot \Gamma(s)\zeta(2s) = \pi^{s-1/2} \cdot \Gamma(\frac{1}{2} - s)\zeta(1 - 2s)$$

except at $s = 0, 1/2$.

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Put

$$\xi(s) := \pi^{-s/2} \cdot \Gamma(\frac{s}{2})\zeta(s).$$

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Put

$$\xi(s) := \pi^{-s/2} \cdot \Gamma(\frac{s}{2})\zeta(s).$$

Then

$$\xi(s) = \xi(1-s),$$

this is the functional equation for the Riemann zeta function.

Special values

Recall,

$$\zeta(2) = \sum \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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How does one compute this?

Bernoulli polynomials

$$B_k(x) \in \mathbb{R}[x]$$

- $B_0(x) = 1$

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- $B_0(x) = 1$
- $B'_k(x) = k \cdot B_{k-1}(x) \quad k \geq 1.$
- $$\int_0^1 B_k(x)dx = 0$$

Bernoulli numbers

$$B_k := B_k(0)$$

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$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_3 = 0,$$
$$B_4 = -1/30, \quad B_5 = 0, \quad B_6 = 1/42$$

Bernoulli polynomials

Theorem

$$\sum_{k\geq 0} B_k(x)\frac{t^k}{k!} = t\cdot\frac{e^{tx}}{e^t - 1}$$

Proof

Put

$$F(x,t) := \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial x}F(x,t) &= \sum_{k\geq 1} B_k'(x)\frac{t^k}{k!}\\ &= \sum_{k\geq 1} k\cdot B_{k-1}(x)\frac{t^k}{k!}\\ &= t\cdot F(x,t)\end{aligned}$$

Proof

This implies

$$\log(F(x,t)) = tx + c(t), \quad F(x,t) = e^{tx+c(t)}.$$

Since

$$\int_0^1 F(x,t)dx = \int_0^1 \left(\sum_{k\geq 0} B_k(x)\frac{t^k}{k!} \right) dx$$

we have

$$1 = \int_0^1 e^{tx+c(t)}dx = e^{c(t)} \cdot \frac{e^{tx}}{t}\Big|_0^1,$$

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$$e^{c(t)} = \frac{t}{e^{tx} - 1}.$$

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Bernoulli polynomials

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Proof:

$$t \cdot e^{tk} = t\left(\sum_{k\geq 0}\frac{t^k}{k!(k+1)}\right)\cdot\left(\sum_{k\geq 0}\frac{B_k(x)t^k}{k!}\right)$$

Looking at the t^k coefficient:

$$t^k\frac{x^k}{k!} = \sum_{n=0}^{k}\frac{k!}{n!}\cdot\frac{B_n(x)}{(k-n+1)!}\cdot\frac{t^k}{k!}$$

Proof

For $n = k$, using that

$$B_k(x) = x^k + \cdots$$

i.e., the leading coefficient is 1, we find

$$x^k = x^k + \beta_{k-1}x^{k-1} + \cdots + \frac{k}{2}x^{k-1} + \cdots$$

with the last term coming from $n = k - 1$. This implies that

$$\beta_{k-1} = -\frac{k}{2}.$$

Special values

Theorem

$$\zeta(2k) = (-1)^k \cdot \pi^{2k} \cdot \frac{2^{2k-1}}{(2k-1)!} \cdot \left(-\frac{B_{2k}}{2k}\right)$$

Special values: proof

Recall the formulas:

$$\sinh(\pi x) = \frac{e^{\pi x} - e^{-\pi x}}{2} = \pi x \cdot \prod_{n \geq 1} \left(1 + \frac{x^2}{n^2}\right)$$

$$\sin(\pi x) = \frac{e^{i\pi x} - e^{-i\pi x}}{2i} = \pi x \cdot \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right)$$

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Let us prove the formula for sin.

Proof

Put $n = 2k + 1$, and recall that

- $$\sin(nx) = P_n(\sin(x)), \quad P_n \in \mathbb{Z}[x], \quad \deg(P_n) \le n, \quad P_n(0) = 0$$

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- $$n\cos(nx) = P_n'(x)(\sin(x)) \cdot \cos(x), \quad n = P_n'(0)$$

Proof

It follows that

$$\frac{\sin(nx)}{n\sin(x)} = 1 + a_1\sin(x) + \cdots + a_{2k}\sin^{2k}(x), \quad a_i \in \mathbb{Q} = \tilde{P}_{2k}(\sin(x)).$$

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Comparing zeroes, we find

$$\tilde{P}_{2k}(y) = \prod_{r=1}^{k} \left(1 - \frac{y^2}{\sin^2(\frac{r\pi}{n})}\right).$$

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$$\frac{\sin(\pi x)}{n\sin(\frac{\pi x}{n})} = \prod_{r=1}^{k} \left(1 - \frac{\sin^2(\frac{\pi x}{n})}{\sin^2(\frac{r\pi}{n})}\right).$$

Proof

Passing to the limit $n \to \infty$, we have

$$\frac{\sin(\pi x)}{\pi \sin(x)} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right) = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} \cdots$$

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Similar formulas hold for $\sinh(x)$.

Proof

Taking logarithmic derivatives, we obtain

$$\begin{aligned}\log(\sinh(\pi x)) &= \log(\frac{e^{\pi x} - e^{-\pi x}}{2}) \\ &= \log\left((\frac{e^{\pi x}}{2}) \cdot (1 - e^{-2\pi x}) \right) \\ &= \log(1 - e^{-2\pi x}) + \pi x - \log(2)\end{aligned}$$

Proof

Now we look at the left side,

$$\log(\sinh(\pi x))$$

equals

$$\log(\pi) + \log(x) + \sum_{n\geq 1} \log(1 + \frac{x^2}{n^2}) = \cdots$$

$$\log(\pi) + \log(x) + \sum_{n\geq 1} \sum_{k\geq 1} (-1)^k \frac{x^{2k}}{kn^{2k}} = \cdots$$

$$\log(\pi) + \log(x) + \sum_{k\geq 1} (-1)^k \frac{x^{2k}}{k} \cdot \zeta(2k) = \cdots$$

Proof

Now we take derivatives on both sides:

$$\frac{2\pi e^{-2\pi x}}{1 - e^{-2\pi x}} + \pi = \frac{1}{x} + 2 \cdot \sum_{k \geq 1} (-1)^{k+1} x^{2k-1} \zeta(2k)$$

$$\frac{\pi x}{e^{\pi x} - 1} + \frac{\pi x}{2} = 1 + \sum_{k \geq 1} (-1)^{k+1} \frac{\zeta(2k)}{2^{2k-1}} x^{2k}$$

$$\frac{\pi x}{2} + \sum_{k \geq 0} B_k \frac{(\pi x)^k}{k!} = 1 + \sum_{k \geq 1} (-1)^{k+1} \frac{\zeta(2k)}{2^{2k-1}} x^{2k}$$

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It follows that

$$\frac{\pi^{2k}}{(2k)!} B_{2k} = (-1)^{k+1} \frac{\zeta(2k)}{2^{2k-1}}.$$

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It follows that

$$\frac{\pi^{2k}}{(2k)!}B_{2k} = (-1)^{k+1}\frac{\zeta(2k)}{2^{2k-1}}.$$

$$\frac{\pi^2}{6} = \sum \frac{1}{n^2}$$

Dirichlet L-functions

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- $\chi(a) = \chi(a')$ if $a \equiv a' \pmod{n}$

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Example

$$\chi(p) = \left(\frac{-1}{p}\right) = \begin{cases} 1 & p \equiv 1 \pmod{4} \\ -1 & p \equiv -1 \pmod{4} \end{cases}$$

Characters

- χ is a periodic function; if period $= n$ then χ is called primitive and $n = f_\chi$ is its conductor

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- χ_1, χ_2 primitive, with conductors f_1, f_2, $\Rightarrow \exists!$ a primitive χ such that
  - $\chi(a) = \chi_1(a)\chi_2(a)$ for all a with $(a, f_1 f_2) = 1$
  - $f_\chi \mid f_1 \cdot f_2$

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  - $\chi(a) = \chi_1(a)\chi_2(a)$ for all a with $(a, f_1 f_2) = 1$
  - $f_\chi \mid f_1 \cdot f_2$
- thus, characters form a group.

L-functions

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Poisson summation formula implies:

- $L(s,\chi)$ admits a meromorphic continuation to $\mathbb{C}$; it is holomorphic for $\chi \neq \chi_0$, the trivial character.

L-functions - functional equation

$$\left(\frac{f}{\pi}\right)^{s/2} \cdot \Gamma(\frac{s+\delta}{2}) \cdot L(s,\chi) =$$
$$\omega_\chi \cdot \left(\frac{f}{\pi}\right)^{(1-s)/2} \cdot \Gamma(\frac{1-s+\delta}{2}) \cdot L(1-s,\bar{\chi})$$

where

- $\omega_\chi = \frac{\tau(\chi)}{\sqrt{f} i^\delta} \qquad |\omega_\chi| = 1$
- $\delta = \delta_\chi = \begin{cases} 0 & \chi(-1) = 1 \\ 1 & \chi(-1) = -1 \end{cases}$

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Proof: Poisson summation formula...

Gauss sum

Recall:

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Gauss sum

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We have

$$|\tau(\chi)| = \sqrt{f}$$

Nonvanishing

Theorem

If $\chi \neq \chi_0$ *then* $L(1, \chi) \neq 0$.

Proof of nonvanishing

Put

$$P(s) := \prod_{\chi \pmod{m}} L(s, \chi)$$

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$$P(s) := \prod_{\chi \pmod{m}} L(s, \chi)$$

- Pole at $s = 1$ coming from $\chi = \chi_0$
- $$\log(L(s, \chi)) = \sum_{p,k} \frac{\chi(p^k)}{kp^{ks}}$$
- $$\log(P(s)) = \sum_{p,k} \sum_{\chi} \chi(p^k) \cdots$$

Proof of nonvanishing

Recall

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$$\sum_{n \geq 1} \frac{a_n}{n^s}, \quad a_n = \begin{cases} \frac{\varphi(n)}{k} & n = p^k \equiv 1 \pmod{m} \\ 0 \end{cases}$$

- converges for $\Re(s) > 1$

Proof of nonvanishing

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$$Q(s) := \varphi(m) \sum_{p^k \equiv 1 \pmod{m}} \frac{1}{kp^{ks}}$$

$$\sum_{n \geq 1} \frac{a_n}{n^s}, \quad a_n = \begin{cases} \frac{\varphi(n)}{k} & n = p^k \equiv 1 \pmod{m} \\ 0 \end{cases}$$

- converges for $\Re(s) > 1$
- $a_n \geq 0$ for all n

Proof of nonvanishing

Recall

$$\sum_{\chi} = \begin{cases} \varphi(m) & a \equiv 1 \pmod m \\ 0 & \text{otherwise} \end{cases}$$

$$Q(s) := \varphi(m) \sum_{p^k \equiv 1 \pmod m} \frac{1}{kp^{ks}}$$

$$\sum_{n \geq 1} \frac{a_n}{n^s}, \quad a_n = \begin{cases} \frac{\varphi(n)}{k} & n = p^k \equiv 1 \pmod m \\ 0 \end{cases}$$

- converges for $\Re(s) > 1$
- $a_n \geq 0$ for all n
- $p \nmid m \Rightarrow p^{\varphi(m)} \equiv 1 \pmod m$

Proof of nonvanishing

It follows that

$$Q(s) > \sum_{p \nmid m} \frac{1}{p^{\varphi(m)s}} \quad s \in \mathbb{R}$$

$$\underbrace{\sum_{p} \frac{1}{p^{\varphi(m)s}}}_{\text{diverges for } s = 1/\varphi(m)} \quad - \underbrace{\sum_{p \mid m} \frac{1}{p^{\varphi(m)s}}}_{\text{finite}}$$

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$$P(s) = e^{Q(s)} = 1 + Q(s) + \cdots$$

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$$\underbrace{P(s)}_{\text{convergent}} \Leftrightarrow \underbrace{Q(s)}_{\text{convergent}}$$

The left side should be convergent for $\Re(s) > 0$ and the right side diverges at $s = \frac{1}{\varphi(m)}$, and has nonnegative coefficients.

This is a contradiction.

Dirichlet's theorem

For $m > 0$ and all a with $(a, m) = 1$ there exist infinitely many primes

$$p \equiv a \pmod{m}.$$

Proof

$\Re(s) > 1.$

- $$\log(L(s,\chi)) = \sum_p \frac{\chi(p)}{p^s} + \sum_{p,k\geq 2} \frac{\chi(p^k)}{kp^{sk}}$$

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- Choose b with $ab \equiv 1 \pmod{m}$.
- Consider $\sum_\chi \chi(b)\log(L(s,\chi)) =$

$$\underbrace{\sum_p \sum_\chi \chi(pb)p^{-s}}_{\begin{cases} \varphi(m) & bp \equiv 1 \pmod{m} \\ 0 & \text{otherwise} \end{cases}} + \underbrace{\sum_\chi \chi(b)R(s,\chi)}_{\text{conv. } \Re(s)>1/2}.$$

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$$\sum_{\chi} \chi(b) \log(L(s,\chi)) = \varphi(m) \sum_{p \equiv a \pmod{m}} p^{-s} + \underbrace{\tilde{R}(s)}_{\text{bounded } \Re(s) > 1/2}$$

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The left side goes to ∞ for $s \to 1+0$, from the contribution of $\chi = \chi_0$, the other characters do not vanish at $s = 1$. It follows that

$$\sum_{p \equiv a \pmod{m}} p^{-s} \to \infty \quad \text{for} \quad s \to 1+0.$$

Special values of Dirichlet L-functions

Consider

$$\chi_{12}(n) := \begin{cases} 1 & n \equiv 1, 11 \pmod{12} \\ -1 & n \equiv 5, 7 \pmod{12} \\ 0 & \text{otherwise} \end{cases}.$$

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Theorem

$$L(1, \chi_{12}) = \frac{\log(7 + 4\sqrt{3})}{\sqrt{12}} = \frac{\log((2 + \sqrt{3})^2)}{\sqrt{12}}$$

Special values of Dirichlet L-functions

Note that $1 = 7^2 - 3 \cdot 4^2$.

Proof:

$$f(x) := \sum \frac{\chi(n)}{n} x^n = \sum \frac{x^{12k+1}}{12k+1} - \frac{x^{12k+5}}{12k+5} - \frac{x^{12k+7}}{12k+7} + \frac{x^{12k+11}}{12k+11}$$

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$$\begin{aligned} f'(x) &= (1 - x^4 - x^6 + x^{10}) \sum_{k \geq 0} x^{12k} \\ &= \frac{(1-x)(1+x)}{1 - x^2 + x^4} \cdot \frac{1}{1 - x^{12}} \end{aligned}$$

Proof

For $|x| < 1$ we have:

$$f(x) = \frac{1}{\sqrt{12}} \left(\int \frac{2x + \sqrt{3}}{1 + \sqrt{3}x + x^2} dx - \int \frac{2x - \sqrt{3}}{1 - \sqrt{3}x + x^2} dx \right)$$

$$= \frac{1}{\sqrt{12}} \log \left(\frac{1 + \sqrt{3}x + x^2}{1 - \sqrt{3}x + x^2} \right)$$

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$$= \frac{1}{\sqrt{12}} \log \left(\frac{1 + \sqrt{3}x + x^2}{1 - \sqrt{3}x + x^2} \right)$$

which is continuous for $x \sim 1$. Substituting $x = 1$, we find

$$f(1) = \frac{1}{\sqrt{12}} \log \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right)$$
$$= \frac{1}{\sqrt{12}} \log \left((2 + \sqrt{3})^2 \right)$$

Special values

This is a particular case of a general result of Dirichlet:

Theorem

$$L(1,\chi_d) = \frac{1}{\sqrt{d}} \cdot \begin{cases} h\log(\epsilon) & d > 0 \\ \pi h & d < 0 \end{cases}$$

where

- $\chi_d(n) = \left(\frac{d}{n}\right)$
- *h is the class number of $\mathbb{Q}(\sqrt{d})$*

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where

- $\chi_d(n) = \left(\frac{d}{n}\right)$
- *h is the class number of $\mathbb{Q}(\sqrt{d})$*
- *ϵ is a fundamental solution of Pell's equation $x^2 - dy^2 = \pm 1$.*

Pell's equation

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Theorem

- *there are infinitely many solutions*

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Theorem

- *there are infinitely many solutions*
- *there exists an* $\epsilon = (x_1, y_1)$ *such that all solutions have the form* $\pm(x_n, y_n)$ *with*

$$x_n + \sqrt{d} y_n = (x_1 + \sqrt{d} y_1)^n$$

Pell's equation

Proof:

(1) $\xi \notin \mathbb{Q} \Leftrightarrow$ there exist infinitely many $x/y \in \mathbb{Q}$ such that

$$|\frac{x}{y} - \xi| < \frac{1}{y^2}$$

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$$[0,1] = [0, \frac{1}{n}) \sqcup [\frac{1}{n}, \frac{2}{n}) \cdots$$

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$$\{0\}, \{\xi\}, \ldots, \{n\xi\}.$$

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At least two are in the same interval, i.e., there exist $0 \leq j < k \leq n$ such that

$$|j\xi - [j\xi] - (k\xi - [k\xi])| < \frac{1}{n}$$

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Now choose $m > 1/|\frac{x}{y} - \xi|$. The same procedure gives the existence of (x_1, y_1) such that

$$|\frac{x_1}{y_1} - \xi| < \frac{1}{my_1} < |\frac{x}{y} - \xi| < \cdots$$

with $0 < y_1 < m$. This is a new approximation, keep going.

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Indeed,

$$x^2 - dy^2 = (x + \sqrt{d}y)(x - \sqrt{d}y),$$

and there are infinitely many (x, y) such that

$$|x - \sqrt{d}y| < \frac{1}{y}.$$

Pell's equation

Thus,

$$|x+\sqrt{d}y| < |x-\sqrt{d}y| + 2\sqrt{d}|y| < \frac{1}{y} + 2\sqrt{d}y.$$

It follows that

$$|x^2 - dy^2| < |\frac{1}{y} + 2\sqrt{d}y| \cdot \frac{1}{y} \leq 2\sqrt{d} + 1$$

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Indeed, may assume there are infinitely many solutions to the inequality with distinct x-components, thus there exist distinct x_1, x_2 such that $x_1 \equiv x_2 \pmod{m}$ and $y_1 \equiv y_2 \pmod{m}$.

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Put

$$\alpha := x_1 - y_1\sqrt{d}, \quad \beta := x_2 - y_2\sqrt{d}$$

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The congruence conditions imply that

$$\alpha \cdot \bar{\beta} = A + B\sqrt{d}, \quad m \mid A, m \mid B$$

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and since

$$\alpha\bar{\alpha} = \beta\bar{\beta} = m,$$

we have a solution

$$1 = u^2 - v^2 d$$

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If $v = 0$ then $\alpha \cdot \bar{\beta} = \pm m$, thus $\alpha \cdot m = \pm m \cdot \beta$, thus $\alpha = \pm\beta$ and $x_1 = x_2$, contradiction. So there exists a solution with $xy \neq 0$.

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Now we find the smallest solution:

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all > 0.

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For such a smallest solution ϵ, we have: any $\beta = \epsilon^n$.

Otherwise, choose n such that

$$\epsilon^n < \beta < \epsilon^{n+1},$$

$$1 < \bar{\epsilon}^n \beta < \epsilon$$

... produce a smaller ϵ.