Lecture 5

Plan

- Arithmetic functions and connection to $\zeta(s)$

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- Lattice points

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- Arithmetic functions and connection to $\zeta(s)$
- Lattice points
- Fourier methods in number theory

Multiplicative functions: divisor function

$$\zeta^2(s) = \left(\sum_{n\geq 1} \frac{1}{n^s}\right) \cdot \left(\sum_{m\geq 1} \frac{1}{m^s}\right) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

where

$$\sigma(n) := \sum_{d|n} 1.$$

Moebius function

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}) = \sum \frac{\mu(n)}{n^s},$$

where

$$\mu(n) := \begin{cases} (-1)^r & \text{if} \quad n = p_1 \cdots p_r \quad \text{distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Euler φ-function

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum \frac{\varphi(n)}{n^s},$$

where

$$\varphi(n) := n \cdot \prod_{p|n} (1 - \frac{1}{p}) = \#(\mathbb{Z}/n\mathbb{Z})^\times$$

is the Euler function.

von Mangoldt function

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) := \begin{cases} \log(p) & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

Ramanujan τ-function

$$\sum_{n\geq 1} \tau(n) q^n = q \prod_{n\geq 1} (1-q^n)^{24}, \quad q := e^{2\pi i z}$$

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We have

$$\tau(nm) = \tau(n)\tau(m), \quad \text{when} \quad (n,m) = 1.$$

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We have

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Moreover, we have

Ramanujan's conjecture = Deligne's theorem 1974

$$|\tau(p)| \leq 2p^{11/2}, \quad \text{for all primes } p$$

Ramanujan τ-function

Lehmer's conjecture

$$\tau(n) \neq 0, \quad \text{for all} \quad n \in \mathbb{N}$$

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Let $p(n)$ be the number of different representations of n as a sum of positive integers (in any order).

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Lehmer's conjecture

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Where did the τ-function come from?

Let $p(n)$ be the number of different representations of n as a sum of positive integers (in any order). The generating function for this is

$$\sum_{n \geq 1} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)\cdots$$

Partition function

Ramanujan discovered remarkable divisibility properties:

Congruences

- $p(7n+5) \equiv 0 \pmod{7}$ (Ramanujan)
- $p(11n+6) \equiv 0 \pmod{11}$ (Ramanujan)

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- $p(13 \cdot 11^3 \cdot n + 237) \equiv 0 \pmod{13}$ (Atkin 1960)

Partition function

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- $p(13 \cdot 11^3 \cdot n + 237) \equiv 0 \pmod{13}$ (Atkin 1960)
- Ken Ono (2000): Let $m \geq 5$ be a prime and $k \in \mathbb{N}$. A positive proportion of primes ℓ have the property that

$$p\left(\frac{m^k \ell^3 n + 1}{24}\right) \equiv 0 \pmod{m},$$

for for all n coprime to ℓ.

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$$\square(n) := \begin{cases} 1 & n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

Convolutions

$$f * g(n) := \sum_{d|n} f(d)g(\frac{n}{d}).$$

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Properties:

- $f * \mathbf{e} = f$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$

Convolutions

Theorem

If f, g are multiplicative then so is $f * g$.

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Proof:

$$\sum_{d_n, d_m | mn} f(d_n d_m) \cdot g(\frac{nm}{d_n d_m}) = \sum_{d_n | n} \sum_{d_m | m} f(d_n) \cdot g(\frac{d}{d_n}) \cdot f(d_m) g(\frac{m}{d_m}) = \ldots$$

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Proof: Clear for $n = 1$. for $n > 1$, write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then

$$\sum_{d|n} \mu(d) = \sum_{k=0}^{r} (-1)^k \underbrace{\binom{r}{k}}_{\text{pick } k \text{ out of } r \text{ primes}} = (1-1)^r = 0$$

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Note that the left side equals

$$\prod_{p|n} (1 + \mu(p)) = 0.$$

Moebius inversion

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$$f = g * \mathbf{1} \quad \Leftrightarrow \quad g = f * \mu.$$

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Proof:

$$g = g * \mathbf{e} = g * (\mathbf{1} * \mu) = (g * \mathbf{1}) * \mu$$

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Proof:

$$g = g * \mathbf{e} = g * (\mathbf{1} * \mu) = (g * \mathbf{1}) * \mu = f * \mu$$

Moebius inversion

What does this say?

$$f(n) := \sum_{d|n} g(d) \quad \Rightarrow \quad g(n) = \sum_{d|n} f(d) \cdot \mu(\frac{n}{d})$$

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Examples:

- $g = \varphi$, the Euler function

$$n = f(n) = \sum_{d|n} \varphi(d) \quad \Rightarrow \quad \varphi(n) = \sum_{d|n} d \cdot \mu(\frac{n}{d})$$

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- $g(n) = n$

$$\sigma_1(n) = \sum_{d|n} d \quad \Rightarrow \quad n = \sum_{d|n} \sigma_1(d) \cdot \mu(\frac{n}{d})$$

Moebius inversion

Theorem

$$f = g * \mathbf{1} = \sum_{d|n} g(d) \quad \Rightarrow \quad \sum_{n \le x} f(n) = \sum_{d \le x} g(d) \cdot \left[\frac{x}{d}\right]$$

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Proof: The left side equals

$$\sum_{n \leq x} \sum_{d|n} g(d) = \sum_{d \leq x} g(d) \cdot \underbrace{\left[\frac{x}{d}\right]}_{n \leq x, \text{ with } d|n}$$

Moebius inversion

We apply the formula with $f(n) = \log(n)$:

$$T(x) := \sum_{n \leq x} \log(n) = \sum_{n \geq 1} \Lambda(n) \cdot \left\lfloor \frac{x}{n} \right\rfloor$$

Moebius inversion

We apply the formula with $f(n) = \log(n)$:

$$T(x) := \sum_{n \le x} \log(n) = \sum_{n \ge 1} \Lambda(n) \cdot \left[\frac{x}{n}\right]$$

$$T(x) - 2T(\frac{x}{2}) = \sum_{n} \Lambda(n) \underbrace{\left(\left[\frac{x}{n}\right] - 2\left[\frac{x}{2n}\right]\right)}_{0 \text{ or } 1}$$

$$\le \sum_{\frac{x}{2} \le p \le x} \log(p) \cdot \frac{\log(x)}{\log(p)}$$

Lattice points

We will now discuss two typical number-theoretic problems.

- Lattice points in a circle (Gauss circle problem):

$$N(B) := \#\{(x, y) \in \mathbb{Z}^2 \mid x^2+y^2 \leq B\} \sim \pi B+E(B), \quad B \to \infty$$

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- Lattice points under a hyperbola

$$N_1(B) := \#\{(x,y) \in \mathbb{Z}^2 \mid xy \leq B\} \sim B(\log(B)+2\gamma-1)+E_1(B),$$

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$$N_1(B) := \#\{(x,y) \in \mathbb{Z}^2 \mid xy \leq B\} \sim B(\log(B)+2\gamma-1)+E_1(B),$$

Our goal is to obtain best possible estimates for the error terms $E(B)$ and $E_1(B)$.

Euler-MacLaurin

Let $f \in C^2([a, b])$,

$$\rho(x) := \frac{1}{2} - \{x\}, \quad \underbrace{\sigma(x) := \int_0^x \rho(u)}_{\text{not the divisor function}} \, du$$

Elementary tools

Euler-MacLaurin

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$$\rho(x) := \frac{1}{2} - \{x\}, \quad \underbrace{\sigma(x) := \int_0^x \rho(u)}_{\text{not the divisor function}} \, du$$

Then

$$\sum_{a<x\leq b} f(x) = \int_a^b f(x)\,dx + \rho(b)f(b) - \rho(a)f(a)$$

$$+ \sigma(a)f'(a) - \sigma(b)f'(b) + \int_a^b \sigma(x)f''(x)\,dx$$

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Proof: Integration by parts.

Elementary tools

A simpler version:

Let $f \in \mathrm{C}^1([a,b])$. Then

$$\sum_{a<x\leq b} f(x) = \int_a^b f(x)\,dx + \rho(b)f(b) - \rho(a)f(a) - \int_a^b f(x)f'(x)dx$$

Circle problem

Theorem (Gauss)

$$E(B) = O(\sqrt{B})$$

Consider the domain

$$0 \le x \le \sqrt{B/2}, \quad 0 \le y \le \sqrt{B - x^2}$$

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There are 8 such domains.

Circle problem

$$N(B) = \underbrace{1}_{(0,)} + 4 \underbrace{[\sqrt{B}]}_{\text{one coordinate } = 0} +$$

$$+ 8 \sum_{0<x\leq\sqrt{B/2}} [\sqrt{B-x^2}] - 4([\sqrt{B/2}])^2$$

$$= 8 \sum_{0<x\leq\sqrt{B/2}} \sqrt{B-x^2} - 2B + 4\sqrt{2B}\{\sqrt{B/2}\} + 4\sqrt{B} -$$

$$- 8 \sum_{0<x\leq\sqrt{B/2}} \{\sqrt{B-x^2}\} + O(1)$$

Circle problem

Applying the formula, we obtain

$$\sum_{0<x\le\sqrt{B/2}} \sqrt{B-x^2} = \int_0^{\sqrt{B/2}} \sqrt{B-u^2}\,du +$$

$$+\left(\frac{1}{2}-\{\sqrt{B/2}\}\right)\sqrt{B/2}-\sqrt{B/2}+\sigma(\sqrt{B/2})+$$

$$+\int_0^{\sqrt{B/2}} \sigma(u)(\sqrt{B-u^2})''\,du$$

$$=\frac{\pi B}{8}+\frac{B}{4}+\cdots, \qquad |\sigma(u)|\le\frac{1}{8}$$

Circle problem

$$\sum_{0<x\le\sqrt{B/2}} \sqrt{B-x^2} = \frac{\pi B}{8} + \frac{B}{4} + \\ + \frac{\sqrt{B/2}}{2} - \sqrt{B/2}\{\sqrt{B/2}\} - \sqrt{B/2} + O(1)$$

Circle problem

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$$N(B) = \pi B + E(B)$$

with

$$E(B) = 2\sqrt{2B} - 8 \sum_{0<x\le\sqrt{B/2}} \{\sqrt{B-x^2}\} + O(1) = O(\sqrt{B})$$

Circle problem: better estimate

Exercise: Let $b - a \ll A$ and $f \in \mathrm{C}^2([a, b])$ be such that

$$f''(x) \gg A^{-1}, \quad 0 < f'(x) \ll 1, \qquad \text{for } \; x \in [a, b]$$

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$$\sum_{a<x\leq b} \{f(x)\} = \frac{b-a}{2} + O(A^{2/3})$$

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Then

$$\sum_{a<x\leq b} \{f(x)\} = \frac{b-a}{2} + O(A^{2/3})$$

Apply this to $f(x) = \sqrt{B - x^2}$. It follows that

$$E(B) = 2\sqrt{2B} - 8 \sum_{0<x\leq\sqrt{B/2}} \{\sqrt{B - x^2}\} + O(1)$$

$$2\sqrt{2B} - \frac{8 \cdot \sqrt{B/2}}{2} + O(B^{\frac{1}{2}\cdot\frac{2}{3}}) = O(B^{\frac{1}{3}}).$$

Points under a hyperbola

Theorem (Dirichlet)

$$E_1(B) = O(\sqrt{B})$$

Points under a hyperbola

Consider the domain

$$1 \leq x \leq \sqrt{B}, \quad 0 \leq y \leq B/x.$$

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We find that

$$N_1(B) = 2 \sum_{1 \leq x \leq \sqrt{B}} [\frac{B}{x}] - ([\frac{B}{x}])^2$$

Points under a hyperbola

$$N_1(B) = 2B \sum_{1\le x\le\sqrt{B}} \frac{1}{x} - 2 \sum_{1\le x\le\sqrt{B}} \{\frac{B}{x}\} - B + 2\sqrt{B}\{\sqrt{B}\} + O(1)$$

Applying the formula to the sum over $\frac{1}{x}$, we find

$$\sum = \log\sqrt{B} + (\frac{1}{2} - \{\sqrt{B}\})\frac{1}{\sqrt{B}} - \frac{1}{2} + \\ + \sigma(\sqrt{B})\frac{1}{B} - \sigma(1) + 2\int_1^\infty \sigma(u)\frac{du}{u^3} + O(\frac{1}{B})$$

Points under a hyperbola

Recall that

$$\gamma := \lim_{x \to \infty} (\sum_{1 \leq n \leq x} \frac{1}{n}) - \log(x),$$

thus

$$\gamma = -\frac{1}{2} + 2 \int_1^\infty \sigma(u) \frac{du}{u^3}$$

Points under a hyperbola

$$N_1(B) = B\log(B) + 2\sqrt{B}(\frac{1}{2} - \sqrt{B}) + (2\gamma - 1)B -$$
$$- 2\sum_{1 \le x \le \sqrt{B}} \{\frac{B}{x}\} + 2\sqrt{B}\{\sqrt{B}\} + O(1)$$
$$= B(\log(B) + 2\gamma - 1) + E_1(B)$$

with

$$E_1(B) = \sqrt{B} + 2\sum_{1 \le x \le \sqrt{B}} \{\frac{B}{x}\} + O(1) = O(\sqrt{B}).$$

Fourier methods

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Generally, these are among the most heavily used tools in analytic number theory.

Fourier analysis

$$e(x) := e^{2\pi ix}, \quad f \in \mathrm{L}^1(\mathbb{R})$$

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$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e(-xy)\,dx$$

$$\hat{\hat{f}}(x) = f(-x)$$

Fourier pairs

$f(y)$	$\hat{f}(x)$	
$\max\{1 - \frac{\lvert y\rvert}{Y}, 0\}$	$\left(\frac{\sin(\pi x Y)}{\pi x Y}\right)^2$	Fejer kernel
$\begin{cases} 1 & \lvert y\rvert < 1 \\ \frac{1}{2} & \lvert y\rvert = \frac{1}{2} \\ 0 & \lvert y\rvert > 0 \end{cases}$	$\frac{\sin(2\pi x)}{\pi x}$	
$e^{-2\pi\lvert y\rvert}$	$\frac{1}{\pi}(1+x^2)^{-1}$	
$e^{-\pi y^2}$	e^{-x^2}	self-dual
$\frac{1}{\mathrm{ch}(\pi y)}$	$\frac{1}{\mathrm{ch}(\pi x)}$	

Mellin transform

$$f : \mathbb{R}_{>0} \to \mathbb{C}$$

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There is an integral transform, defined by

$$M(f)(s) := \int_0^\infty f(y) y^{s-1} \, dy$$

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which depends on $\Re(s)$. This is a version of the Fourier transform. Namely, put

$$y = e^x, \quad s := it.$$

Mellin transform

The inverse transform is given by

$$f(y) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma} M(f)(s) y^{-s} \, ds$$

Mellin transform

$f(y)$	$M(f)(s)$	$\sigma := \Re(s)$
e^{-y}	$\Gamma(s)$	$\sigma > 0$
$\cos(y)$	$\Gamma(s)\cos(\frac{\pi s}{2})$	$0 < \sigma < 1$
$\log(1+y)$	$\frac{\pi}{s \cdot \sin(\pi s)}$	$-1 < \sigma < 0$
$\cos(\frac{x}{2} \cdot (y - \frac{1}{y}))$	$2K_s(x)\cos(\frac{\pi s}{2})$	$x > 0$, Bessel function

Poisson summation formula

Theorem

$f, \hat{f} \in \mathrm{L}^1$, *and bounded variation (or continuous).*

Poisson summation formula

Theorem

$f, \hat{f} \in \mathrm{L}^1$, and bounded variation (or continuous). Then

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Poisson summation formula

Proof: Put

$$\begin{aligned} F(x) &:= \sum_{m\in\mathbb{Z}} f(x+m) \quad \in \mathrm{L}^1(\mathbb{R}/\mathbb{Z}) \\ &= \sum_{n\in\mathbb{Z}} c_F(n)\cdot e(nx) \end{aligned}$$

where

$$c_F(n) := \int_0^1 F(t)e(-nt)\,dt = \int_{\mathbb{R}} f(t)e(-nt)\,dt = \hat{f}(n)$$

Poisson summation formula

Proof: Put

$$F(x) := \sum_{m \in \mathbb{Z}} f(x+m) \quad \in \mathrm{L}^1(\mathbb{R}/\mathbb{Z})$$
$$= \sum_{n \in \mathbb{Z}} c_F(n) \cdot e(nx)$$

where

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The series converges uniformly. Now put $x := 0$.

Poisson summation formula

Corollaries:

- $$\sum_{m\in\mathbb{Z}} f(vm+u) = \frac{1}{v}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{v})e(\frac{un}{v})$$

Poisson summation formula

Corollaries:

- $$\sum_{m\in\mathbb{Z}} f(vm+u) = \frac{1}{v}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{v})e(\frac{un}{v})$$

- Let $\chi : \mathbb{Z} \to \mathbb{S}^1$ be a Dirichlet character $\pmod q$, e.g., $\chi(\cdot) = \left(\frac{\cdot}{q}\right)$. Then

  $$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}} \hat{f}(\frac{n}{q})\bar{\chi}(n)$$

  where

  $$\tau(\chi) := \sum_{b \pmod q} \chi(b)e(\frac{b}{q})$$

  is the corresponding Gauss sum.

Poisson summation formula

A similar formula holds in higher dimensions:

$$\sum_{m \in \mathbb{Z}^k} f(m) = \sum_{n \in \mathbb{Z}^k} \hat{f}(n).$$

Let us apply this for $k = 2$. Assume that f is invariant under $\mathbb{S}^1$ (rotation invariant).

Poisson summation formula

A similar formula holds in higher dimensions:

$$\sum_{m \in \mathbb{Z}^k} f(m) = \sum_{n \in \mathbb{Z}^k} \hat{f}(n).$$

Let us apply this for $k = 2$. Assume that f is invariant under $\mathbb{S}^1$ (rotation invariant). Then $\hat{f}$ is also invariant.

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Then

$$\sum_{m=0}^{\infty} r_2(m) g(m) = \pi \int_0^\infty g(x)\, dx + \sum_{n=1}^{\infty} r_2(n) h(n)$$

where ...

Poisson summation formula

$$h(y) := \pi \int_0^\infty g(x) J_0(2\pi\sqrt{xy})\, dx, \quad \text{Hankel transform}$$

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and

$$\pi J_0(z) = \int_0^\pi \cos(z \sin(\theta)), d\theta, \quad \text{Bessel function}$$

Analysis of the asymptotic

1. $$\pi J_0(z) \sim \left(\frac{2\pi}{z}\right)^{1/2} \cdot \left(\cos(z - \frac{\pi}{4}) + \frac{\sin(z - \frac{\pi}{4})}{8z} + O(\frac{1}{z^2})\right), \ z > 0$$

   pause

2. $$\begin{aligned} h(y) &= \int_0^\infty (xy)^{-1/4} g(x) \cos(2\pi\sqrt{xy} - \frac{\pi}{4})\, dx \quad (= I_1) \\ &+ \int_0^\infty (xy)^{-3/4} g(x) \sin(2\pi\sqrt{xy} - \frac{\pi}{4})\, dx \quad (= I_2) \\ &+ O\left(\int_0^\infty (xy)^{-5/4} |g(x)|\, dx\right) \quad (= I_3) \end{aligned}$$

Analysis of the asymptotic

Integration by parts, applied to both I_1 and I_2, yields

$$h(y) = -\frac{1}{\pi y} \int_0^\infty (xy)^{1/4} g'(x) \sin(2\pi\sqrt{xy} - \frac{\pi}{4})\, dx +$$
$$O\left(\int_0^\infty (xy)^{-5/4} (|g(x)| + x \cdot |g'(x)|)\, dx\right)$$

Analysis of the asymptotic

Now assume that $1 \leq A \leq B^{1/2}$ and put

$$g(x) := \begin{cases} \min(1, x, (A+B-x)A^{-1}) & 0 \leq x \leq A+B \\ 0 & \text{otherwise} \end{cases}$$

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We have

$$h(y) \ll y^{-3/4} B^{1/4} (1 + \frac{y}{C})^{-1/2}, \quad C := \frac{B}{A^2}$$

Analysis of the asymptotic

We continue:

$$\sum_{n\geq 1} r_2(n)h(n) = \sum_{a,b} h(a^2+b^2) \ll (BC)^{1/4} = \left(\frac{B}{A}\right)^{1/2}$$

and

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The error term is optimal when $A = B^{1/3}$. This gives

$$E(B) = O(B^{1/3}).$$

Analysis of the asymptotic

The same arguments apply to

$$r_3(m),$$

(sum of three squares). One has

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There are hundreds of papers improving these error terms.

Riemann zeta function

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Proof:

$$\frac{1}{|\zeta(s)|} = |\prod_p \left(1 - \frac{1}{p^s}\right)| \leq \prod_p \left(1 + \frac{1}{p^\sigma}\right) < \sum_{n \geq 1} \frac{1}{n^\sigma}$$

$$< 1 + \int_1^\infty \frac{du}{u^\sigma} = \frac{\sigma}{\sigma - 1}.$$

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Thus

$$|\zeta(s)| > \frac{\sigma - 1}{\sigma} > 0$$

Riemann zeta function

- For $\Re(s) > 0$, we have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\int_N^\infty \frac{\rho(u)}{u^{s+1}}\,du$$

where

$$\rho(u) = \frac{1}{2} - \{u\}.$$

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  I.e., we have an isolated pole at $s = 1$, with residue 1.

Riemann zeta function

Proof: We use the Euler-MacLaurin formula:

$$\sum_{N+1/2<n\leq M+1/2} \frac{1}{n^s} = \int_{N+1/2}^{M+1/2} \frac{du}{u^s} + s\int_{N+1/2}^{M+1/2} \frac{\rho(u)}{u^{s+1}}\, du$$

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- $H \subseteq G$ a closed subgroup

Abstract Fourier analysis

Theorem (Poisson formula)

$$\int_H f \, dh = \int_{(G/H)^\perp} \hat{f} \, d\hat{g}$$

where

$$\hat{f}(\chi) := \int_G f(g)\chi(g)dg$$

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Issues:

- measure
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Fourier analysis

Main example: $G = (\mathbb{R}, +)$:

- $\chi = \chi_a : x \mapsto e^{2\pi i x a}$, with $a \in \mathbb{R}$.
- $\hat{G} = \mathbb{R}$ (self-dual)
- for $H = G$ get

$$\int_G f dg = \hat{f}(\chi_0)$$

- for $H = 0$ get Fourier inversion

$$f = \int_{\hat{G}} \hat{f} d\hat{g}$$

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- for $H = G$ get
$$\int_G f dg = \hat{f}(\chi_0)$$
- for $H = 0$ get Fourier inversion
$$f = \int_{\hat{G}} \hat{f} d\hat{g}$$
- for $H = \mathbb{Z} \subset \mathbb{R}$ get the classical Poisson summation formula:
$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$
note that $\mathbb{Z}$ is self-dual in $\mathbb{R}$.

Self-dual functions

If

$$f := e^{-x^2}$$

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The theta-function is defined by

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n z} \cdot e^{\pi i n^2 \tau}$$

(whenever this converges).

Theta-function

Consider the special case:

$$\theta(y) := \theta(iy, 0) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \cdot \sum_{n\in\mathbb{Z}} e^{-\pi n^2/y},$$

(the series converges e.g., for $y \in \mathbb{R}_{>0}$).

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(the series converges e.g., for $y \in \mathbb{R}_{>0}$). Thus

$$\theta(y) = \frac{1}{\sqrt{y}} \cdot \theta(\frac{1}{y}).$$

Theta-function

Consider a related function

$$\omega(s) = \sum_{n\geq 1} e^{-n^2\pi s} = \frac{\theta(s) - 1}{2}$$

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Using the functional equation for the theta-function, we obtain

$$\omega(s) = s^{-1/2} \cdot \omega\left(\frac{1}{s}\right) - \frac{1}{2} + \frac{s^{-1/2}}{2}.$$

Mellin transform

We compute the Mellin transform:

$$M(\omega) = \int_0^\infty \omega(t) t^{s-1}\, dt = \sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t} t^{s-1}\, dt$$

which converges for $\Re(s) > 1/2$.

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$$\begin{aligned}\int_0^\infty e^{-\pi n^2 t} t^{s-1}\, dt &= \frac{1}{\pi^s n^{2s}} \int_0^\infty t^{s-1} e^{-t}\, dt \\ &= \frac{\Gamma(s)}{\pi^s n^{2s}}\end{aligned}$$

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It follows that

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$$\begin{aligned} M(\omega)(s) &= \int_0^1 \omega(t) t^{s-1}\, dt + \int_1^\infty \omega(t) t^{s-1}\, dt \\ &= \int_0^1 t^{s-1} \underbrace{\left(t^{-1/2}\omega(\frac{1}{t}) - \frac{1}{2} + \frac{t^{-1/2}}{2} \right)}_{\omega(t)} dt \\ &+ \int_1^\infty \omega(t) t^{s-1}\, dt \end{aligned}$$

Mellin transform

We change variables, putting $u = 1/t$. This turns the first summand into

$$\int_1^\infty u^{-1-s}\left(u^{1/2}\omega(u) - \frac{1}{2} + \frac{u^{1/2}}{2}\right) du$$
$$= \int_1^\infty u^{-1/2-s}\omega(u)\, du - \frac{1}{2s} - \frac{1}{1-2s}$$

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Since

$$\omega(t) < \sum_{n \geq 1} e^{-\pi n t} < 2e^{-\pi t}$$

the integral converges for all s.

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It has a simple pole at $s = 0, 1/2$.

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It follows that

$$\pi^{-s} \cdot \Gamma(s)\zeta(2s) = \pi^{s-1/2} \cdot \Gamma(\frac{1}{2} - s)\zeta(1 - 2s)$$

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Then

$$\xi(s) = \xi(1-s),$$

this is the functional equation for the Riemann zeta function.