Lecture 4

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$$\mathbb{Q} \hookrightarrow \mathbb{Q}_p \qquad \mathbb{C}(x) \hookrightarrow \mathbb{C}((x - \alpha))$$

Functions

We started investigating functions

$$f : \mathbb{Q}_p \to \mathbb{Q}_p$$

- rational functions
- series $\sum a_n x^n$, e.g.,

$$e^x = \sum \frac{x^n}{n!}, \quad \log_p(1+x) = \sum_{n \geq 1} \frac{x^n}{n}(-1)^{n+1}, \quad \text{for } |x|_p < p^{-\frac{1}{p-1}},$$

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$$(1+x)^a = \sum_{n\geq 0} \frac{a(a-1)\cdots(a-n+1)}{n!} x^n =: B_{a,p}(x) \in \mathbb{Z}_p[[x]]$$

Interpolation over $\mathbb{R}$

Given a finite set of pairs

$$(x_j, y_j), \quad j = 0, \ldots, m,$$

find a function (e.g., polynomial) f such that $f(x_j) = y_j$ for all j.

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$$f(x) := \sum_{k=0}^{m} y_k \cdot \frac{\prod_{j \neq k}(x - x_j)}{\prod_{j \neq k}(x_k - x_j)}$$

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This is a polynomial interpolation of a finite set of points. Another instance of interpolation is approximation via continuity: how does one define a^x? First for $x \in \mathbb{Q}$, then by continuity, since $\mathbb{Q}$ is dense in $\mathbb{R}$.

Interpolation over $\mathbb{Q}_p$

Recall that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

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When is this possible? How does one achieve this?

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Let us try

$$a^x, \quad a \in \mathbb{Z},$$

p-adically.

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$$|a^x - a^{x'}|_p = |1 - p^{p^N}|_p = 1, \quad \forall N.$$

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Not good, we are not getting closer.

Interpolation over $\mathbb{Q}_p$: a^x

Assume that $1 < a < p$. Then

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Again, we have a problem.

Interpolation over $\mathbb{Q}_p$: a^x

However, let $a \equiv 1 \pmod{p}$, $a = 1 + bp$ and $x' = x + x'' p^N$. Then

$$|x' - x|_p \leq \frac{1}{p^N},$$

$$|a^x - a^{x'}|_p = |a^x|_p \cdot |1 - a^{x'-x}|_p = |1 - (1+bp)^{x'' p^N}|_p \leq |p^{N+1}|_p = \frac{1}{p^{N+1}}$$

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It follows that for $a \equiv 1 \pmod{p}$, the function

$$f(x) = a^x$$

is well-defined and continuous for $x \in \mathbb{Z}_p$.

Interpolation over $\mathbb{Q}_p$: a^x

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$$S := \{x \in \mathbb{N} \mid x \equiv x_0 \pmod{p-1}\} \subset \mathbb{Z}_p$$

This set is dense. Thus, any

$$f : S \to \mathbb{Z}_p$$

will have a unique continuous extension to $\mathbb{Z}_p$.

Interpolation: the Γ-function

Recall

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$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s \, dx, \quad s \in \mathbb{C}$$

interpolates (over $\mathbb{C}$) between the values $n!$

Interpolation: the Γ-function

Note, there does not exist a continuous function

$$f : \mathbb{Z}_p \to \mathbb{Z}_p, \qquad f(n) = n!, \quad \forall n \in \mathbb{N}.$$

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Why? $n!$ is too divisible by p.

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Try:

$$\prod_{1 \le j \le n,\ p \nmid j} j$$

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Does not work either.

Interpolation: the Γ-function

Theorem

Let $p \geq 3$ be a prime. The function

$$n \mapsto (-1)^n \prod_{j \leq n,\, p \nmid j} j$$

admits a continuous extension to

$$\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$$

Γ_p

Proof: We need to show that

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First, observe that

- $\Gamma_p(n) \in \mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$

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$$1 \equiv \frac{\Gamma_p(n')}{\Gamma_p(n)} = (-1)^n \cdot \prod_{n \le j < n'} j \pmod{p^N}$$

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Indeed, assume first $n_1 = 1$. Note that $(-1)^{p^N} = -1$, thus we need to show that

$$\prod_{n \leq j < n+p^N} j \equiv -1 \pmod{p^N}$$

$$\equiv \prod_{0<j<p^N,\, p\nmid j} j$$

$$\equiv \prod \underbrace{jj'}_{1} \cdot 1 \cdot (-1)$$

the only solutions to $j^2 = 1$ are $j = 1, -1 \pmod{p^N}$.

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A similar argument works for arbitrary n_1.

Γ_p: Properties

- $$\frac{\Gamma_p(a+1)}{\Gamma_p(a)} = \begin{cases} -a & a \in \mathbb{Z}_p^\times \\ -1 & a \in p\mathbb{Z}_p \end{cases}$$

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  Then apply induction:

  $$\frac{\Gamma_p(a+1) \cdot \Gamma_p(1-(a+1))}{\Gamma_p(a) \cdot \Gamma_p(1-a)} = \begin{cases} -a/(-(-a)) = -1 & a \in \mathbb{Z}_p^\times \\ -1/(-1) = 1 & a \in p\mathbb{Z}_p \end{cases}$$

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Recall:

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi.$$

Artin-Hasse exponential

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Theorem

This converges for $|x|_p < 1$ *(better than* $\exp(x)$*).*

Artin-Hasse exponential

Proof:

$$\mu(n) := \begin{cases} (-1)^r & \text{if} \quad n = p_1 \cdots p_r \quad \text{distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

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Properties:

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2. $\sum_{d|n} |\mu(d)| = 2^k$, where $k = \#$ of distinct primes dividing n

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3. $\sum_{n\geq 1} -\frac{\mu(n)}{n} \cdot \log(1 - x^n) = x$

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3. $\sum_{n \geq 1} -\frac{\mu(n)}{n} \cdot \log(1 - x^n) = x$
4. $\sum_{n \geq 1,\, p \nmid n} -\frac{\mu(n)}{n} \cdot \log(1 - x^n) = x + \frac{x^p}{p} + \cdots$

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As formal power series.

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Theorem

$$E_p(x) \in \mathbb{Z}_p[[x]]$$

and thus converges for $|x|_p < 1$.

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Proof: For $p \nmid n$, we have $-\frac{\mu(n)}{n} \in \mathbb{Z}_p$.

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Thus

$$\prod_{n,\, p \nmid n} (\cdots) \in \mathbb{Z}_p[[x]]$$

Dieudonné-Dwork theory

Theorem

Let $f \in 1 + x\mathbb{Q}_p[[x]]$*. Then*

$$f \in \mathbb{Z}_p[[x]] \quad \Leftrightarrow \quad f(x)^p / f(x^p) \in 1 + p\mathbb{Z}_p[[x]]$$

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Proof: $\Rightarrow \quad f(x)^p \equiv f(x^p) \pmod{p}$.

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It follows that

$$\frac{f(x)^p}{f(x^p)} \in 1 + px\mathbb{Z}_p[[x]].$$

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$\Leftarrow$ Let
$$f(x) = 1 + \sum_{i \geq 1} a_i x^i, \quad a_i \in \mathbb{Q}_p.$$

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Assume that

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We see that

$$a_0 = 1$$
$$a_1 = b_1 \in \mathbb{Z}_p$$

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Now we proceed by induction, assuming that $a_i \in \mathbb{Z}_p$ for all $i < n$. Comparing coefficients at x^n:

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On the left: $\left(\sum_{i \le n} a_i x^i\right)^p$ **On the right:** $f(x^p) \cdot (1 + p \sum b_j x^j)$

$$= \sum_{i \le n} a_i^p x^{ip} + p \underbrace{(\cdots)}_{a_{i1} \cdots a_{ip} x^{i_1 + \cdots i_p}} \qquad \sum_{i \le \frac{n}{p}} a_i x^{pi} (1 + p \sum b_j x^j)$$

$$= \underbrace{a_i^p}_{ip=n} + pa_n + p\mathbb{Z}_p \qquad = \underbrace{a_{\frac{n}{p}}}_{\mathbb{Z}_p} + p\mathbb{Z}_p - \text{terms}$$

$$= \underbrace{a_{\frac{n}{p}}^p}_{\mathbb{Z}_p} + pa_n + p\mathbb{Z}_p \qquad \text{Have: } a_{\frac{n}{p}}^p \equiv a_{\frac{n}{p}} \pmod p$$

Dieudonné-Dwork theory

Thus,

$$pa_n \in p\mathbb{Z}_p \Rightarrow a_n \in \mathbb{Z}_p$$

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Apply: Since

$$E_p(x)^p = e^{px}E_p(x^p) \quad \text{and} \quad e^{px} \in 1 + px\mathbb{Z}_p[[x]]$$

we conclude

$$E_p(x) \in \mathbb{Z}_p[[x]],$$

i.e., converges for $|x|_p < 1$ (alternative proof of the previous theorem).

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$$E_p(x)^p = e^{px} E_p(x^p) \quad \text{and} \quad e^{px} \in 1 + px\mathbb{Z}_p[[x]]$$

we conclude

$$E_p(x) \in \mathbb{Z}_p[[x]],$$

i.e., converges for $|x|_p < 1$ (alternative proof of the previous theorem). Here we used that

$$\nu_p\left(\frac{p^n}{n!}\right) \geq n - \frac{n-1}{p-1} = \frac{p-2}{p-1}n + \frac{1}{p-1} \geq 1, \quad \forall n.$$

Binomial polynomials

Recall,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \Rightarrow \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} \in \mathbb{Q}[x].$$

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In particular, this extends to a continuous function $\mathbb{Z}_p \to \mathbb{Z}_p$.

Proof: OK for $x \in \mathbb{N}$, note that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Binomial polynomials

Theorem

Let $\mathcal{L}$ be the $\mathbb{Z}$-module of all functions $f \in \mathbb{Q}[x]$ such that

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Binomial polynomials

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Let $\mathcal{L}$ be the $\mathbb{Z}$-module of all functions $f \in \mathbb{Q}[x]$ such that

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Then $\mathcal{L}$ is free, with basis $\binom{x}{k}$, i.e.,

$$f(x) = \sum_{k \geq 0} m_k \binom{x}{k}, \quad m_k \in \mathbb{Z}.$$

Binomial polynomials

The proof uses the difference operator:

$$\Delta f(x) := f(x+1) - f(x).$$

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This is the analog of

$$\partial : \frac{x^k}{k!} \to \frac{x^{k-1}}{(k-1)!}$$

Binomial polynomials

The proof proceeds by induction,

$$f(0) := m_0$$

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The existence follows by setting

$$m_k := (\Delta^k f)(0), \qquad \text{i.e., a Taylor expansion}$$

$$f(x) = \sum_k \frac{(\Delta^k f)(0)}{k!} \cdot x(x-1)\cdots(x-k+1)$$

Binomial polynomials

Assume that

$$\left(\sum a_n \frac{x^n}{n!}\right) \cdot \left(\sum c_n \frac{x^n}{n!}\right) = \left(\sum b_n \frac{x^n}{n!}\right).$$

Then

$$\sum \binom{n}{k} a_k c_{n-k} = b_n.$$

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Proof: Compare coefficients at x^n.

Binomial polynomials

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k$$

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Proof: Apply to

$$\left(\sum a_n \frac{x^n}{n!} \right) \cdot e^x = \left(\sum b_n \frac{x^n}{n!} \right).$$

p-adic interpolation

Mahler 1961

Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a continuous function. Put

$$a_n(f) := \sum (-1)^{n-k} \binom{n}{k} f(k), \qquad \text{this is a finite sum}$$

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Then

$$\sum_{k=0}^{\infty} \binom{x}{k} a_k(f) \to f(x)$$

converges uniformly.

p-adic interpolation

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p-adic interpolation

- The sum is finite on $\mathbb{Z}$
- $|a_k|_p \to 0$, so that the series converges to a continuous function
- Every continuous function has such a representation, and it is unique (since determined by restriction to $\mathbb{N}$)

Mahler's theory

Let K be a field of characteristic zero, e.g., $\mathbb{Q}$ or $\mathbb{Q}_p$. Introduce the following operators on $K[x]$:

- translation operator: for $a \in K$

$$\begin{aligned} \tau_a : & K[x] \to K[x] \\ (\tau_a f)(x) := & f(x+a) \end{aligned}$$

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$$\begin{aligned} \tau_a :& K[x] \to K[x] \\ (\tau_a f)(x) :=& f(x+a) \end{aligned}$$

- δ-operator: a linear endomorphisms $\delta : K[x] \to K[x]$, which commutes with τ_a for all $a \in K$, i.e.,

$$\delta \circ \tau_a = \tau_a \circ \delta,$$

  and satisfies

$$\delta(x) = c \in K^{\times}.$$

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It follows that

$$\delta(a) = 0, \quad \forall a \in K^\times, \quad \deg(\delta f) = deg(f) - 1.$$

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A basis system $\{q_n = q_{n,\delta}\}_{n\in\mathbb{N}}$ is a collection of polynomials such that

- $\deg(q_n) = n$, for all n
- $\delta q_n = nq_{n-1}$, for $n \geq 1$,
- $q_0 = 1, q_n(0) = 0$, for $n \geq 1$.

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This is uniquely determined, by induction.

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Examples:

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- $\tau_a - \tau_b$, for $a \neq b$
- Any formal power series of order 1 in $\frac{\partial}{\partial x}$:

$$\delta := \sum_{i \geq 1} c_i \left(\frac{\partial}{\partial x}\right)^i \in K[[\frac{\partial}{\partial x}]], \quad c_1 \neq 0$$

Mahler's theory

For all $f \in K[x]$, we have

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as if we were computing

$$q_n(x+y)" = "(q(x)+q(y))^n$$

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$$T := K[x] \to K[x]$$

be an endomorphism. The following properties are equivalent:

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- $T = \phi(\frac{\partial}{\partial x}) \in K[[\frac{\partial}{\partial x}]]$
- $T \circ \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \circ T$
- $T \circ \delta = \delta \circ T$, for all δ-operators

Mahler's theory

Proof: Based on the identities:

$$T := \sum_{k \geq 0} \frac{(Tq_k)(0)}{k!} \delta^k$$

$$\tau_a = \sum_{k \geq 0} \frac{q_k(0)}{k!} \delta^k$$

which means that if T commutes with δ then also with τ_a.

Mahler's theory

Consider the Banach space (complete normed vector space)

$$\mathcal{C}(\mathbb{Z}_p) = \{f : \mathbb{Z}_p \to \mathbb{Q}_p\}$$

of continuous functions on $\mathbb{Z}_p$.

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be a continuous endomorphism (note that $\frac{\partial}{\partial x}$ is not continuous). We can define its norm

$$\|T\| := \sup_{\|f\|=1} \|Tf\|.$$

Mahler's theory

Assume that T commutes with τ_1 (or $\Delta = \tau_1 - \mathrm{Id}$). Then T preserves

$$K[x] \subset \mathcal{C}(\mathbb{Z}_p)$$

and the restriction of T to $K[x]$ can be written as

$$\sum \alpha_n \Delta^n \in K[[\Delta]]$$

Mahler's theory

Assume that $T(1) = 0$, and $\|T\| = 1$. Let $\{q_n\}$ be a basis system for T, $Tq_n = nq_{n-1}$. Then

$$\|\frac{q_n}{n!}\| = 1.$$

Every $f \in \mathcal{C}(\mathbb{Z}_p)$ admits a representation (generalized Mahler series):

$$f(x) = \sum c_n \frac{q_n}{n!},$$

with

$$c_n := (T^n f)(0) \to 0$$

and

$$\|f\| = \sup_{n \geq 0} |c_n|_p.$$

Number-theoretic functions

Next, we will discuss various functions arising in arithmetic.

- They are multiplicative.

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$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad \Re(s) > 1.$$

Number-theoretic functions

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- They are multiplicative.
- Many of them are related to the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \Re(s) > 1.$$

- There are deep conjectures concerning statistical behavior of these functions.

Divisor function

$$\zeta^2(s) = \left(\sum_{n\geq 1} \frac{1}{n^s}\right) \cdot \left(\sum_{m\geq 1} \frac{1}{m^s}\right) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

where

$$\sigma(n) := \sum_{d|n} 1.$$

is the number of different representations of n as a product of two integers.

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$$D(x) = x \log(x) + x(2\gamma - 1) + E(x), \qquad \text{error term }.$$

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Conjecture

$$E(x) = O(x^{\frac{1}{4}+\epsilon}), \qquad \text{for all} \qquad \epsilon > 0.$$

Divisor function

More generally,

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We have

$$\sigma_r(nm) = \sigma_r(n) \cdot \sigma_r(m), \quad \text{when} \quad (n, m) = 1.$$

Moebius function

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}) = \sum \frac{\mu(n)}{n^s},$$

where

$$\mu(n) := \begin{cases} (-1)^r & \text{if} \quad n = p_1 \cdots p_r \quad \text{distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

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Titchmarsh 1951

Riemann hypothesis is equivalent to

$$\sum_{n \leq x} \mu(n) = O(x^{\frac{1}{2}+\epsilon}), \quad \text{for all} \quad \epsilon > 0.$$

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$$\sum_{n \leq x} \mu(n) = O(x^{\frac{1}{2}+\epsilon}), \quad \text{for all} \quad \epsilon > 0.$$

I.e., $\mu(n)$ is a random sequence.

Euler φ-function

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum \frac{\varphi(n)}{n^s},$$

where

$$\varphi(n) := n \cdot \prod_{p|n} (1 - \frac{1}{p}) = \#(\mathbb{Z}/n\mathbb{Z})^\times$$

is the Euler function.

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Lehmer's conjecture 1932

There are no composite n such that $\varphi(n) \mid (n-1)$.

Dedekind ψ-function

$$\frac{\zeta(s)\cdot\zeta(s-1)}{\zeta(2s)} = \sum_{n\geq 1}\frac{\psi(n)}{n^s},$$

where

$$\psi(n) := n\cdot\prod_{p|n}(1+\frac{1}{p})$$

is the Dedekind ψ-function.

von Mangoldt function

$$-\frac{\zeta'(s)}{\zeta(s)} = -\log(\zeta(s))' = \sum_p \log(1-\frac{1}{p^s})' = \sum_p \frac{1}{1-p^{-s}}(p^{-s})' \cdot (-1)$$

Since

$$(p^{-s})' = (e^{-s\log(p)})' = \log(p)e^{-s\log(p)}$$

we find

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{1}{1-p^{-s}} \cdot \log(p) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$$

where

$$\Lambda(n) := \begin{cases} \log(p) & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

von Mangoldt function

We have

$$\sum_{d|n} \Lambda(d) = \log(n).$$