Lecture 1

Organizational matters

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Course webpage:
`cims.nyu.edu/~tschinke/teaching/Spring24/number.html`

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Grading:

- Homework: 20%
- Midterm March 11: 30%
- Final May 6 50%

Syllabus

- Elementary number theory, residues, quadratic reciprocity

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- Diophantine equations

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- Basic algebraic number theory: number fields, rings of integers, class groups
- Transcendence of e and π

Integers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

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Leopold Kronecker (1823-1891)

Integers were created by God, the rest is human labor.

Substance of Number Theory

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Seems trivial: what is there to study?

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Trivially, multiplying two n-digit integers takes $O(n^2)$ steps. Best known bound is

$$O(n \log(n) \log(\log(n))).$$

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Why?

- Elegant, simply stated problems
- Deep, difficult proofs that stimulate the development of new areas of mathematics

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- Harmonic analysis over $\mathbb{R}$ (and $\mathbb{Q}_p$), special functions

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- Complex variables
- Approximation theory

Primes

Definition

A positive integer $p \geq 2$ is called prime if it is divisible only by itself and 1.

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Example

primes: $2, 3, 5, 7, 11, 13, \ldots$.

not primes: $1, 4, 6, 8, 9, 10, 12, 15, \ldots$.

Primes are atoms in the ring of integers, a free basis of the multiplicative semigroup.

Properties

Theorem

Every positive integer is a product of prime factors

$$n = p_1 \cdots p_r.$$

These prime factors are uniquely determined, up to permutation.

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Proof: Use Euclidean algorithm.

Euclid, 325 BC

Theorem

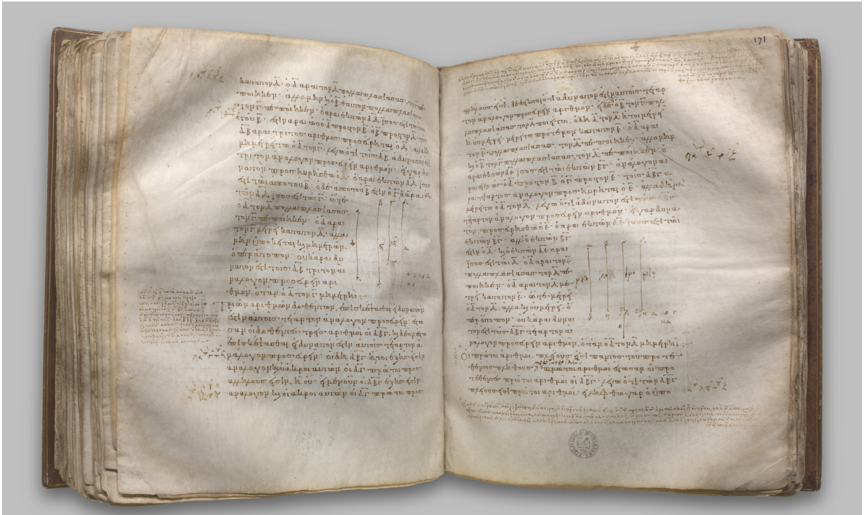
There are infinitely many primes.

Proof by contradiction. Assume that there are finitely many primes $p_1, \ldots, p_n$. None of these can divide the number

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1.$$

Thus there exists another prime. Contradiction to the assumption.

888 AD., Bodleian Library, Oxford



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- Congruences: $a \equiv b \pmod{m}$ iff $m \mid (a - b)$
- Solving congruences $ax \equiv b \pmod{m}$
- Euler function $\varphi(m) := \#(\mathbb{Z}/m\mathbb{Z})^\times$
  - $\varphi(p^n) = (p-1)p^{n-1}$
  - $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ if $(a, b) = 1$

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    - $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ if $(a, b) = 1$
    - $$\sum_{d|m} \varphi(d) = m$$

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$$(a, m) = 1 \Rightarrow a^{\varphi(m)} \equiv 1 \pmod{m}$$

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In particular, if p is a prime, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Carmichael numbers

The converse is not true! There exist composite numbers m such that for all a one has

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Theorem (Alford, Granville, Pomerance 1994)

There are infinitely many Carmichael numbers.

First results about primes

Theorem

Assume that $p > 2$ *satisfies* $p \mid a^2 + b^2$*. Then* $p \equiv 1 \pmod 4$*.*

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Assume that $p > 2$ *satisfies* $p \mid a^2 + b^2$. *Then* $p \equiv 1 \pmod 4$.

Proof:

- $(p, a) = (p, b) = 1$
- $$a^2 \equiv -1 \cdot b^2 \Rightarrow (a^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \cdot (b^2)^{\frac{p-1}{2}} \pmod p$$
- $$1 \equiv (-1)^{\frac{p-1}{2}} \pmod p$$
- Thus, $\frac{p-1}{2}$ is even, and $p \equiv 1 \pmod 4$. □

First results about primes

Corollary: There are infinitely many primes $\equiv 1 \pmod 4$.

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Proof: Assume that there are finitely many $p_1, \ldots, p_r$. Consider

$$N := 4\prod_{j=1}^{r} p_j^2 + 1$$

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Corollary: There are infinitely many primes $\equiv 1 \pmod 4$.

Proof: Assume that there are finitely many $p_1, \dots, p_r$. Consider

$$N := 4\prod_{j=1}^{r} p_j^2 + 1$$

Every prime dividing N is of the form $4m + 1$, so must be one of the listed primes, contradiction. □

Representations as sums of squares

Theorem

$$p \equiv 1 \pmod 4 \Rightarrow p = x^2 + y^2, \quad x, y \in \mathbb{Z}$$

Proof

(1) Given $r \not\equiv 0 \pmod{p}$ and e, f with $ef > p$ there exists a representation

$$r = \pm x/y, \quad 1 \leq y < f, \quad 1 \leq x < e.$$

Indeed, we have $e \cdot f$ numbers of the form $y \cdot r + x$, at least two have to be equal $\pmod{p}$

$$\begin{aligned} y'r + x' &\equiv y''r + x'' \pmod{p} \\ (y' - y'')r &\equiv x'' - x' \pmod{p} \\ r &\equiv \pm\frac{x'' - x'}{y'' - y'} \pmod{p} \end{aligned}$$

Proof

(2) Assume there exists an r with $r^2 \equiv -1 \pmod{p}$. We can represent it as

$$r \equiv \pm \frac{x}{y}, \quad 1 \le x, y \le [\sqrt{p}] + 1.$$

Then we have

$$x^2 + y^2 \equiv 0 \pmod{p} \quad \Rightarrow x^2 + y^2 = mp, \quad \text{with} \quad m = 1, 2$$

If $m = 1$ we are done, if $m = 2$, we can write

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = p.$$

Representations by polynomials

Theorem (Friedlander-Iwaniec 1998)

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Theorem (Heath-Brown 2001)

There are infinitely many primes of the form $x^3 + 2y^3$.

Integral part

- $[x]$ - integral part:
$$[x] \leq x < [x] + 1$$
- $\{x\}$ - fractional part, $\{x + 1\} = \{x\}$,
$$x = [x] + \{x\}$$
- $$[\frac{[x]}{m}] = [\frac{x}{m}] \quad \forall m \in \mathbb{N}, x \in \mathbb{R}.$$
(prove this at home)

Applications

For an integer $m > 1$ write

$$m! = p_1^{\nu_1} \cdots p_s^{\nu_s}.$$

Then

$$\nu_j = [\frac{m}{p_j}] + [\frac{m}{p_j^2}] + \cdots$$

Applications

Proof: List

$$1, 2, \ldots, m,$$

$$p, 2p, \ldots, k_1 p,$$

we have

$$k_1 p \leq m < (k_1 + 1)p \Rightarrow k_1 = [\frac{m}{p}].$$

Then

$$\nu = k_1 + \cdots$$

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List

$$1, 2, \ldots, k_1$$

and repeat the process: $k_2 = [\frac{k_1}{p}]$.

Applications

Inductively,

$$\nu_p(m!) = [\frac{m}{p}] + [\frac{k_1}{p}] + \cdots + [\frac{k_n}{p}], \quad \text{with} \quad [\frac{k_n}{p}] = [\frac{[\frac{k_{n-1}}{p}]}{p}].$$

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Now apply the lemma.

Applications

How to use this?

Example: Find the maximal ν such that

$$3^{\nu} \mid 1000!$$

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Solution:

$$\begin{aligned}\nu &= [\frac{1000}{3}] + [\frac{1000}{9}] + [\frac{1000}{27}] + [\frac{1000}{81}] + [\frac{1000}{243}] + [\frac{1000}{729}] \\ &= 333 + 111 + 37 + 12 + 4 + 1 \\ &= 498\end{aligned}$$

Tricks

Let $n \geq 9$ be an odd integer. Put

$$N_0 := n, \quad N_1 = N_0 + 1, \quad N_2 = N_1 + 3, \quad N_s := N_{s-1} + (2s - 1)$$

with

$$1 \leq s \leq [\frac{n-9}{6}].$$

Tricks

Theorem

n is composite iff for some s, $N_s = t^2$.

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Proof:

$$N_s = n + 1 + 3 + \cdots (2s - 1)$$
$$n + s^2$$

If $N_s = n + s^2 = t^2$ then

$$n = t^2 - s^2 = (t - s)(t + s).$$

Tricks

Since $s \leq (n-9)/6$ we have

$$n \geq 6s + 9 > 2s + 1, \quad t^2 = n^2 + s^2 > (s+1)^2,$$

$$t > (s+1) \Rightarrow (t-s) > 1,$$

i.e., n is composite.

Tricks

Conversely, suppose $n = a \cdot b$, with a, b odd. Then

$$3 \leq a \leq \sqrt{n}, \quad \sqrt{n} \leq b \leq n/3.$$

Put

$$s := \frac{b-a}{2} \Rightarrow N_s = n + s^2 = ab + \left(\frac{b-a}{2}\right)^2 = \left(\frac{b+a}{2}\right)^2$$

with

$$0 \leq \frac{b-a}{2} \leq \frac{1}{2}(\frac{n}{3} - 3) \Rightarrow \frac{b-a}{2} \leq [\frac{n-9}{6}].$$

Tricks

How to use this? Let's try to factor 391:

$$N_0 = 391, \quad N_1 = 392, \quad N_2 = 395, \quad N_3 = 400 = 20^2.$$

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$$N_0 = 391, \quad N_1 = 392, \quad N_2 = 395, \quad N_3 = 400 = 20^2.$$

Thus

$$s = 3, t = 20 \Rightarrow 391 = (20 - 3)(20 + 3) = 17 \cdot 23.$$

Residues modulo primes: reminder

- $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic, $\varphi(p-1)$ primitive roots (generators)

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-

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

Residues modulo primes: reminder

- Quadratic reciprocity

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) \cdot (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$$

Proof of quadratic reciprocity

Let p and q be odd primes. Put $\zeta := e^{\frac{2\pi i}{p}}$ and consider the Gauss sum:

$$\tau_p := \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \zeta^j.$$

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One has

$$\tau_p^2 = \sum_{j,k} \left(\frac{jk}{p}\right) \zeta^{j+k} = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \underbrace{\sum_{j=0}^{p-1} \zeta^{j(1+k)}}_{\begin{cases} 0 & k \neq -1 \\ p & k = -1 \end{cases}}$$

$$= \left(\frac{-1}{p}\right) p = (-1)^{\frac{p-1}{2}} p$$

Proof of quadratic reciprocity

$$\tau_p^q = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \cdot p^{\frac{q-1}{2}} \cdot \tau_p \equiv (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \left(\frac{p}{q}\right) \tau_p \pmod q$$

$$\equiv \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \zeta^{jq} \equiv \left(\frac{q}{p}\right) \tau_p \pmod q$$

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$$\equiv \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) \zeta^{jq} \equiv \left(\frac{q}{p}\right) \tau_p \pmod{q}$$

It follows that

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \left(\frac{p}{q}\right).$$

Logarithmus: John Napier(1550-1617)

Mirifici Logarithmorum Canonis Descriptio (1614)

$$\log(a \cdot b) = \log(a) + \log(b)$$

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Napier: Logarithms are artificial numbers and antilogarithms - natural numbers.

Tables for $\log(n)$

- 1617 H. Briggs: $n \leq 1000$
- 1628 A. Vlacq: $n \leq 100000$
- 1795 G. Vega: logarithms of primes ≤ 10000

JOHANN CARL SCHULZE

wirklichen Mitgliedes der Königl. Preussischen Academie der Wissenschaften

NEUE UND ERWEITERTE

SAMMLUNG

LOGARITHMISCHER,

TRIGONOMETRISCHER

und anderer

ZUM GEBRAUCH DER MATHEMATIK

UNENTBEHRLICHER

TAFELN.

I. BAND.

BERLIN, 1778.

BEY AUGUST MYLIUS, BUCHHÄNDLER

IN DER BRÜDERSTRASSE.

Abb. 10
Gauß' erste Logarithmentafel, ein Geschenk des braunschweigischen Herzogs Carl Wilhelm Ferdinand an den Vierzehnjährigen (D 2)

G. Vega: *Thesaurus logarithmorum completus*

Bestseller: more than 100 editions.

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Gauss (1851)

Vega scheint sich nun mit der Hoffnung geschmeichelt zu haben, dass seine Tafeln fast fehlerfrei geworden seien, und verspricht, für die erste Anzeige eines Fehlers eine Prämie von einem Dukaten zu bezahlen. ...

Unter 568 geprüften irrationalen Logarithmen haben sich 301 als richtig und 267 als unrichtig ausgewiesen. Dürfte man dies Verhältnis als durchschnittlich zutreffend betrachten, so würden unter den 68038 irrationalen Logarithmen des Vega'schen Thesaurus ... nach der Wahrscheinlichkeit etwa 31983 fehlerhafte anzunehmen sein.

Distribution of primes

Definition

$\pi(x) :=$ number of primes less than or equal to x.

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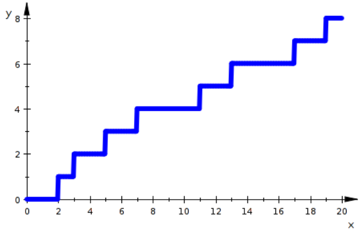
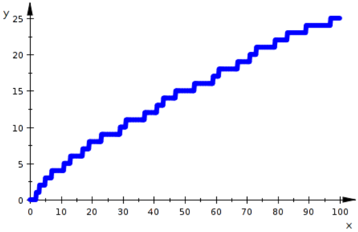
The pi-function for small values

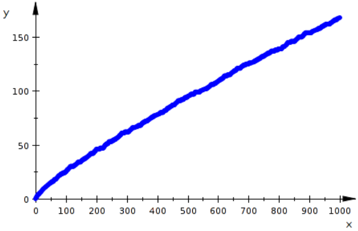


x	1	2	3	4	5	6	7	8	9	10	20
$\pi(x)$	0	1	2	2	3	3	4	4	4	4	8

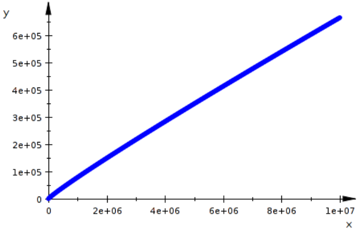
The pi-function for $x \leq 100$



The pi-function for $x \leq 1000$



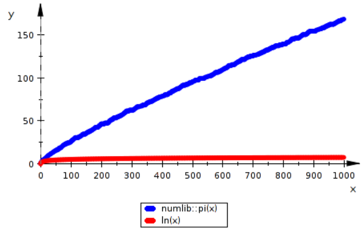
The pi-function for $x \leq 10$ million



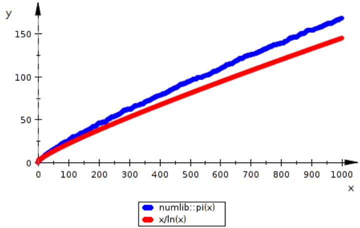
Problem

Goal: Find a simple formula for the pi-function.

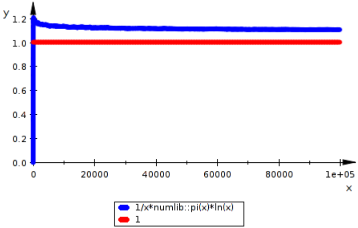
pi-function and the logarithm



pi-function and $x/\log(x)$

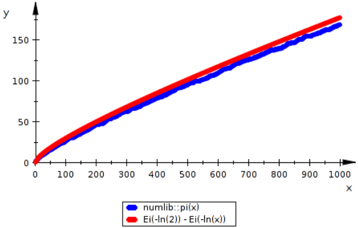


pi-function divided by $x/\log(x)$

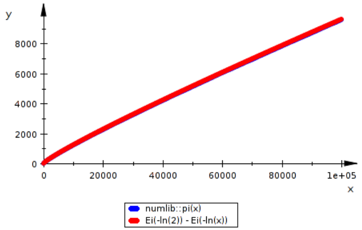


pi-function and integral-log for $x \leq 1000$

$$\text{Intlog}(x) := \int_2^x \frac{1}{\log(t)} \text{dt}$$



pi-function and integral-log for $x \leq 10000$



Hochzuverehrender Freund.

Vor allem statte ich Ihnen für die gewogentliche Übersendung des Jahrbuchs von 1852 meinen verbindlichsten Dank ab.

Die gütige Mittheilung Ihrer Bemerkungen über die Frequenz der Primzahlen ist mir in mehr als einer Beziehung interessant gewesen. Sie haben mir meine eignen Beschäftigungen mit demselben Gegenstande in Erinnerung gebracht, deren erste Anfänge in eine sehr entfernte Zeit fallen, ins Jahr 1792 oder 1793, wo ich mir die Lambertschen Supplemente zu den Logarithmentafeln angeschafft hatte. Es war noch ehe ich mit feinern Untersuchungen aus der höhern Arithmetik mich befasst hatte eines meiner ersten Geschäfte, meine Aufmerksamkeit auf die abnehmende Frequenz der Primzahlen zu richten, zu welchem Zweck ich dieselben in den einzelnen Chiliaden abzählte, und die Resultate auf einem der angehefteten weissen Blätter verzeichnete. Ich erkannte bald, dass unter allen Schwankungen diese Frequenz durchschnittlich nahe dem Logarithmen verkehrt proportional sei, so dass die Anzahl aller Primzahlen unter einer gegebenen Grenze n nahe durch das Integral

$$\int \frac{dn}{\log n}$$

ausgedrückt werde, wenn der hyperbolische Logarithm verstanden werde. In späterer Zeit, als mir die in Vegas Tafeln (von 1796) ~~bat~~ abgedruckte Liste bis 400031 bekannt wurde, dehnte ich meine Abzählung weiter aus, ~~die~~ was jenes Verhältniss bestätigte. Eine grosse Freude machte mir 1811 die Erscheinung von Chernacs cribrum, und ich habe (da ich zu einer anhaltenden Abzählung der Reihe nach keine Geduld hatte) sehr oft einzelne unbeschäftigte Viertelstunden verwandt, um bald hie bald dort eine Chiliade abzuzählen; ich liess jedoch zuletzt es ganz liegen, ohne mit der Million ganz fertig zu werden. Erst später benutzte ich Goldschmidts Arbeitsamkeit, theils die noch gebliebenen Lücken in der ersten Million auszufüllen, theils nach Burckhardts Tafeln die Abzählung weiter fortzusetzen. So sind (nun schon seit vielen Jahren) die drei ersten Millionen abgezählt, und mit dem Integralwerthe verglichen. Ich setze hier nur einen kleinen Extract her:

Unter	gibt es Primzahlen	Integral $\int \frac{dn}{\log n}$	Differ.	Ihre Formel	Abweich.
500 000	41 556	41606,4	+50,4	41596,9	+40,9
1000 000	78 501	78627,5	+126,5	78672,7	+171,7
1500 000	114 112	114263,1	+151,1	114374,0	+264,0
2000 000	148883	149054,8	+171,8	149233,0	+350,0
2500 000	183016	183245,0	+229,0	183495,1	+479,1
3000 000	216745	216970,6	+225,6	217308,5	+563,6

Dass Legendre sich auch mit diesem Gegenstande beschäftigt hat, war mir nicht bekannt; auf Veranlassung Ihres Briefes habe ich in seiner Théorie des Nombres nachgesehen, und in der zweiten Ausgabe einige darauf bezügliche Seiten gefunden, die ich früher übersehen (oder seitdem vergessen) haben muss. Legendre gebraucht die Formel

$$\frac{n}{\log n - A}$$

wo A eine Constante sein soll, für welche er 1,08366 setzt. Nach einer flüchtigen Rechnung finde ich danach in obigen Fällen die Abweichungen

−23,3
+42,2
+68,1
+92,8
+159,1
+167,6

Diese Differenzen sind noch kleiner als die ~~nach~~ mit dem Integral, sie scheinen aber bei zunehmendem n ~~stärker~~ schneller zu wachsen als diese, so dass leicht möglich wäre, dass bei viel weiterer Fortsetzung jene die letztern überträfen. Um Zählung und Formel in Übereinstimmung zu bringen müsste man respective anstatt $A = 1{,}08366$ setzen

1,09040
1,07682
1,07582
1,07529
1,07179
1,07297

Es scheint, dass bei wachsendem n der (Durchschnitts-) Werth von A abnimmt; ob aber die Grenze beim Wachsen des n ins Unendliche 1 oder eine von 1 verschiedene Grösse sein wird, darüber wage ich keine Vermuthung. Ich kann nicht sagen, dass eine Befugniss da ist, einen ganz einfachen Grenzwerth zu erwarten; von der andern Seite ~~scheint~~ könnte der Überschuss des A über 1 ganz füglich ein Grösse von der Ordnung $\frac{1}{\log n}$ sein. Ich würde geneigt sein zu glauben, dass das Differential der betreffenden Function einfacher sein muss, als die Function selbst; Indem ich früher $\frac{dn}{\log n}$ vorausgesetzt habe, würde Legendre's Formel eine Differentialfunction voraussetzen, die etwa $\frac{dn}{\log n - (A-1)}$ wäre. Ihre Formel übrigens würde für ein sehr grosses n als mit

$$\frac{n}{\log n - \frac{1}{2k}} =$$

übereinstimmend betrachtet werden können, wo k der Modulus der Briggischen Logarithmen ist, also mit Legendres Formel, wenn man

$$A = \frac{1}{2k} = 1{,}1513 \text{ setzt.}$$

Endlich will ich noch bemerken, dass ich zwischen Ihren Abzählungen und den meinigen ein Paar Differenzen bemerkt habe.

Zwischen 59000	u. 60000	haben Sie	95	ich 94
101000	102000		94	93

Die erste Differenz hat vielleicht ihren Grund darin, dass in Lamberts Suppl. die Primzahl 59023 zweimahl aufgeführt ist. Die Chiliade von 101000 — 102000 wimmelt in Lamberts Supplementen von Fehlern; ich habe in meinem Exemplare 7 Zahlen ausgestrichen, die keine Primzahlen sind, u. dagegen 2 fehlende eingeschaltet. Könnten Sie nicht den jungen Dase veranlassen, dass er die Primzahlen in den folgenden Millionen aus denjenigen bei der Akademie befindlichen Tafeln abzählte, die wie ich fürchte das Publicum nicht besitzen soll? Für diesen Fall bemerke ich, dass in der 2. u. 3 Million die Abzählung auf meine Vorschrift nach einem besondern Schema gemacht ist, welches ich selbst auch schon bei einem Theile der ersten Million angewandt hatte. Die Abzählungen von je 100000 stehen auf einer (klein) Octavseite in 10 Columnen, jede sich auf Eine Myriade beziehend; dazu kommt noch eine Columne davor (links) und eine dahinter (rechts); ~~jene zeigten~~ als Beispiel hier eine Verticalcolumne u. die beiden Zusatzcolumnen aus dem Intervall 1000000 — 1100000

	0	1	2	......	9	
1	1					1
2						4
3						21
4	2					54
5	11					114
6	14					171
7	26					217
8	19					164
9	11					126
10	8					71
11	6					39
12	1					12
13	1					6
14						
15						
16						
	752					7210

Zur Erläuterung diene z. B. die 1te ~~horizontal~~ Verticalreihe. In der Myriade 1000000 bis 1010000 sind 100 Hecatontaden; darunter ist 1 die nur eine Primzahl enthält; gar keine mit 2 oder 3; 2 Stück mit je 4 Primzahlen; 11 Stück mit je 5 u.s.w. alle zusammen geben 752. = 1.1 + 4.2 + 5.11 + 6.14 + u.s.w. Die letzte Columne enthält die Aggregate aus den 10 einzelnen. Die Zahlen 14.15.16 in der ersten Verticalreihe stehen hier nur zum Überfluss, da keine Hecatontaden mit so vielen Primzahlen vorkommen; ~~sie~~ aber auf den folgenden Blättern bekommen sie Geltung. Zuletzt werden wieder die 10 Seiten in 1 vereinigt, u. umfassen so die ganze 2te Million.

Doch es ist Zeit abzubrechen. Ich sage noch meinen herzlichen Dank für Ihre Mittheilungen über die Färbungen der dortigen öffentl. Zustände. Noch sieht man keinen Ausweg aus dem Labyrinth, in das uns die Nachäfferei der Franzosen gezerrt hat. Unter herzlichen Wünschen für Ihr Wohlbefinden

Stets der Ihrige
C. F. Gauss

Göttingen 24 ~~Decbr~~ December 1849

Letter to Encke / 1849

My distinguished friend,

Your remarks concerning the frequency of primes were of interest to me in more ways than one. You have reminded me of my own endeavors in this field which began in the very distant past, in 1792 or 1793, after I had acquired the Lambert supplements to the logarithmic tables. Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end I counted the primes in several chiliads and recorded the results on the attached white pages. I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm, so that the number of primes below a given bound n is approximately equal to

$$\int \frac{dn}{\log(n)},$$

where the logarithm is understood to be hyperbolic.

Letter to Encke / 1849

Later on, when I became acquainted with the list in Vega's tables (1796) going up to 400031, I extended my computation further, confirming that estimate. In 1811, the appearance of Chernau's cribrum gave me much pleasure and I have frequently (since I lack the patience for a continuous count) spent an idle quarter of an hour to count another chiliad here and there; although I eventually gave it up without quite getting through a million. Only some time later did I make use of the diligence of Goldschmidt to fill some of the remaining gaps in the first million and to continue the computation according to Burkhardt's tables. Thus (for many years now) the first three million have been counted and checked against the integral.

Chiliades.

1	168	51	89	101	81	151	85	201	77	251	71	301	85	351	74	401	70	451	92
2	135	52	97	102	93	152	90	202	87	252	88	302	83	352	80	402	71	452	76
3	127	53	89	103	87	153	88	203	78	253	78	303	72	353	82	403	76	453	63
4	120	54	92	104	80	154	77	204	78	254	81	304	84	354	76	404	75	454	72
5	119	55	90	105	91	155	84	205	77	255	76	305	88	355	87	405	70	455	74
6	114	56	93	106	82	156	85	206	85	256	87	306	80	356	79	406	83	456	82
7	117	57	99	107	92	157	76	207	83	257	72	307	82	357	67	407	67	457	73
8	107	58	91	108	76	158	88	208	87	258	78	308	73	358	80	408	81	458	77
9	110	59	90	109	91	159	87	209	85	259	86	309	76	359	83	409	79	459	75
10	112	60	94	110	88	160	85	210	88	260	76	310	80	360	71	410	82	460	68
11	106	61	88	111	83	161	85	211	84	261	77	311	79	361	68	411	73	461	77
12	103	62	87	112	84	162	84	212	86	262	73	312	69	362	79	412	81	462	69
13	109	63	88	113	81	163	81	213	69	263	79	313	86	363	76	413	74	463	74
14	105	64	93	114	88	164	83	214	81	264	84	314	86	364	84	414	69	464	77
15	102	65	80	115	82	165	77	215	86	265	80	315	76	365	77	415	90	465	85
16	108	66	98	116	93	166	80	216	74	266	78	316	77	366	77	416	80	466	74
17	98	67	84	117	81	167	81	217	76	267	87	317	84	367	85	417	67	467	69
18	104	68	99	118	90	168	83	218	80	268	94	318	84	368	79	418	82	468	83
19	94	69	80	119	79	169	73	219	84	269	75	319	81	369	72	419	85	469	85
20	102	70	81	120	87	170	87	220	91	270	78	320	86	370	68	420	75	470	72
21	98	71	98	121	88	171	87	221	78	271	84	321	79	371	70	421	75	471	87
22	104	72	95	122	86	172	81	222	80	272	78	322	80	372	76	422	73	472	78
23	100	73	90	123	88	173	89	223	81	273	83	323	81	373	81	423	77	473	73
24	104	74	83	124	88	174	79	224	80	274	71	324	71	374	73	424	83	474	78
25	94	75	92	125	83	175	83	225	83	275	80	325	87	375	82	425	81	475	80
26	98	76	91	126	84	176	75	226	84	276	83	326	85	376	85	426	74	476	86
27	101	77	83	127	83	177	95	227	76	277	83	327	73	377	80	427	71	477	75
28	94	78	95	128	86	178	73	228	80	278	74	328	86	378	71	428	78	478	69
29	98	79	84	129	89	179	89	229	89	279	81	329	73	379	77	429	71	479	85
30	92	80	91	130	83	180	94	230	88	280	73	330	81	380	83	430	89	480	71
31	95	81	88	131	85	181	71	231	84	281	87	331	80	381	72	431	76	481	77
32	92	82	92	132	83	182	79	232	78	282	85	332	82	382	76	432	79	482	78
33	106	83	89	133	87	183	91	233	76	283	77	333	72	383	74	433	84	483	82
34	100	84	84	134	82	184	79	234	71	284	72	334	80	384	81	434	80	484	75
35	94	85	87	135	80	185	83	235	87	285	90	335	77	385	78	435	85	485	65
36	92	86	85	136	89	186	91	236	73	286	77	336	77	386	80	436	82	486	63
37	99	87	88	137	96	187	79	237	76	287	71	337	84	387	78	437	73	487	82
38	94	88	93	138	80	188	87	238	73	288	71	338	80	388	69	438	70	488	78
39	90	89	76	139	85	189	80	239	87	289	85	339	77	389	75	439	75	489	83
40	96	90	94	140	84	190	88	240	79	290	84	340	68	390	84	440	75	490	78
41	88	91	89	141	87	191	75	241	80	291	84	341	84	391	81	441	79	491	78
42	101	92	85	142	87	192	89	242	91	292	77	342	77	392	79	442	72	492	76
43	102	93	97	143	82	193	89	243	76	293	78	343	77	393	86	443	85	493	67
44	85	94	86	144	77	194	84	244	77	294	68	344	80	394	87	444	88	494	82
45	96	95	87	145	79	195	74	245	78	295	85	345	80	395	75	445	82	495	80
46	86	96	95	146	85	196	85	246	80	296	75	346	76	396	72	446	68	496	87
47	90	97	84	147	84	197	76	247	84	297	82	347	80	397	75	447	68	497	68
48	95	98	82	148	83	198	87	248	79	298	73	348	82	398	75	448	73	498	81
49	89	99	87	149	83	199	96	249	88	299	73	349	77	397	82	449	70	499	72
50	98	100	87	150	91	200	77	250	80	300	78	350	82	400	81	450	80	500	81

In ~~chiliade~~ centade 268300 – 268400 unicus primus
155900 – 156000 item
414300 – 414400 item

501	78	551	79	601	75	651	61	701	75	751	68	801	85	851	70	901	74	951	76
502	74	552	75	602	73	652	74	702	71	752	85	802	66	852	77	902	73	952	70
503	67	553	71	603	83	653	85	703	81	753	73	803	70	853	74	903	70	953	78
504	76	554	80	604	76	654	69	704	71	754	71	804	69	854	66	904	63	954	65
505	76	555	77	605	73	655	78	705	87	755	83	805	78	855	71	905	81	955	73
506	83	556	61	606	74	656	73	706	68	756	70	806	79	856	73	906	70	956	76
507	76	557	88	607	72	657	71	707	82	757	66	807	68	857	78	907	80	957	58
508	71	558	68	608	78	658	70	708	74	758	68	808	70	858	76	908	80	958	69
509	76	559	74	609	78	659	79	709	77	759	79	809	69	859	69	909	79	959	77
510	75	560	77	610	80	660	73	710	77	760	77	810	78	860	71	910	82	960	69
511	72	561	86	611	73	661	83	711	78	761	77	811	78	861	77	911	62	961	68
512	82	562	61	612	71	662	70	712	76	762	80	812	72	862	74	912	81	962	88
513	70	563	83	613	76	663	74	713	72	763	68	813	69	863	83	913	71	963	71
514	77	564	67	614	79	664	77	714	73	764	79	814	72	864	60	914	54	964	74
515	81	565	77	615	71	665	77	715	66	765	72	815	78	865	80	915	73	965	74
516	66	566	78	616	75	666	77	716	83	766	82	816	69	866	80	916	70	966	70
517	85	567	72	617	85	667	73	717	69	767	78	817	75	867	68	917	72	967	73
518	83	568	72	618	81	668	73	718	65	768	68	818	75	868	78	918	79	968	66
519	76	569	71	619	67	669	66	719	67	769	77	819	62	869	80	919	75	969	75
520	78	570	80	620	73	670	74	720	74	770	74	820	83	870	73	920	71	970	73
521	73	571	85	621	77	671	75	721	78	771	77	821	75	871	79	921	72	971	76
522	83	572	72	622	70	672	76	722	77	772	75	822	72	872	58	922	72	972	78
523	79	573	85	623	74	673	76	723	73	773	70	823	84	873	76	923	72	973	74
524	69	574	72	624	75	674	77	724	86	774	76	824	78	874	65	924	81	974	63
525	77	575	70	625	68	675	69	725	75	775	72	825	71	875	75	925	76	975	85
526	79	576	77	626	69	676	75	726	69	776	67	826	81	876	80	926	80	976	70
527	84	577	78	627	70	677	74	727	76	777	70	827	78	877	75	927	74	977	66
528	72	578	77	628	70	678	63	728	75	778	76	828	69	878	67	928	63	978	60
529	70	579	76	629	71	679	82	729	76	779	81	829	69	879	68	929	70	979	80
530	78	580	77	630	75	680	83	730	75	780	71	830	68	880	75	930	80	980	65
531	80	581	73	631	67	681	75	731	69	781	70	831	76	881	80	931	69	981	67
532	68	582	79	632	81	682	78	732	76	782	82	832	79	882	69	932	69	982	75
533	79	583	73	633	77	683	66	733	71	783	68	833	82	883	72	933	76	983	70
534	74	584	78	634	70	684	78	734	75	784	74	834	68	884	73	934	68	984	70
535	72	585	72	635	82	685	72	735	74	785	75	835	67	885	69	935	81	985	74
536	71	586	81	636	78	686	74	736	79	786	77	836	73	886	77	936	68	986	76
537	87	587	79	637	73	687	74	737	69	787	70	837	71	887	76	937	71	987	76
538	67	588	87	638	74	688	82	738	78	788	73	838	64	888	71	938	71	988	63
539	78	589	73	639	59	689	74	739	70	789	80	839	80	889	77	939	72	989	72
540	71	590	68	640	72	690	79	740	81	790	68	840	69	890	68	940	68	990	72
541	73	591	71	641	77	691	60	741	67	791	78	841	70	891	68	941	74	991	71
542	77	592	67	642	71	692	79	742	74	792	71	842	69	892	80	942	79	992	79
543	78	593	80	643	68	693	77	743	73	793	82	843	83	893	69	943	72	993	68
544	81	594	77	644	70	694	73	744	67	794	71	844	68	894	72	944	76	994	68
545	68	595	78	645	86	695	76	745	64	795	73	845	78	895	74	945	73	995	78
546	68	596	77	646	75	696	77	746	67	796	79	846	70	896	80	946	66	996	69
547	73	597	79	647	74	697	73	747	76	797	77	847	69	897	64	947	72	997	69
548	76	598	73	648	79	698	79	748	71	798	72	848	77	898	75	948	66	998	83
549	77	599	72	649	73	699	62	749	76	799	71	849	75	899	76	949	67	999	74
550	78	600	73	650	84	700	72	750	72	800	81	850	68	900	61	950	75	1000	65

Primzahlen

von 1000000 bis 1100000.

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	
1.	1.										1.
2.		1.				1.		1.	1.		4.
3.		4.	2.	2.	3.	1.	2.	3.	3.	1.	21.
4.	2.	8.	5.	4.	3.	6	9.	4.	5.	8.	54.
5	11.	10.	8.	18.	12.	10.	10.	12.	15.	8	114
6	14.	14.	18.	21.	16.	22.	19.	15.	17.	15.	171.
7.	26	17.	23.	23.	24.	24.	17.	22.	20.	21.	217.
8.	19.	19.	21.	7.	14.	15.	20.	17.	15.	17.	164.
9.	11.	13.	9.	13.	14.	14.	12.	13.	11.	16.	126.
10.	8.	6.	8.	5.	9.	5.	5.	9.	7.	9.	71.
11.	6.	6.	4.	6.	3.	1.	3.	1.	4.	5.	39.
12.	1.	1.	2.	1.	1.	1.	2.	2.	1.		12.
13.	1.	1.			1.		1.	1.	1.		6.
14											
15.											
16.											
	752	719	732.	700.	731.	698.	713.	722.	706.	737.	7210.

$\int \frac{dx}{lx} = 7212{,}99$

I

Primzahlen

von 1100000 bis 1200000.

	0.	1.	2.	3.	4.	5	6.	7.	8.	9.		
0.											0	
1.					1.						1.	
2.			1.				1.	1.		2.	5	
3.	4.	3.	3	3.	3.	3.	3.	2.		1.	25.	
4.	5.	6.	7.	5.	9.	4.	4.	5.	6.	6.	57	
5.	8.	13.	10.	12.	11.	11.	9.	12.	12.	9.	107.	
6.	14.	20.	17.	20.	17.	17	18	16	17	14.	170.	
7.	21.	19.	22.	19.	21.	21.	20.	18	20.	27	217	
8.	22.	13.	10.	12.	18	20.	17	19.	9.	20	160	
9.	13.	14	16	17.	7.	11.	16	14	12	11.	131	
10.	9.	6.	10.	10.	8.	9.	6.	8.	3.	8.	77	
11.	1.	4.	2.	2.	2.	1.	4.	5.	9.	2.	32.	
12	1.	1.	1.		2.	3.	1.		2.		11.	
13	1.	1.	1.				1.		1.		5.	
14.	1.				1.						2.	
	736.	710.	716.	713.	697.	725.	729.	723.	725.	710.		7194.

$\int \frac{\partial x}{\log x}$ .... 7165.9!!

438.

Math 18 ④ ~~44a/4~~

Primzahlen

von 1200000 bis 1300000.

	0.	1.	2.	3.	4.	5.	6.	7.	8	9.		
0.												
1								1	1.		2.	
2.	2.		2.	1.						1.	6.	
3.	3.	2.	4.	5.	4.	3.	1.	4	3.	3.	32.	
4.	~~9.~~ 7	7.	7.	3.	~~6~~ 5	~~8.~~ 7	12.	2.	3.	10	~~67.~~ 65	63
5.	~~14.~~ 15	~~13.~~ 12	12.	15	~~9.~~ 10	~~13.~~ 14	9.	15.	6	12	~~118~~ 119	120
6.	~~17.~~ 16	14	~~14.~~ 13	19.	~~19.~~ 17	~~17.~~ 16	16	15.	~~21.~~ 20	14.	166. 165	160
7.	~~23.~~ 24	15.	~~24.~~ 25	24.	~~19.~~ 21	~~19.~~ 20	~~16.~~ 15	22.	~~23.~~ 24	24	~~209~~ 210	214
8	17.	~~18.~~ 19	16.	11.	~~18~~ 17	15.	22.	18.	~~20.~~ 19	14	~~169.~~	168
9	8.	~~13.~~ 12	~~8.~~ 7	10.	~~11.~~ 12	13.	~~12.~~ 14	13.	~~12.~~ 13	9	110.	111
10.	~~4.~~ 3	~~13.~~ 11	~~9.~~ 10	8	10.	4.	5.	~~4.~~ 3	9.	10.	~~76.~~ 75	73
11.	~~2.~~ 3	~~2.~~ 6	3.	2.	3.	5.	3.	~~5.~~ 6	1	3.	~~36.~~ 33	35
12.	1.	1.	1.			1.	3.	1.	1.		9.	
13	1	1.		1.		2.					~~4.~~ 5	
14				1.							1	
16.					1.						1.	
	~~664~~ 676	~~737~~ 744	~~694~~ 693	693	~~720~~ 724	~~714~~ 713	~~716~~ 718	~~708~~ 709	~~720~~ 722	689	~~7049.~~	
	?									r	7081	

$\int \frac{dx}{\log x} = 7123,35.$

1200 000 — 1300 000

	0	1	2	3	4	5	6	7	8	9	
0											
1								1	1		2
2	2		2	1						1	6
3	3	2	4	5	4	3	1	4	3	3	32
4	7	7	7	3	5	7	12	2	3	10	63
5	15	12	12	15	10	14	9	15	6	12	120
6	16	14	13	19	17	16	16	15	20	14	160
7	24	15	25	24	21	20	15	22	24	24	214
8	17	19	16	11	17	15	22	18	19	14	168
9	8	12	7	10	12	13	14	13	13	9	111
10	3	11	10	8	10	4	5	3	9	10	73
11	3	6	3	2	3	5	3	6	1	3	35
12	1	1	1			1	3	1	1		9
13	1	1		1		2					5
14				1							1
16					1						1
	676	744	693	693	724	713	718	709	722	689	7081

$$\int \frac{dx}{\log x} = 7123.35$$

1300 000 — 1400 000

	0	1	2	3	4	5	6	7	8	9	
1						1					1
2		1	1		1	1	1	2	1	1	9
3	3	1	1	3	2	2	2		5		19
4	3	10	7	11	6	8	7	6	6	5	69
5	17	13	11	11	15	12	8	8	14	10	119
6	15	14	17	14	20	18	23	17	16	19	173
7	22	18	26	18	21	16	16	28	19	23	207
8	14	22	14	16	14	13	21	17	15	15	161
9	17	11	12	14	12	9	12	7	13	13	120
10	5	6	2	11	5	15	5	8	6	7	70
11	4	2	7	1	1	3	3	6	3	3	33
12		1	2	1	2	2	2	1	1	3	15
13		1		~~7~~	1				1		3
14										1	1
	709	702	713	705	692	713	709	723	695	737	7098

$$\int \frac{dx}{\log x} = 7084.48$$

Primzahlen von 1300000 bis 1400000

	0.	1.	2.	3.	4.	5.	6	7.	8.	9.		
1.						1.					1.	
2.		1.	1.		1.	1.	1.	2.	1.	1.	9	
3.	3.	1.	1.	3.	2.	2.	2.		5		19	
4	3	10.	7.	11.	6.	8.	7.	6.	6.	5.	69	
5	11.	[illegible]	11.	11.	15	12.	8.	8.	14.	10.	110	
6	15.	14.	18. [17]	14	20.	18.	23.	17.	16.	[illegible]	174.	−1
7	[illegible]	18.	25.	18.	21.	16	16.	[23] 24.	19.	[illegible]	207.	+1 −1
8	14.	[illegible]	14.	16.	11.	15.	21	16.	15.	15	16[illegible].	+1
9.	17.	11.	19.	14.	17.	[9] 11.	1[illegible].	7.	13.	[illegible]	12[illegible].	−2
10.	5.	4	2.	11.	3.	12. [15]	3.	8.	6.	7.	68	+2
11.	4.	2	7.	1.	1.	3.	2.	6.	3.	3	33.	
12.		1	2.	1.	2.	2.	[illegible]	1.	1.	[illegible]	15.	
13.		1.			1.				1.		3.	
14.										1.	1	
	709	70[illegible]	71[illegible]	705	692	714	7[illegible]	72[illegible]	695	707	7094 (5)	
			713			713		723				

$\int \frac{dx}{lx} = 7084,48$

Math 18 (7) ~~44a~~ 7

1400000 bis 1500000

	141	142	143	144	145	146	147	148	149	150	
0	–	–	–	–	–	–	–	–	–	–	0
1	–	–	1	–	1	1	1	–	–	1	5
2	–	1	1	–	2	–	–	–	2	1	7
3	3	3	0	2	1	2	2	4	2	0	19
4	8	8	8	4	6	7	9	9	5	8	72
5	17	9	7	14	13	11	14	15	16	13	129
6	21	23	20	18	20	19	11	16	16	19	183
7	17	23	18	18	13	24	18	11	15	22	179
8	12	16	28	17	24	14	17	18	23	14	183
9	12	11	4	15	6	10	13	12	7	8	98
10	7	2	7	7	9	7	8	9	8	9	73
11	2	2	2	4	2	3	6	5	4	4	34
12	1	2	3	1	2	2	1	1	2	1	16
13	–	–	1	–	1	–	–	–	–	–	2
	679	680	717	723	703	701	716	705	706	698	7028

$$\int_{1400000}^{1500000} \frac{dx}{\log x} = 7048.78186$$

1500000 bis 1600000

	151	152	153	154	155	156	157	158	159	160	
0	–	–	–	–	–	–	–	–	–	–	0
1	–	–	–	–	–	1	1	–	–	–	2
2	–	–	1	1	2	–	2	–	1	3	10
3	2	–	4	2	2	3	5	3	6	1	28
4	8	5	5	7	9	13	6	10	7	7	77
5	8	19	9	13	11	9	12	15	11	17	124
6	16	20	25	21	20	12	26	14	23	22	199
7	19	21	18	19	18	15	12	19	11	20	172
8	19	12	15	18	15	17	10	17	15	11	149
9	16	14	16	12	8	17	15	6	11	9	124
10	8	3	5	4	9	7	5	10	10	2	63
11	–	5	2	2	3	3	3	2	4	5	29
12	4	–	–	1	3	1	1	4	1	2	17
13	–	1	–	–	–	2	2	–	–	1	6
	731	702	691	686	698	714	680	701	693	675	6971

$$\int_{1500000}^{1600000} \frac{dx}{\log x} = 7015,78776$$

1600000 bis 1700000

	161	162	163	164	165	166	167	168	169	170	
0	–	–	–	–	–	–	–	1	–	–	1
1	–	–	–	–	–	2	–	–	–	–	2
2	1	3	1	–	1	1	2	–	–	2	11
3	3	3	3	4	4	2	2	4	4	–	29
4	7	4	9	7	7	10	4	4	10	6	68
5	10	11	8	12	11	11	13	12	18	14	120
6	18	22	15	19	15	11	14	19	10	16	159
7	22	15	14	21	25	18	22	24	24	18	203
8	14	23	25	15	17	16	17	15	13	19	174
9	8	12	18	12	12	21	12	8	14	13	130
10	7	2	5	8	4	6	11	7	4	9	63
11	7	3	1	1	3	1	3	2	2	3	26
12	2	1	1	1	–	–	–	4	0	–	9
13	1	0	–	–	1	–	–	–	1	–	3
14	–	1	–	–	–	–	–	–	–	–	1
15	–	–	–	–	–	1	–	–	–	–	1
	719	694	710	692	692	700	716	702	675	712	7012

$$\int_{1600000}^{1700000} \frac{dx}{\log x} = 6985,13714$$

1700000 bis 1800000

	171	172	173	174	175	176	177	178	179	180	
0	–	–	–	–	–	–	–	–	–	–	0
1	–	–	–	1	–	–	–	–	–	–	1
2	1	–	–	–	–	2	1	–	–	1	5
3	3	4	–	3	3	4	3	5	3	2	30
4	7	9	6	6	5	8	6	6	10	7	70
5	13	15	19	16	12	15	21	13	13	15	152
6	17	16	22	22	20	14	15	13	18	17	174
7	23	21	22	15	22	19	19	21	17	15	194
8	11	16	11	15	16	15	13	18	19	13	147
9	18	11	8	11	15	10	12	14	10	15	124
10	3	1	8	7	2	9	6	9	5	11	61
11	1	3	3	1	3	4	4	1	4	2	26
12	2	3	–	1	1	–	–	–	1	2	10
13	1	1	1	2	–	–	–	–	–	–	5
14	–	–	–	–	–	–	–	–	–	–	–
15	–	–	–	–	1	–	–	–	–	–	1
	695	685	691	689	706	684	679	700	689	713	6931

$$\int_{1700000}^{1800000} \frac{dx}{\log x} = 6956,53562$$

Math 18
1800000 bis 1900000 ⑨ ~~44a/~~ 9

	181	182	183	184	185	186	187	188	189	190	
0	–	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–	–	–
2	1	1	2	1	–	2	–	–	–	3	10
3	3	2	1	5	1	1	1	2	3	3	22
4	6	5	10	12	7	6	5	5	8	7	71
5	14	15	10	11	11	12	19	17	12	14	135
6	13	20	14	15	21	19	16	19	23	15	175
7	25	26	18	17	21	21	20	22	19	17	206
8	15	19	13	18	22	15	19	15	11	14	161
9	10	7	19	13	9	8	10	12	10	15	113
10	9	4	8	6	6	13	4	6	8	10	74
11	2	–	4	–	–	3	5	2	5	2	23
12	2	1	1	2	2	–	1	–	1	–	10
13	–	–	–	–	–	–	–	–	–	–	–
14	–	–	–	–	–	–	–	–	–	–	–
	704	672	718	674	700	707	703	689	697	691	6955

$$\int_{1800000}^{1900000} \frac{dx}{\log x} = 6929{,}73917$$

1900000 bis 2000000

	191	192	193	194	195	196	197	198	199	200	
0	–	–	–	–	–	–	–	–	–	–	–
1	1	–	–	–	–	–	–	–	–	–	1
2	–	–	1	–	–	1	2	–	1	–	5
3	4	3	1	2	10	1	3	4	4	2	34
4	5	4	4	6	4	9	7	10	11	7	67
5	12	18	15	18	11	12	11	16	11	12	136
6	19	20	18	16	17	24	20	20	18	10	182
7	21	20	23	27	20	16	25	17	21	31	221
8	16	10	16	14	14	18	17	15	8	20	148
9	15	14	8	8	9	12	8	8	15	6	103
10	5	6	8	6	11	4	5	5	6	6	62
11	2	4	6	2	3	2	1	3	2	5	30
12	–	1	–	1	–	0	0	2	1	–	5
13	–	–	–	–	1	0	1	–	1	–	3
14	–	–	–	–	–	1	–	–	1	1	3
	689	697	711	683	692	685	673	670	688	714	6902

$$\int_{1900000}^{2000000} \frac{dx}{\log x} = 6904,54424$$

Anzahl der Primzahlen zwischen 2000000 und 2100000.

	201	202	203	204	205	206	207	208	209	210	
0	–	–	–	–	–	–	–	–	–	–	–
1	–	–	–	–	2	1	–	–	–	–	3
2	–	–	2	2	1	–	–	2	2	1	10
3	3	3	5	2	2	3	4	5	2	3	32
4	7	8	9	4	8	5	6	7	9	6	69
5	13	10	9	15	13	10	12	13	9	15	119
6	25	20	13	16	26	23	25	14	17	18	197
7	10	22	23	25	13	23	18	23	25	22	204
8	13	17	15	22	12	13	12	16	18	19	157
9	16	15	11	4	11	12	11	14	11	10	115
10	10	2	8	6	11	7	7	3	3	6	63
11	2	1	5	3	–	1	3	3	3	–	21
12	–	2	–	–	1	2	2	–	1	–	8
13	1	–	–	1	–	–	–	–	–	–	2
	705	691	693	690	671	696	694	674	686	674.	6874. Li = 6880.780

Li =

440

III

Anzahl der Primzahlen zwischen 2100000 und 2200000

	211	212	213	214	215	216	217	218	219	220	
0	–	–	–	–	–	–	–	–	–	–	–
1	–	1	–	–	–	–	1	–	–	–	2
2	1	1	1	–	–	1	1	1	1	2	9
3	5	3	3	1	4	–	2	3	2	4	27
4	7	7	2	13	5	3	9	6	9	8	69
5	12	14	20	16	16	12	17	13	12	14	146
6	12	20	17	14	16	25	16	23	21	19	183
7	19	14	18	23	26	22	18	22	22	17	201
8	22	21	20	20	12	16	20	10	12	15	168
9	12	10	8	7	9	10	7	13	16	17	109
10	6	5	6	2	7	7	8	5	2	4	52
11	3	2	3	1	4	2	1	–	2	–	18
12	1	1	1	1	1	2	–	2	–	–	9
13	–	1	1	1	–	–	–	–	1	–	4
14	–	–	–	1	–	–	–	2	–	–	3
	699	683	697	697	693	712	666	691	679	664	6857. Li = 6858.292

440

15

Anzahl der Primzahlen zwischen 2200000 und 2300000

	221	222	223	224	225	226	227	228	229	230	
1	–	–	–	–	1	–	–	–	1	–	2
2	–	–	1	1	–	2	–	–	3	2	9
3	5	2	4	7	2	1	2	2	2	2	29
4	8	9	5	5	10	7	7	7	6	9	73
5	12	24	16	13	11	10	14	14	15	9	138
6	17	18	12	18	18	20	16	17	21	22	179
7	19	17	25	21	18	20	23	25	19	18	205
8	12	12	19	15	18	24	13	17	16	22	168
9	14	9	6	9	19	11	16	9	13	7	113
10	7	6	6	6	1	3	7	5	–	3	44
11	5	2	4	4	2	2	2	2	2	5	30
12	1	1	2	1	–	–	–	2	2	1	10
	701	660	695	680	683	688	701	694	662	685	6849

L.i = 6836.977

I

Anzahl der Primzahlen zwischen 2300000 und 2400000

	231	232	233	234	235	236	237	238	239	240	
1	1	1	1	–	1	–	–	–	–	–	4
2	0	–	1	–	2	2	1	2	1	2	11
3	4	1	2	3	4	3	2	6	4	3	32
4	8	12	10	9	8	7	3	13	7	9	86
5	13	18	13	14	17	12	10 (10)	13	10	16	136
6	15	16	21	20	16	21	26	11	14	16	176
7	22	25	20	20	13	16	26	18	19	15	194
8	13	9	16	21	17	15	15	15	21	16	158
9	13	9	6	7	14	14	8	13	14	14	112
10	7	5	3	2	5	5	8	7	8	5	55
11	3	3	4	3	3	5	1	2	2	2	28
12	1	–	3	1	–	–	–	–	–	2	7
13	–	1	–	–	–	–	–	–	–	–	1
	690	662	672	671	666	690	691	660	705	680	6787

L.i = 6816.706

II

Anzahl der Primzahlen zwischen 2400000 und 2500000 ~~44 of~~ 11

	241	242	243	244	245	246	247	248	249	250	
1	—	2	—	—	—	—	—	—	1	—	1
2	2	1	—	—	—	—	1	1	2	3	9
3	4	6	4	4	1	5	3	5	3	2	37
4	12	8	7	7	9	7	10	8	6	4	78
5	13	14	17	19	15	12	18	11	11	17	147
6	18	16	21	20	18	22	18	18	20	22	193
7	21	16	19	20	17	17	21	22	19	17	189
8	10	16	14	14	21	17	10	17	16	16	151
9	8	13	9	11	7	12	6	11	12	13	102
10	9	6	5	4	6	4	8	6	7	3	58
11	2	4	3	1	4	1	4	1	2	1	23
12	1	—	1	—	1	1	1	—	—	2	7
13	—	1	—	—	1	2	—	—	1	—	5
	660	690	672	657	701	687	666	672	687	674	6766

L.i. = 6797.394

Anzahl der Primzahlen zwischen 2500000 und 2600000

	251	252	253	254	255	256	257	258	259	260	
1	—	1	1	—	—	—	—	—	—	1	3
2	1	2	—	1	1	—	—	—	—	—	5
3	4	—	6	6	2	2	1	2	8	4	35
4	7	18	9	7	8	6	10	7	8	8	88
5	10	11	7	15	15	23	16	16	10	13	136
6	22	14	20	21	20	15	18	19	20	25	194
7	24	17	12	20	22	22	23	15	13	12	180
8	18	15	20	9	16	18	16	20	19	19	170
9	8	10	13	7	9	7	6	12	6	10	88
10	2	5	8	10	5	4	7	4	9	4	58
11	1	5	3	2	2	1	2	3	5	—	24
12	1	1	1	2	—	2	1	2	1	2	13
13	2	1	—	—	—	—	—	—	1	2	6
	677	675	696	670	670	671	678	698	693	676	6804

L.i. = 6778.960

Unzahl der Primzahlen zwischen 2600000 und 2700000

	261	262	263	264	265	266	267	268	269	270	
0	—	—	—	1	—	—	—	—	—	—	1
1	—	—	2	—	—	—	—	2	—	—	4
2	1	2	1	—	2	—	1	2	1	—	10
3	3	6	2	2	4	—	3	3	3	2	28
4	9	6	6	12	3	7	7	8	6	7	71
5	11	15	14	11	13	17	22	16	18	21	158
6	26	17	14	18	23	19	20	24	19	15	195
7	23	11	27	20	21	21	21	14	22	21	201
8	14	23	16	13	15	16	12	8	12	13	142
9	9	10	13	16	10	10	5	10	5	8	96
10	3	6	5	4	2	8	2	5	8	10	53
11	1	1	—	1	3	1	4	4	5	2	22
12	—	3	—	2	4	1	1	4	1	1	17
13	—	—	—	—	—	—	1	—	—	—	1
14	—	—	—	—	—	—	1	—	—	—	1
	653	681	672	680	689	695	660	665	681	686	6762

Li 6761.332

Unzahl der Primzahlen zwischen 2700000 und 2800000

	271.	272	273	274	275	276	277	278	279	280	
1	—	—	—	—	—	1	—	—	1	—	2
2	2	—	2	—	—	2	—	—	1	—	7
3	4	5	6	5	4	4	4	3	3	5	43
4	9	7	16	7	8	9	8	8	12	11	95
5	10	14	13	14	13	12	13	17	11	18	135
6	24	18	15	28	19	20	15	21	18	17	195
7	18	22	15	20	24	16	23	19	22	9	188
8	13	10	13	12	15	13	20	19	15	15	145
9	9	9	10	4	11	12	9	7	8	8	87
10	6	9	6	7	5	8	7	2	7	10	67
11	3	3	4	2	—	2	—	3	1	6	24
12	1	3		1	1	1	—	—	1	1	9
13	—	—	—	—	—	—	1	1	—	—	2
14	—	—	—	—	—	—	—	—	—	—	—
15	—	—	—	—	—	—	—	—	—	—	—
16	—	—	—	—	—	—	—	—	—	—	—
17	1	—	—	—	—	—	—	—	—	—	1
	679	695	644	657	672	671	684	666	662	684	6714

Li 6744.430

Anzahl der Primzahlen zwischen 2800000 und 2900000

	281	282	283	284	285	286	287	288	289	290	
1	—	—	—	—	1	—	1	—	—	—	2
2	—	2	4	—	5	2	—	1	1	—	15
3	2	4	4	3	4	3	1	3	2	4	30
4	9	7	6	9	7	8	10	11	10	8	85
5	14	7	14	14	7	16	19	16	18	15	140
6	18	17	20	13	23	18	16	18	15	21	179
7	24	22	21	20	20	27	22	23	21	22	222
8	13	18	9	20	13	12	9	12	12	14	132
9	10	17	12	12	8	6	14	9	10	11	109
10	7	4	6	3	5	7	5	5	8	3	53
11	—	1	3	3	5	—	1	2	1	2	18
12	2	1	—	—	—	1	2	—	2	—	8
13	—	—	1	2	2	—	—	—	—	—	5
14	1	—	—	1	—	—	—	—	—	—	2
	690	695	667	704	671	654	672	653	676	662	6744.

Li 6728.220

Anzahl der Primzahlen zwischen 2900000 und 3000000

	291	292	293	294	295	296	297	298	299	300	
1	—	1	—	—	—	—	—	—	1	—	2
2	—	1	2	1	2	2	—	1	1	3	13
3	3	3	5	2	8	6	4	4	5	4	44
4	7	7	6	6	7	9	6	6	6	4	64
5	20	11	14	18	12	15	17	19	16	11	153
6	17	21	27	18	18	16	11	26	21	17	187
7	19	30	18	22	22	25	27	13	15	23	214
8	14	11	12	12	17	11	13	11	14	19	134
9	10	9	12	13	9	6	12	12	9	11	103
10	6	4	5	6	3	6	8	7	9	4	58
11	2	1	3	1	—	2	1	—	1	4	15
12	2	1	—	1	2	2	—	1	2	—	11
13	—	—	1	—	—	—	—	—	—	—	1
14	—	—	—	—	—	—	—	—	—	—	—
15	—	—	—	—	—	—	1	—	—	—	1
	680	663	671	680	649	652	694	658	671	687	6705

Li 6712.64

Anzahl der Primzahlen zwischen 2000000 und 3000000

	210	220	230	240	250	260	270	280	290	300	
0	—	—	—	—	—	—	1	—	—	—	1
1	3	2	2	4	1	3	4	2	2	2	25
2	10	9	9	11	9	6	10	7	15	13	98
3	32	27	29	32	37	35	28	43	30	44	337
4	69	69	73	86	78	88	71	95	85	64	778
5	119	146	138	136	147	136	158	135	140	153	1408
6	197	183	179	176	193	194	195	195	179	187	1878
7	204	201	205	194	189	180	201	188	222	214	1998
8	157	168	168	158	151	170	142	145	132	134	1525
9	115	109	113	112	102	88	96	87	109	103	1034
10	63	52	44	55	58	58	53	67	53	58	561
11	21	18	30	28	23	24	22	24	18	15	223
12	8	9	10	7	7	13	17	9	8	11	99
13	2	4	—	1	5	6	1	2	5	1	27
14	—	3	—	—	—	—	1	—	2	—	6
15	—	—	—	—	—	—	—	—	—	1	1
16	—	—	—	—	—	—	—	—	—	—	—
17	—	—	—	—	—	—	—	1	—	—	1
	6874	6857	6849	6787	6766	6804	6762	6714	6744	6705	67862

$$\int_{2000000}^{3000000} \frac{dx}{\log x} = 67915{,}733$$

Die 26379te Centade enthält keine Primzahl

Die 27050te Centade enthält 17 Primzahlen.

Gauß, Math 18

1000000 bis 2000000 (13) ~~44a~~ 13

	110	120	130	140	150	160	170	180	190	200	
0	0	0	–	–	–	–	1	–	–	–	1
1	1	1	2	1	5	2	2	1	–	1	16
2	4	5	6	9	7	10	11	5	10	5	72
3	21	25	32	19	19	28	29	30	22	34	259
4	54	57	63	69	72	77	68	70	71	67	668
5	114	107	120	119	129	124	120	152	135	136	1256
6	171	170	160	173	183	199	159	174	175	182	1746
7	217	217	219	207	179	172	203	194	206	221	2030
8	164	160	168	161	183	149	174	147	161	148	1615
9	126	131	111	120	98	124	130	124	113	103	1180
10	71	77	73	70	73	63	63	61	74	62	687
11	39	32	35	33	34	29	26	26	23	30	307
12	12	11	9	15	16	17	9	10	10	5	114
13	6	5	5	3	2	6	3	5	–	3	38
14	–	2	1	1	–	–	1	–	–	3	8
15	–	–	–	–	–	–	1	1	–	–	2
16	–	–	1	–	–	–	–	–	–	–	1
Summe	7210	7194	7081	7098	7028	6971	7012	6931	6955	6902	70382
~~Log int.~~	~~7212.99~~	~~7165.94~~	7123.34	7084.48	7048.78	~~7015.79~~	~~6985.14~~	~~6956.54~~	6929.74	6904.54	70427.78

$$\int \frac{dx}{\log x} = 70427.78$$

Math 18

(14) ~~[illegible]~~ 17

	1210	1220	1230	124	125	126	127	128	129	130	
1								1	1		2
2	2		2	1	–	–	–	–	–	1	6
3	3	2	4	5	4	3	1	4	3	3	32
4	7	7	7	3	5	7	12	2	3	10	63
5	15	12	12	17	10	14	9	15	6	12	120
6	16	14	13	19	17	16	16	15	20	14	160
7	24	15	25	24	21	20	15	22	24	24	214
8	17	19	16	11	17	15	22	18	19	14	168
9	8	12	7	10	12	13	14	13	13	9	111
10	3	11	10	8	10	4	5	3	9	10	73
11	3	6	3	2	3	5	3	6	1	3	35
12	1	1	1	–	–	1	3	1	1		9
13	1	1		1	–	2	–				5
14				1	–	–	–				1
15					–	–	–				
16					1	–	–				1
	676	744	693	693	724	713	718	709	722	689	7081

Chiliades Myriades

1820 Jun 20	1821 März 31	1822 April 5	Nov. 6	Dec. 31	1825 März 5	Sept 25	1826 Juli 24	1827 Febr. 27
20 1	23 1	23 1	23 1	25 1	29 2	31. 2	31.2	35 2
34 3	34 3	45 4	45 4	45 4	45 4	47. 4	47.4	49.4
87 8	87 8	87 8	91 9	100 10	100 10	100. 10	100.10	100.10
45 4	45 4	49 4	50 4	50 4	50 4	50. 4	50.4	52.4
45 4	45 4	45 4	45 4	45 4	45. 4	47. 4	47.4	48.4
17 1	20 1	20 1	22 1	22 1	26 1	28 1	28.1	34.2
19 1	21 1	21 1	21 1	26 2	26 2	28 2	28.2	29 2
18 1	20 1	26 2	26 2	26 2	26 2	26 2	32.2	33.2
19 1	21 1	21 1	21 1	25 1	27 1	33 2	33.2	34.2
100 10	100 10	100 10	100 10	100 10	100 10	100 10	100.10	100.10
404 34	416 34	437 36	444 37	464 39	474 40	490 41	496.41	514.42

1827 May. 2	Aug. 14	1828 März 2	März 17	März 25	Mai 6	Mai 13	Mai 18	Mai 22	Jun 14	Jun 29
38. 2	44. 3	46.3	48 3	58. 4	60. 4	62.4	66.4	72.5	74.5	74.5
52. 4	54.4	56.4	59 4	60. 4	60. 4	60.4	66.4	66.4	68.5	70.5
100. 10	100.10	100.10	100. 10	100. 10	100. 10	100.10	100.10	100.10	100.10	100.10
54 4	56.5	58.5	58. 5	59.5	60 5	60.5	64.5	68.5	68.5	70.5
49 4	49.4	51.4	51. 4	53.4	60.4	60.4	60.4	66.4	70.5	70.5
36 2	40 3	42.3	42. 3	52.4	60.4	60.4	60.4	60.4	64.5	70.5
31 2	34 2	40 2	46 3	54.4	60.4	60.4	60.4	60.4	64.5	70.5
34.2	36 2	38 2	44 3	50.4	52.4	60.4	60.4	60.4	62.5	62.5
35.2	36.2	38.2	44 3	50.4	52.4	60.4	60.4	60.4	66.5	66.5
100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10
529.42	549.45	569.45	592.48	636.53	664.53	682.53	696.53	712.54	736.60	752.60

Math 18

(15)

1829 Jul. 22	Aug 6	Aug 26	Sept 15	1830 Jan 20	Apr 27	Mai 17	Mai 26	Jun 14	Jun 18	Jul 9	Aug 2	1835 Spt 13
88.8	90.8	94.8	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100 10
70.5	70.5	70.5	70.5	80.6	84.6	86.6	92.6	92.6	94.7	96.8	100.10	100 10
100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100 10
70.5	70.5	70.5	70.5	72.5	72.5	76.5	78.5	82.5	86.5	86.5	88.6	88 6
70.5	70.9	70.5	70.5	72.5	72.5	76.5	76.5	80.5	80.5	86.5	86.5	86 5
70.5	70.5	70.5	70.5	72.5	72.5	76.5	76.5	78.5	78.5	82.6	82.6	82 6
70.5	70.5	70.5	70.5	72.5	72.5	76.5	76.5	76.5	78.5	78.5	78.5	78 5
68.5	68.5	70.5	70.5	72.5	74.5	76.5	76.5	76.5	78.5	78.5	78.5	82 6
66.5	68.9	72.5	72.5	74.5	76.5	76.5	76.5	76.5	78.5	78.5	78.5	78 5
100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.10	100.0
772.63	776.63	786.65	792.65	814.66	822.66	842.66	850.66	860.66	872.67	884.69	890.72	894.73

Primzahlen

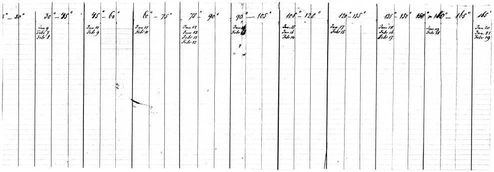
~~[illegible]~~

1000000	
1100000	7210
1200000	7194
1300000	7081
1400000	7098
1500000	7028
1600000	6971
1700000	7012
1800000	6931
1900000	6955
2000000	6982
	70382

d.i 70427,77844

(16)

0.7133 b. a.. 8.7966
0.2301 s.. 9.7563 9.6056
− 0.2384 + 0.4832 + 0.1512 − 0.0253 | 0.7102 8.6193 0.2484 9.7802 9.6149
− 0.1585 + 0.4618 + 0.1583 − 0.0171 | 0.7122 8.7078 0.2405 9.7775 9.6087
− 0.1944 + 0.4717 + 0.1855 − 0.0207 | 0.7128 0.2437
− 0.1603 + 0.4691



°– 30°	30° – 45°	45° – 60°	60° – 75°	75° – 90°	90° – 105°	105° – 120°	120° – 135°	135° – 150°	150° – 165°	165°
	Jan 9	Jan 10	Jan 11	Jan 12	Jan [illegible]	Jan. 15	Jan. 17	Jan. 18	Jan 19	Jan 20
	Febr 7	Febr 9	Febr 10	Jan 13	Febr [illegible]	Jan 16	Febr 15	Febr 16	Febr 18	Jan. 21
	Febr 8			Febr 11		Febr 14		Febr 17		Febr 19
				Febr 12						

100000	9629,80904							
200000	18036,05216	+8406.24312	-355.60305					
300000	26086,69223	8050.64007	214.71030	+140.89275	-78.32915			
400000	33922,62200	7835.92977	152.14670	62.56360	27 77793	+50.55122	-34.78461	
500000	41606,40507	7683.78307	117.36103	34.78567	12.01132	15.76661	11.14331	
600000	49172,82711	7566.42204	94.58668	22.77435	7.38802	4.62330	1.29900	
700000	56644,66247	7471.83536	79.20035	15.38633	4.06372	3.32430	1.97819	
800000	64037,29748	7392.63501	67.88774	11.32261	2.70761	1.35611	0.50092	
900000	71362,05475	7324.75727	59.26274	8.61500	1.85242	0.85519	0.22424	
1000000	78627,54928	7265.49453	52.50016	6.76258	1.32147	0.53095	0,18252	
1100000	85840,54365	7212.99437	47.05905	5.44111	0,97304	0,34843	0,11273	
1200000	93006.47897	7165.93532	42,59098	4,46807	0,73734	0,23570	0,07107	
1300000	100129,82331	7123,34434	38.86025	3,73073	0,57271	0,16463	0,04181	
1400000	107214,30740	7084,48409	35.70223	3,15802	0.44989	0,12282	0.03758	
1500000	114263,08926	7048.78186	32,99410	2,70813	0.36465	0.08524	0.01497	
1600000	121 278,87702	7015,78776	30.65062	2.34348	0.29438	0.07027	0.01992	
1700000	128 264,01416	6985,13714	28.60152	2.04910	0.24403	0.05035	0.00987	
1800000	135220,54978	6956.53562	26,79645	1.80507	0 20355	0.04048		
1900000	142150,28895	6929.73917	25,19493	1.60152				
2000000	149054,83319	6904.54424						

69100 – 1851·0
8.6199
9.7726
9.6870

Gauss?, Math 18 (17) ~~44a~~ 20

Anzahl der Primzahlen von 0

		+3		$\int \frac{dx}{\log x}$	+1	+1	Legend.
bis	100000	9590	9594				
	200000	18002					
	300000	26014					
	400000	33875					
	500000	41553	41596.9	41606.4	+[illegible].4	37.9	−23.3
	600000	49111		49172.8	+57.8		
	700000	56540		56644.7	100.7		
	800000	63957		64037.3	76.3		
	900000	71270		71362.1	98.1		
	1000000	78498	78672.7	78627.5	125.5	170.7	+42.2
	1100000	85708		85840.5	128.[illegible]		
	1200000	92902		93006.5	100.5		
	1300000	99983					
	1400000	107081					
	1500000	114109	114374.0	114263.1	150.1	[illegible]0	+68.1
	1600000	121080		121278.9	1[illegible]4.9		
	1700000	128092					
	1800000	135023					
o	1900000	141978					
	2000000	148880	149233.0	149054.833	170.8	349.0	+92.8
	2100000	155754		155935.613	177.6		
	2200000	162611		162793.905	178.9		
	2300000	169460		169630.882	166.9		
	2400000	176247		176447.588	196.6		
	2500000	183013	183495.1	183244.982	228.0	478.1	+159.1
	2600000	189817		190023.940	202.9		
	2700000	196579		196785.272	202.3		
	2800000	203293		203529.702	232.7		
	2900000	210037		210257.922	216.9		
	3000000	216742	217308.5	216970.566	224.6	562.5	+167.6

B. Riemann 1849

$$\zeta(s) := 1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\frac{1}{n^s}+\ldots = \prod_p (1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\ldots) = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

B. Riemann 1849

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots \frac{1}{n^s} + \dots = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Analytic properties of this function carry information about the distribution of primes.

B. Riemann 1849

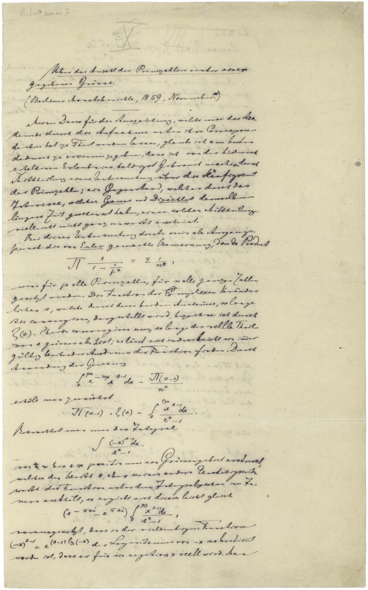
$$\zeta(s) := 1+\frac{1}{2^s}+\frac{1}{3^s}+\dots\frac{1}{n^s}+\dots = \prod_p (1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots) = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

Analytic properties of this function carry information about the distribution of primes.

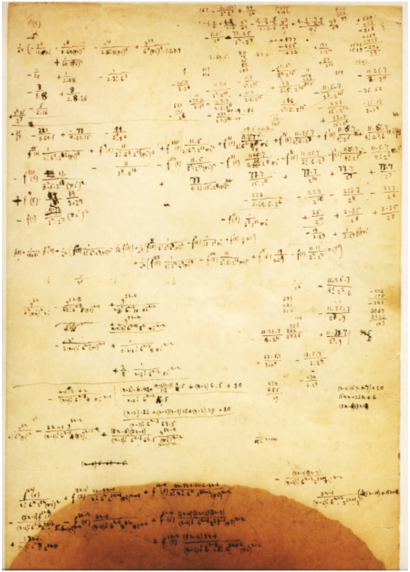
Riemann Conjecture

All nontrivial zeroes of this function have real part $\Re(s) = 1/2$.

Manuscript by B. Riemann



Manuscript by B. Riemann



Primes

$$\pi(x) := \{p \le x\} \sim \frac{x}{\log(x)}, \quad x \to \infty$$

- Gauss conjecture
- Riemann's approach via the zeta function
- Hadamard, de la Vallee-Poussin
- Selberg – "elementary" proof

Primes

Proof of $\gg$

$$\nu_p(n) := \text{ } p\text{-power dividing } n$$

Primes

Proof of $\gg$

$$\nu_p(n) := \ p\text{-power dividing } n$$

$$\nu_p(n!) = \sum_{k \geq 1} \left[\frac{n}{p^k} \right]$$

Proof of $\gg$

$$\nu_p(n) := \text{ } p\text{-power dividing } n$$

$$\nu_p(n!) = \sum_{k \geq 1} \left[\frac{n}{p^k} \right]$$

Apply to

$$N = \binom{2m}{m} = \frac{(2m)!}{(m!)^2}$$

$$\nu_p(N) = \sum_{k \geq 1} \left[\frac{2m}{p^k} \right] - 2 \left[\frac{m}{p^k} \right]$$

$$N = (\frac{m+1}{1})(\frac{m+2}{2})\cdots(\frac{m+m}{m})$$

Thus

$$N \geq 2^m, \quad p \mid N \Rightarrow p \leq 2m.$$

The summand in $\nu_p(N)$ vanishes if $k > \frac{\log(2m)}{\log(p)}$, and is at most 1, in other cases. It follows that

$$\nu_p(N) \leq \frac{\log(2m)}{\log(p)}$$

Primes

We find

$$\pi(2m)\log(2m) = \sum_{p\le 2m} \frac{\log(2m)}{\log(p)}\cdot\log(p)$$

$$\ge \sum_{p\le 2m} \nu_p(N)\cdot\log(p) = \log(N)$$

$$\ge m\log(2)$$

Thus

$$\pi(2m) \ge \frac{1}{2}\log(2)\frac{2m}{\log(2m)}.$$

Primes in arithmetic progressions

$$(a, m) = 1 \quad \Rightarrow$$

$$\#\{p \equiv a \pmod{m}, p \leq x\} \sim \frac{1}{\varphi(m)} \frac{x}{\log(x)}$$

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E.g.

$$q \mid (\prod p_j)^2 + 1 \Rightarrow q \equiv 1 \pmod{4}$$

Theorems of Green-Tao

- 2
- 2,3
- 3,5,7
- 5,11,17,23
- 5,11,17,23,29

Theorems of Green-Tao

- 2
- 2,3
- 3,5,7
- 5,11,17,23
- 5,11,17,23,29

- no infinitely long arithmetic progressions in primes (trivial)
- van der Corput 1939: $\exists$ infinitely many arithmetic progressions of length 3 in primes
- Green-Tao 2004: there exist arbitrarily long arithmetic progressions in primes
- Tao-Ziegler 2006: $P_1, \ldots, P_k \in \mathbb{Z}[x]$, $\mathrm{P}_j(0) = 0$, $\Rightarrow \Pi \supset$ infinitely many progressions of the form

$$n + P_1(r), \ldots, n + P_k(r)$$

Open problems

- Goldbach conjecture (1742): every even number ≥ 4 is a sum of two primes.

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- $p = n^2 + 1$?

Open problems

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- Schinzel's hypothesis = same for systems of equations $f_1, \ldots, f_r \in \mathbb{Z}[x]$... $\Rightarrow \exists\, \infty$-many $\quad n \mid f_j(n) = p_j$ e.g., $x, x + 2$ ...