Homework 7 / due November 14

1. Let $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt[3]{a}$, with $a \in \mathbb{Z}$, a squarefree. Show that if $a \neq \pm 1 \mod 9$ then $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2$.

2. Find an integral basis for $\mathcal{O}_K$, where $K = \mathbb{Q}(\alpha)$ and $\alpha^3 - \alpha + 1 = 0$.

3. Let $K = \mathbb{Q}(\zeta_\ell)$ be the ℓ-th cyclotomic field. Show that the discriminant

$$D_K = (-1)^{\frac{\ell-1}{2}} \ell^{\ell-2}.$$

4. Let $K = \mathbb{Q}(\sqrt{-5})$. Show that $\mathfrak{a} := (4 + \sqrt{-5}, 1 + 2\sqrt{-5})$ is not a principal ideal in $\mathcal{O}_K$.

5. Let $\mathfrak{a}$ be an integral ideal in $\mathcal{O}_K$. Then

$$\cap_{n=1}^{\infty} \mathfrak{a}^n = \begin{cases} \mathcal{O}_K & \text{if } \mathfrak{a} = \mathcal{O}_K \\ (0) & \text{otherwise} \end{cases}$$