ALGEBRA: HOMEWORK 8

Problem 1. Let $K/\mathbb{Q}$ be a cubic extension and $(p) = p_1 \cdot p_2 \cdot p_3$ a product of distinct prime ideals. Let $\alpha \in \mathcal{O}_K$ be an integer with $Tr_{K/\mathbb{Q}}(\alpha) = 0$. Show that if $p_1 \cdot p_2 | (\alpha)$ then $p_3 | (\alpha)$.

Proof. □

Problem 2. Assume that Cl_K contains distinct nontrivial classes C_1, C_2. Show that there exist integral ideals $I_i \in C_i$ which are coprime.

Proof. Assume $I_1 \in C_1$, $J \in C_3 := C_2^{-1}$ are two ideals. Let $p_1, ..., p_r$ be the prime divisors of J. For each n, let v_n be the largest power of p_n that divides I_1. Choose an element $a_n \in p_n^{v_n} - p_n^{v_n+1}$. By Chinese remainder theorem (a generalized version of problem 4, homework 7), there exists $a \in \mathcal{O}_K$ such that

$$a \equiv a_n \pmod{p_n^{v_n+1}}$$

for all $n = 1, ..., r$ and also

$$a \equiv 0 \pmod{I_1 / \prod p_n^{v_n}}.$$

Then $(a) \subset I_1$ and let $I_2 = (a)/I_1 \in C_2$. It suffices to prove that I_1 and I_2 are coprime, which can be done directly by counting the powers of prime divisors in I_1 and I_2. □

Problem 3. Show that $\varepsilon_1 := 1 + \zeta_7 + \zeta_7^6, \varepsilon_2 := 1 + \zeta_7^3 + \zeta_7^4$ are multiplicatively independent units, i.e.,

$$\varepsilon_1^a \cdot \varepsilon_2^b = 1 \Rightarrow a = b = 0.$$

Proof. Assume $\sigma : \zeta_7 \mapsto \zeta_7^2$ is the automorphism on $\mathbb{Q}(\zeta_7)$, and $\varepsilon_3 := 1 + \zeta_7^2 + \zeta_7^5$. Then we have

$$1 = \varepsilon_1^a \varepsilon_2^b = \sigma(\varepsilon_1^a \varepsilon_2^b) = \sigma^2(\varepsilon_1^a \varepsilon_2^b),$$

i.e.,

$$\varepsilon_1^a \varepsilon_2^b = 1, \varepsilon_3^a \varepsilon_1^b = 1, \varepsilon_2^a \varepsilon_3^b = 1.$$

Eliminate unknowns ε_2 and ε_3, we get $\varepsilon_1^{a^3+b^3} = 1$. Since $\varepsilon_1 \neq 1$, we have $a + b = 0$, and then $\varepsilon_1^a \varepsilon_2^{-a} = 1$. Since $\varepsilon_1 / \varepsilon_2 \neq 1$, $a = b = 0$. □

Problem 4. Let $K = \mathbb{Q}(\eta), \eta^3 = 6$. Show that $h_K = 1$.

Proof. First by Homework 6, #2 we know that $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{6}]$. The Minkowski's bound of K is

$$(\frac{4}{\pi})^{r_2}\frac{n!}{n^n}\sqrt{disc(K)} = (\frac{4}{\pi})^1\frac{3!}{3^3}\sqrt{27\cdot 6^2} \approx 8.82.$$

So we need to factor primes 2,3,5,7.

(i) modulo (2), we have

$$x^3 - 6 \equiv x^3 \pmod 2,$$

and $(2) = p_2^3$, where $p_2 = (2, \sqrt[3]{6}) = (\sqrt[3]{6} - 2)$ is principal.

(ii) modulo (3), we have

$$x^3 - 6 \equiv x^3 \pmod 3,$$

and $(3) = p_3^3$, where $p_3 = (3, \sqrt[3]{6}) = (3 + 2\sqrt[3]{6} + \sqrt[3]{36})$ is principal.

(iii) modulo (5), we have

$$x^3 - 6 \equiv (x-1)(x^2+x+1) \pmod 5,$$

and $(5) = p_5 \cdot p_5'$, where $p_5 = (5, \sqrt[3]{6} - 1) = (\sqrt[3]{6} - 1)$ is principal and thus p_5' is also principal.

(iv) modulo (7), we have

$$x^3 - 6 \equiv (x+1)(x+2)(x-3) \pmod 5,$$

and $(7) = p_7 \cdot p_7' \cdot p_7''$.

Here $p_7 = (7, \sqrt[3]{6} + 1) = (\sqrt[3]{6} + 1)$ and $p_7' = (7 + 4\sqrt[3]{6} + 2\sqrt[3]{36})$ are principal, and thus p_7'' is principal.

□

Problem 5. Compute the class number of $\mathbb{Q}(\sqrt{-13})$.

Proof. The Minkowski's bound of $K = \mathbb{Q}(13)$ is

$$(\frac{4}{\pi})^{r_2}\frac{n!}{n^n}\sqrt{disc(K)} = (\frac{4}{\pi})^1\frac{2!}{2^2}\sqrt{4\cdot 13} < 5.$$

So we want to factor prime 2,3.

(i) modulo (2), we have

$$x^2 + 13 \equiv (x-1)^2 \pmod 2,$$

and $(2) = p_2 = (2, \sqrt{-13} - 1)^2$, and $N(p_2) = \sqrt{N(2)} = 2$. Since $x^2 + 13y^2 = 2$ has no integer solutions, p_2 is not principal.

(ii) modulo (3), we have that $x^2 + 13$ is irreducible. So (3) itself is a prime principal ideal. □