Homework 8

1. Let $K/\mathbb{Q}$ be a cubic extension and $(p) = \wp_1 \cdot \wp_2 \cdot \wp_3$ a product of distinct prime ideals. Let $\alpha \in \mathcal{O}_K$ be an integer with $Tr_{K/\mathbb{Q}}(\alpha) = 0$. Show that if $\wp_1 \cdot \wp_2 \mid (\alpha)$ then $\wp_3 \mid (\alpha)$.

2. Assume that Cl_K contains distinct nontrivial classes C_1, C_2. Show that there exist integral ideals $\mathfrak{a}_i \in C_i$ which are coprime.

3. Show that $\epsilon_1 := 1+\zeta_7+\zeta_7^6, \epsilon_2 := 1+\zeta_7^3+\zeta_7^4$ are multiplicatively independent units, i.e.,
$$\epsilon_1^a \cdot \epsilon_2^b = 1 \Rightarrow a = b = 0.$$

4. Let $K = \mathbb{Q}(\eta), \eta^3 = 6$. Show that $h_K = 1$.

5. Compute the class number of $\mathbb{Q}(\sqrt{-13})$.