Homework 7

1. Let $K = \mathbb{Q}(\sqrt{-5})$ and $\mathfrak{a} := (4+\sqrt{-5}, 1+2\sqrt{-5}) \subset \mathcal{O}_K$. Show that $\mathfrak{a}$ is not a principal ideal and that it is a prime ideal.
2. Let $\alpha \in \mathcal{O}_K$ be an element such that $K = \mathbb{Q}(\alpha)$. Show that

$$\operatorname{disc}(\alpha) = \operatorname{disc}(K/\mathbb{Q}) \cdot t^2,$$

   for some $t \in \mathbb{Z}$. This t is called the *index* of α.
3. Show that if $\alpha \in \mathcal{O}_K$ has index one then $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
4. Given coprime integral ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_K$ and $\alpha, \beta \in \mathcal{O}_K$ show that there exists a $\lambda \in \mathcal{O}_K$ such that

$$\lambda \equiv \alpha \pmod{\mathfrak{a}}, \quad \lambda \equiv \beta \pmod{\mathfrak{b}}.$$

5. Let $K = \mathbb{Q}(\zeta_7)$. Find a unit of infinite order.