Homework 1

1. Prove that $15x^2 - 7y^2 = 9$ has no solutions in $\mathbb{Z}$.

2. Prove that an integer of the form $8n + 7$ cannot be written as a sum of three integer squares.

3. Show that if $x^2 = a$ is solvable modulo p then it is also solvable modulo p^n, for all positive integers n.

4. Show that $(2, 3, 7)$ is the only triple of integers > 1 such that
$$c \mid (ab + 1), \quad b \mid (ac + 1) \quad \text{and} \quad a \mid (bc + 1).$$

5. Let $f : \mathbb{N} \to \mathbb{C}$ be given by
$$\sum_{d \mid n} f(d) = \phi(n), \quad \text{(the Euler function)},$$
for all $n \in \mathbb{N}$. Find all such f.