BOUNDS FOR MATRIX COEFFICIENTS AND ARITHMETIC APPLICATIONS

by

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Abstract. — We explain an important result of Hee Oh [16] on bounding matrix coefficients of semi-simple groups and survey some applications.

Contents

1. Introduction

Let G be a connected reductive group over a local field F. We denote by $\widehat{\mathsf{G}(F)}$ the unitary dual of $\mathsf{G}(F)$, that is the collection of equivalence classes of irreducible unitary representations of $\mathsf{G}(F)$. $\widehat{\mathsf{G}(F)}$ has a natural topology known as the *Fell Topology* which is described as follows. We will introduce a basis of neighborhoods. Let ρ be any element of $\widehat{\mathsf{G}(F)}$, ϵ a positive number, $\phi_1, \dots, \phi_n$ diagonal matrix coefficients of ρ, and K a compact subset of $\mathsf{G}(F)$. We define the open set

$$W(\phi_1, \dots, \phi_n, K; \epsilon; \rho)$$

to be the set of all $\eta \in \widehat{\mathsf{G}(F)}$ such that there exist $\phi'_1, \dots, \phi'_n$ each of which is a sum of diagonal matrix coefficients of η satisfying

$$|\phi'_i(x) - \phi_i(x)| < \epsilon$$

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for all $x \in K$ and all $i = 1, \dots, n$. For more details see [9, 22]. We say $\mathsf{G}(F)$ has *Property T* if the trivial representation is isolated in $\widehat{\mathsf{G}(F)}$. In concrete terms this means that if ρ is a non-trivial irreducible unitary representation of $\mathsf{G}(F)$, and ϕ is a K-finite matrix coefficient of ρ, then ϕ has exponential decay in the sense that will be made explicit in the next section. Kazhdan [12] has shown that if G is simple has F-rank at least two, then it has property T. The purpose of this note is to discuss a uniform quantitative version of Kazhdan's theorem due to Oh [16], and describe some recent applications to arithmetic problems. Oh's theorem is uniform in several aspects, and this will be essential in applications. We will make this precise in the text.

At least for $\mathrm{GL}(n)$, property T is intimately related to the generalized Ramanujan conjecture. Let me make this a little more explicit. For simplicity suppose we are working with $\mathrm{PGL}(n)$ over a number field F, and suppose T is the split torus consisting of the diagonal matrices. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{PGL}(n)$ over F. Let v be a place such that π_v is an unramified principal series representation. Suppose π_v is induced from a character χ_v of the torus $\mathsf{T}(F_v)$. Let $\mathcal{C}(\chi_v)$ be the Langlands class of π_v, and suppose ϕ_v is the normalized spherical function associated with χ_v. Let ϖ_v be a local uniformizer for F_v. Then the matrices

$$t_v^k := \begin{pmatrix} I_k & \\ & \varpi_v I_{n-k} \end{pmatrix} \quad k = 1, \dots, n-1$$

generate $T(F_v)/T(\mathcal{O}_v)$. Then one can show that

$$\phi_v(t_v^k) = \frac{q_v^{\frac{k(n-k)}{2}}}{\mathrm{vol}(\mathsf{K} t_v^k \mathsf{K})} \,.\, \mathrm{tr}(\wedge^k \mathcal{C}(\chi_v)^{-1}).$$

Now one can use global results on the description of the automorphic spectrum such as the theorem of Moeglin and Waldspurger, and bounds towards the Ramanujan conjecture due to Luo, Rudnick, and Sarnak to get non-trivial bounds for the spherical function. Note that this is pretty specific to PGL_n. There are of course two parts to this procedure. The first part is expressing the value of the spherical matrix coefficients. This part is perfectly general. One can in fact describe the value of the matrix coefficients explicitly in terms of the fundamental representations of the L-group of the group; this is done in Satake's paper [19]. The next step would be somewhat problematic. For example even for a small group such as the symplectic group of order four, Howe and Piatetski-Shapiro have constructed automorphic cuspidal representations whose local components are not tempered. For this reason it is not clear how one can get enough cancelation to obtain non-trivial bounds for the value of the spherical matrix coefficients. For a survey of known results about the Ramanujan conjecture, see Sarnak's notes on the Generalized Ramanujan Conjecture. What is surprising in Oh's theorem is that one can in fact get non-trivial bounds for matrix coefficients, spherical or not, without using explicit formulae for matrix coefficients.

Furthermore, Oh's theorem applies even when the representation under consideration is not the local component of an automorphic representation, as long as it is infinite dimensional and the local rank of the group is at least two.

Let us make some remarks on the situation where the rank of the group is one. This is related to the so-called the *Property* τ. For groups of rank one, the trivial representation may not be isolated in the unitary dual of the local group; however, not all is lost. To describe what is known in this case, let G be a connected reductive group over a number field F and let $\widehat{\mathsf{G}(F_v)}_{aut}$ be the closure in the Fell topology of $\widehat{\mathsf{G}(F_v)}$ of the set of all π_v which occur as the v-component of some automorphic representation π of G over F. We say the pair (G, v) has *Property* τ if the trivial representation is isolated in $\widehat{\mathsf{G}(F_v)}_{aut}$. Lubotzky and Zimmer have conjectured, and Clozel has proved [4], that if G is semi-simple, then for all v, (G, v) has property τ. Before saying anything about the proof of Clozel's theorem, let us recall a principle due to Burger, Li, and Sarnak ([2, 3] for the archimedean place and [6] for the general F_S case). Let H be a semi-simple subgroup of G defined over F. Then for all places v, if $\sigma \in \widehat{\mathsf{H}(F_v)}_{aut}$ then

$$Ind_{\mathsf{H}(F_v)}^{\mathsf{G}(F_v)}\sigma \subset \widehat{\mathsf{G}(F_v)}_{aut}$$

and if $\rho, \pi \in \widehat{\mathsf{G}(F_v)}_{aut}$, then

$$Res_{\mathsf{H}(F_v)}^{\mathsf{G}(F_v)}\rho \subset \widehat{\mathsf{H}(F_v)}_{aut}$$

and

$$\rho \otimes \pi \subset \widehat{\mathsf{G}(F_v)}_{aut}.$$

Here Ind means unitary induction. In these equations, the inclusion should be understood as saying if a representation is weakly contained in the left hand side then it is contained in the right hand side. Recall that if we say a representation ρ_1 is weakly contained in ρ_2 if every diagonal matrix coefficient of ρ_1 can be approximated uniformly on compact sets by convex combinations of diagonal matrix coefficients of ρ_2. Incidently, the proof of this principle as explained in [3] uses the equidistribution of certain Hecke points. Clozel's idea to prove the Property τ is to use the Burger-Li-Sarnak principle in the following fashion. If G is isotropic, then we let H be a root subgroup isomorphic to either SL_2 or PGL_2 which one can use what's known about the Ramanujan conjecture. If G is anisotropic, then Clozel shows that G contains certain special subgroups for which Property τ can be verified by transferring automorphic representations to the general linear group via the trace formula. Sarnak has conjectured if $\mathsf{G}(F_v)$ has rank one, then every non-tempered point of $\widehat{\mathsf{G}(F_v)}_{aut}$ is isolated. We refer the reader to Sarnak's survey [17] for various examples and further explanations.

This paper is organized as follows. In Section 2 we review Oh's important theorem, and give a rough sketch of the proof. In Section 3 we discuss an application to the equidistribution of Hecke points from [5, 10]. The goal of this of these papers, especially [10], was to apply the equidistribution of Hecke points to problems of distribution of points of interest on algebraic varieties. Here I would like to emphasize on the Hecke points themselves, and for that reason I will not discuss these applications. In the last chapter we will explain a recent theorem on the distribution of rational points on wonderful compactifications of semi-simple groups of adjoint type, verifying a conjecture of Manin. We will sketch two proofs for this theorem, one due to this author joint with Shalika and Tschinkel, and the other due to Gorodnick, Maucourant, and Oh; both argument rely on Oh's theorem.

I first learned of Oh's work from Sarnak in 2001. This paper, especially the introduction has been influenced greatly by his ideas especially those expressed in [17]. Also I must confess that there is nothing new in these notes, and everything here has been taken from the sources listed in the bibliography. I wish to thank Hee Oh for a careful reading of a draft of this paper and pointing out various inaccuracies and suggesting improvements. Here I thank Gan, Oh, Sarnak, Shalika, and Tschinkel for many useful discussions over the years.

2. Uniform Pointwise Bounds for Matrix Coefficients

2.1. A general theorem of Oh. — Let k be a non-archimedean local field of $\mathrm{char}(k) \neq 2$, and residual degree q. Let H be the group of k-rational points of a connected reductive split or quasi-split group with $\mathsf{H}/\mathsf{Z}(\mathsf{H})$ almost k-simple. Let S be a maximal k-split torus, B a minimal parabolic subgroup of H containing S and K a good maximal compact subgroup of H with Cartan decomposition $\mathsf{G} = \mathsf{K}\mathsf{S}(k)^+\mathsf{K}$. Let Φ be the set of non-multipliable roots of the relative root system $\Phi(\mathsf{H},\mathsf{S})$, and Φ^+ the set of positive roots in Φ. A subset $\mathcal{S}$ of Φ^+ is called a strongly orthogonal system of Φ if any two distinct elements α and α' of $\mathcal{S}$ are strongly orthogonal, that is, neither of $\alpha \pm \alpha'$ belongs to Φ. Define a bi-K-invariant function $\xi_{\mathcal{S}}$ on H as follows: first set

$$n_{\mathcal{S}}(g) = \frac{1}{2}\sum_{\alpha\in\mathcal{S}} \log_q |\alpha(g)|,$$

then

$$\xi_{\mathcal{S}}(g) = q^{-n_{\mathcal{S}}(g)} \prod_{\alpha\in\mathcal{S}} \left(\frac{(\log_q |\alpha(g)|)(q-1) + (q+1)}{q+1}\right).$$

The following is a special case of Theorem 1.1 of [16].

Theorem 2.1. — *Assume that the semi-simple k-rank of H is at least 2. Let $\mathcal{S}$ be any strongly orthogonal system of Φ. Then for any unitary representation ϱ of H without an invariant vector and with K-finite unit vectors ν and ν',*

$$|(\varrho(g)\nu,\nu')| \leq (\dim(\mathsf{K}\nu)\dim(\mathsf{K}\nu'))^{\frac{1}{2}} \cdot \xi_{\mathcal{S}}(g),$$

for any $g \in \mathsf{H}$.

Here and elsewhere it is not necessary to assume that the groups are quasi-split, and one just needs to assume that the given representation does not have an invariant vector under the action of the subgroup H^+ generated by all one parameter unipotent subgroups. If the group H is simply-connected, then $\mathsf{H} = \mathsf{H}^+$, but in general they may not be the same. The special case considered above is for simplicity. In order to prove the theorem Oh constructs a subgroup H_α isomorphic to $\mathrm{SL}_2(k)$ or $\mathrm{PGL}_2(k)$ associated to each root α of a strongly orthogonal system, and then show that every representation ϱ of H restricted to H_α is a direct integral of tempered representations. Note that this gives a bound for matrix coefficient of $\varrho|_{\mathsf{H}_\alpha}$ in terms of Harish-Chandra functions for SL_2. Then one uses an idea of Howe to glue the information coming from the various H_α. Roughly the idea is this. Suppose we have a group G and a subgroup H which contains the maximal split torus of G. Suppose π is a representation of G. If we know bounds for K-finite matrix coefficients of π when restricted to H, then since H contains A, we get bounds for matrix coefficients of π. Oh's insight is that in the setup of the theorem the subgroups H_α provide the framework for applying such an idea. The proposition that makes this possible is the following general fact:

Proposition 2.2. — *Let* G *be a connected reductive group over a local field* F. *Let* $\mathsf{A}, \mathsf{B}, \mathsf{K}$ *be respectively a maximal split torus, a minimal parabolic subgroup containing* A, *and a good maximal compact subgroup of* $\mathsf{G}(F)$. *Further, for* $1 \leq i \leq k$, *let* H_i *be a connected reductive subgroup of* G *such that* $\mathsf{H}_i \cap \mathsf{A}$, $\mathsf{H}_i \cap \mathsf{B}$, *and* $\mathsf{H}_i(F) \cap \mathsf{K}$ *are respectively a maximal split torus, a minimal parabolic subgroup, and a good maximal compact subgroup of* H_i. *Suppose*

– *for all* $i \neq j$, $\mathsf{H}_i \leq C_{\mathsf{G}}(\mathsf{H}_j)$ *and* $\mathsf{H}_i(F) \cap \mathsf{H}_j(F)$ *is a finite subset of* $\mathsf{H}_i(F) \cap \mathsf{K}$.

– *for each* i, *there is a bi-*$\mathsf{H}_i(F) \cap \mathsf{K}$*-invariant function* ϕ_i *on* $\mathsf{H}_i(F)$ *such that for each non-trivial irreducible unitary representation* σ *of* $\mathsf{G}(F)$, *the bi-*$\mathsf{H}_i(F) \cap \mathsf{K}$*-finite matrix coefficients of* $\sigma|_{\mathsf{H}_i(F)}$ *are bounded by* ϕ_i.

Then for any unitary representation ρ *of* $\mathsf{G}(F)$ *without a non-zero invariant vector under* $\mathsf{G}^+(F)$ *and* K*-invariant unit vectors* v, w

$$\left|\langle \rho\left(c\prod_{i=1}^{k} h_i\right) v, w\rangle\right| \leq \prod_{i=1}^{k} \phi_i(h_i)$$

for $h_i \in \mathsf{H}_i(F)$ *and* $c \in \cap_{i=1}^{k} C_{\mathsf{G}(F)}(\mathsf{H}_i(F))$.

For applications of a similar strategy to related problems see [**13, 14**]. A couple of remarks are in order. Notice that this is a perfectly local statement and has nothing to do with automorphic representations. The theorem applies to infinite-dimensional representations. For this reason in practice when applying the result to local components of automorphic representations, one needs to make sure that for $v \notin S$, the local representations are in fact infinite-dimensional (for example see Proposition 4.4 of [20] where the required dimension result follows from the Strong Approximation). Second remark is that typically one also needs a similar bound on the spherical functions when the semi-simple rank is equal to one. In this case, local considerations do not suffice, as the trivial representation may not be isolated in the unitary dual of the local group. Given a semi-simple group H as above, we know there is a quasi-split group H' which is a global inner form of H. Note that by standard theorem in Galois cohomology [18], $\mathsf{H}(F_v)$ will be isomorphic to $\mathsf{H}'(F_v)$ for v outside of a finite set of places. In the application discussed in the introduction we need bounds for the matrix coefficients for almost all places. The point is that given a group H the local groups $\mathsf{H}(F_v)$ are rank one at a positive proportion of places only when H is related to either PGL_2 or $\mathsf{U}(2,1)$. For automorphic representations of $\mathsf{U}(2,1)$ we can use the results of Rogawski to transfer the representations to $\mathrm{GL}(3)$ where one can apply Oh's theorem. Also see the remarks regarding the Property τ in the introduction.

The application considered in [16] was to calculating Kazhdan constants of semi-simple groups. In the subsequent sections we will consider applications of more immediate arithmetic interest.

3. Applications to equidistribution of Hecke points

This is worked out in a wonderful paper of Clozel, Oh, and Ullmo [5], and was later generalized by Gan and Oh [10]. Let us first explain the setup of [5]. Let G be a connected almost simple simply connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. Let $a \in \mathsf{G}(\mathbb{Q})$. For $x \in \Gamma\backslash\mathsf{G}(\mathbb{R})$, we set $T_a x = \{[\Gamma a \Gamma x] \in \Gamma\backslash\mathsf{G}(\mathbb{R})\}$. We also define a *Hecke operator* T_a on $L^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$ as follows: for any $f \in L^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$,

$$T_a(f)(x) = \frac{1}{|T_a x|} \sum_{y \in T_a x} f(y).$$

Alternatively $T_a(f)$ can be described by

$$T_a(f)(x) = \frac{1}{\deg a} \sum_{i=1}^{\deg a} f(a_i x)$$

where $a_1, \dots, a_{\deg a}$ are representatives for the left action of Γ on $\Gamma a \Gamma$. The purpose of [5] is to obtain an estimate for the L^2-norm of the restriction of T_a to the orthogonal complement of constant functions, and to prove the equidistribution of the sets $T_a x$ as $\deg a \to \infty$ with rate estimates. For an explanation of the relevance of these results to concrete arithmetic problems, the reader is referred to [6]. For a more thorough exposition of the results of this section, and applications to the distribution of points on spheres, see [15]. For an alternative approach, see [8].

3.1. Adelic Hecke operators. — Our first purpose here is to give an adelic interpretation of the Hecke operators described above. Let G be a connected almost simple simply connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. . The adele group $\mathsf{G}(\mathbb{A})$ will be the restricted topological product of the group $\mathsf{G}(\mathbb{Q}_p)$ with respect to a collection of compact-open subgroups K_p for each finite prime p. If p is an unramified prime for G, we can take K_p to be a hyper-special maximal compact open subgroup of $\mathsf{G}(\mathbb{Q}_p)$. We may assume that for almost all p, $\mathsf{K}_p = \mathsf{G}(\mathbb{Z}_p)$ for a smooth model of G over $\mathbb{Z}[1/N]$ for some integer N. For each finite p, let U_p be a compact-open subgroup of $\mathsf{G}(\mathbb{Q}_p)$. We assume that for almost all p, $\mathsf{U}_p = \mathsf{K}_p$. Set $\mathsf{U}_f = \prod_p \mathsf{U}_p$. A subgroup Γ of the form $\mathsf{G}(\mathbb{Q}) \cap (\mathsf{G}(\mathbb{R}) \times \mathsf{U}_f)$ is called a *congruence subgroup*. It follows from the *strong approximation* that

$$\mathsf{G}(\mathbb{A}) = \mathsf{G}(\mathbb{Q})\mathsf{G}(\mathbb{R})\mathsf{U}_f.$$

As a result the spaces $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/\mathsf{U}_f$ and $\Gamma\backslash\mathsf{G}(\mathbb{R})$ are naturally identified, and there is a natural isomorphism

$$\phi : \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f} \to \mathsf{L}^2(\Gamma\backslash\mathsf{G}(\mathbb{R}))$$

given by $\phi(f)(x) = f(x, (e_p)_p)$. Here $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$ is the space of U_f-invariant functions.

For $a \in \mathsf{G}(\mathbb{Q})$, we set $\deg_p(a) = |\mathsf{U}_p\backslash\mathsf{U}_p a \mathsf{U}_p|$. Then it is seen that $\deg a = \prod_p \deg_p a$. We will now define a local Hecke operator $T_{a(p)}$ acting on the space of right U_f-invariant function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. If $\{a_1, \ldots, a_n\}$, $n = \deg_p(a)$, is a collection of representatives for $\mathsf{U}_p\backslash\mathsf{U}_p a \mathsf{U}_p$, we set

$$T_{a(p)}(f)((x_q)_q) = \frac{1}{\deg_p(a)} \sum_{i=1}^{\deg_p(a)} f((x_q)_{q\neq p}, (x_p a_i^{-1})).$$

Clearly $T_{a(p)}$ is independent of the choice of the representatives. Furthermore, given $a \in \mathsf{G}(\mathbb{Q})$, for almost all p, $T_{a(p)}$ is the identity operator. Then one can consider the operator $\hat{T}_a = \prod_p T_{a(p)}$ acting on $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$. Then one can show that for any $a \in \mathsf{G}(\mathbb{Q})$, we have

$$\phi(\hat{T}_a(f)) = T_a(\phi(f)).$$

3.2. Equidistribution. — As before let G be a connected almost simple simply connected linear algebraic group over $\mathbb{Q}$ with $\mathsf{G}(\mathbb{R})$ non-compact and $\Gamma \subset \mathsf{G}(\mathbb{Q})$ a congruence subgroup. Our equidistribution statement in this adelic language is the following statement:

Assertion 3.1. — *For* $f \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$, *we have*

$$\lim_{\deg a \to \infty} \hat{T}_a(f) = \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f \, d\mu.$$

Let $\mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$ be the orthogonal complement of the space of constant functions. The Hecke operators preserve this subspace. Let T^0_a be the restriction of $\hat{T}_a$ to L^2_0, and

$$\|T^0_a\| = \sup\{|\langle\hat{T}_a f, h\rangle| \mid f, h \in \mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}, \|f\| = \|h\| = 1\}.$$

Then for any $f \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{U}_f}$, we have

$$\|\hat{T}_a(f) - \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f\, d\mu\| \leq \|T^0_a\| \, . \, \|f - \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} f\, d\mu\| \leq \|T^0_a\| \, . \, \|f\|.$$

Consequently, in order to prove the assertion, it would suffice to find a bound for $\|T^0_a\|$ that would go to zero as $\deg a$ gets large. This also gives a rate for the convergence. There are also pointwise convergence statements which we will not discuss here.

We now describe the result that gives the connection between bounds for norms of Hecke operators and bounds for matrix coefficients. First a couple of definitions. If ρ_1, ρ_2 are two representations of $\mathsf{G}(\mathbb{Q}_v)$ for some place v, we say ρ_1 is *weakly contained in* ρ_2 if every diagonal matrix coefficient of ρ_1 can be uniformly approximated on compact sets by convex combinations of diagonal matrix coefficients of ρ_2. For each prime p, we let $\widehat{\mathsf{G}}_p$ be the unitary dual of $\mathsf{G}(\mathbb{Q}_p)$, and $\widehat{\mathsf{G}}^{aut}_p \subset \widehat{\mathsf{G}}_p$ the set of unitary representations that are weakly contained in the representations that occur as the p-components of $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})/\mathsf{O}_f)$ for some compact open subgroup $\mathsf{O}_f \subset \mathsf{G}(\mathbb{A}_f)$. Then we have the following elementary but crucial proposition:

Proposition 3.2. — *Let* G *be a connected almost simple simply-connected* $\mathbb{Q}$*-group with* $\mathsf{G}(\mathbb{R})$ *non-compact and* $\Gamma \subset \mathsf{G}(\mathbb{Q})$ *a congruence subgroup of the form* $\Gamma = \mathsf{G}(\mathbb{Q}) \cap (\mathsf{G}(\mathbb{R}) \times \prod_p \mathsf{U}_p)$*. Suppose that for each finite p, there exists a bi-*K_p*-invariant positive function* F_p *on the group* $\mathsf{G}(\mathbb{Q}_p)$ *such that for any non-trivial* $\rho_p \in \mathsf{G}^{aut}_p$ *with* K_p*-finite unit vectors* v, w*,*

$$|\langle\rho_p(g)v, w\rangle| \leq (\dim\langle\mathsf{K}_p v\rangle \dim\langle\mathsf{K}_p w\rangle)^{1/2} F_p(g)$$

for all $g \in \mathsf{G}(\mathbb{Q}_p)$*. Assume moreover that* $F_p(e) = 1$ *for almost all p. Then for any* $a \in \mathsf{G}(\mathbb{Q})$*,*

$$\|T^0_a\| \leq C \prod_p F_p(a),$$

with $C = \prod_p [\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p]$*.*

Before we give a sketch of the proof of the proposition, we note that Oh's result provides a function F_p whenever the $\mathbb{Q}_p$-rank of the group G is larger than or equal to two. For the places where the $\mathbb{Q}_p$-rank is one we need to use results towards the Ramanujan conjecture as usual.

We now give a sketch of the proof of the proposition. Let S be a large set of places that contains the archimedean place and such that for $v \notin S$, G is unramified over

$\mathbb{Q}_v$, and $\mathsf{U}_v = \mathsf{K}_v$. Let $\mathsf{G}_S = \prod_{v\in S} \mathsf{G}(\mathbb{Q}_v)$. As a G_S-module we write

$$\mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})) = \int_X m_x\rho_x \, d\nu(x)$$

where $X = \widehat{\mathsf{G}}_S$, $\rho_x = \prod_{v\in S} \rho_{x(v)}$ is irreducible, m_x is a multiplicity for each $x \in X$, and ν is a measure on X. Further, each $\rho_{x(v)}$ is an irreducible unitary representation of $\mathsf{G}(\mathbb{Q}_v)$. It follows from the strong approximation that for each $v \in S$, $\rho_{x(v)}$ is non-trivial for almost all $x \in X$. Set $\mathcal{L}_0 = \mathsf{L}^2_0(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$ as a $\mathsf{G}(\mathbb{A}_S)$-space. For each $x \in X$, we let $\mathcal{L}_x = \rho_x^{\oplus x}$ be the ρ_x-isotypic component of ρ_x. Then if $v = (v_x)_{x\in X}$ and $w = (w_x)_{x\in X}$, $v_x, w_x \in \mathcal{L}_x$, are elements of $\mathcal{L}_0$, we have

$$\langle v, w\rangle = \int_X \langle v_x, w_x\rangle \, d\nu(x).$$

The Hecke operator T^0_a acts on $\mathcal{L}_0^{\mathsf{U}_f}$ by the product $\prod_{p\in S} T_{a(p)}$ where each $T_{a(p)}$ acts as a local Hecke operator on the p-factor $\rho_{x(p)}^{\mathsf{U}_p}$ of ρ_x as follows: if v is U_p-invariant, then

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)}\rho_{x(p)}(\chi_{\mathsf{U}_p a \mathsf{U}_p})(v) = \frac{1}{\deg_p(a)} \int_{\mathsf{G}(\mathbb{Q}_p)} \chi_{\mathsf{U}_p a\mathsf{U}_p}(g)\rho_{x(p)}(g)(v)\, d\mu_p(g)$$

where μ_p is the Haar measure on $\mathsf{G}(\mathbb{Q}_p)$ with $\mu_p(\mathsf{U}_p) = 1$. Consequently, if $\{a_1, \ldots, a_r\}$, $r = \deg_p(a)$, is a collection of representatives for $\mathsf{U}_p\backslash\mathsf{U}_p a\mathsf{U}_p$, then

$$T_{a(p)}(v) = \frac{1}{\deg_p(a)} \sum_{i=1}^{r} \rho_{x(p)}(a_i)v.$$

Clearly, in order to prove the proposition it would suffice to show that for each finite $p \in S$, and for any U_p-invariant vectors v, w in the space $\rho_{x(p)}$

$$\langle T_{a(p)}v, w\rangle \leq [\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p]\, F_p(a)\, \|v\| \,.\, \|w\|.$$

But it is easy to see that $\langle T_{a(p)}v, w\rangle = \langle \rho_{x(p)}v, w\rangle$. Since for almost all x, $\rho_{x(p)}$ is non-trivial, and the dimension of $\mathsf{K}_p v$ and $\mathsf{K}_p w$ are bounded by $[\mathsf{K}_p : \mathsf{K}_p \cap \mathsf{U}_p]$, we get the result.

3.3. A generalization. — We now explain the results and methods of [10]. The setup is as follows: Let G be a connected reductive linear algebraic group over $\mathbb{Q}$, and let Z be the connected component of the center of G. We assume that $\mathsf{Z}\backslash\mathsf{G}$ is absolutely simple with $\mathbb{Q}$-rank at least one. Let $\overline{G} = \mathsf{Z}(\mathbb{R})^0\backslash\mathsf{G}(\mathbb{R})^0$, and $\mathsf{G}_{\mathbb{Q}} = \mathsf{G}(\mathbb{Q}) \cap \mathsf{G}(\mathbb{R})^0$. Let $\Gamma \subset \mathsf{G}_{\mathbb{Q}}$ be an arithmetic subgroup of $\mathsf{G}(\mathbb{R})^0$ such that $\Gamma = \mathsf{G}_{\mathbb{Q}} \cap \mathsf{U}$ for some compact open subgroup $\mathsf{U} = \prod_p \mathsf{U}_p$ of $\mathsf{G}(\mathbb{A}_f)$. We let $\overline{\Gamma}$ be the image of Γ in $\overline{G}$. $\overline{\Gamma}$ is a lattice in $\overline{G}$.

Assumption 3.3. — *We assume that*

$$\mathsf{G}(\mathbb{A}) = \mathsf{G}(\mathbb{Q})\mathsf{G}(\mathbb{R})^0\mathsf{U};$$

$$\mathsf{Z}(\mathbb{A}) = \mathsf{Z}(\mathbb{Q})\mathsf{Z}(\mathbb{R})^0(\mathsf{U} \cap \mathsf{Z}(\mathbb{A}_f)).$$

This assumption is satisfied if for example G is simply-connected and Γ is a congruence subgroup, or when G is $\mathbb{Q}$-split and $\Gamma = \mathsf{G}(\mathbb{R})^0 \cap \mathsf{G}(\mathbb{Z})$. Note that in the last example, G is canonically defined over $\mathbb{Z}$, and for that reason $\mathsf{G}(\mathbb{Z})$ makes sense.

Via the diagonal embedding $\mathsf{G}_\mathbb{Q}$ is viewed as a subgroup of $\mathsf{G}(\mathbb{A}_f)$. For $a \in \mathsf{G}_\mathbb{Q}$, we set

$$\mathsf{G}[a] = \mathsf{G}_\mathbb{Q} \cap \mathsf{U}a\mathsf{U}.$$

There is an obvious map from $\Gamma\backslash\mathsf{G}[a]$ to $\mathsf{U}\backslash\mathsf{U}a\mathsf{U}$, and this maps turns out to be a bijection. If we set

$$\deg(a) = |\Gamma\backslash\mathsf{G}[a]|,$$

$$\deg_p(a) = |\mathsf{U}_p\backslash\mathsf{U}_p a\mathsf{U}_p|,$$

then $\deg(a) = \prod_p \deg_p(a) < \infty$. For any function f on $\overline{\Gamma}\backslash\overline{G}$, we set

$$T_a(f)(g) = \frac{1}{\deg a}\sum_{y \in \Gamma\backslash\mathsf{G}[a]} f(yg).$$

$T_a(f)$ is independent of the choice of representatives for $\Gamma\backslash\mathsf{G}[a]$, and is again a function on $\overline{\Gamma}\backslash\overline{G}$. It is again seen that there are local and adelic Hecke operators with compatibility relations as above. As before we have an equidistribution statement as follows: For any $f \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ and $x \in \overline{\Gamma}\backslash\overline{G}$, we have

$$\lim_{\deg a \to \infty} T_a(f)(x) = \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_G(g).$$

It is also possible to give a rate for this. For simplicity we will describe the rate of convergence in the L^2-sense. Let R_1 (resp. R_2) be the collection of places where the $\mathbb{Q}_p$-rank of $\mathsf{Z}\backslash\mathsf{G}$ is equal to (resp. greater than) one. For each place p, let $\mathcal{S}_p$ be a maximal strongly orthogonal system of positive root for $\mathsf{G}(\mathbb{Q}_p)$ with respect to some maximal split torus A_p. Define a real valued function ξ on $\mathsf{G}(\mathbb{Q})$ by

$$\xi(g) = \prod_{p \in R_1} \xi_{\mathcal{S}_p}(g)^{\frac{1}{2}}.\prod_{p \in R_2} \xi_{\mathcal{S}_p}(g).$$

Then the first part of Theorem 3.7 of [10] asserts that there is a constant $C > 0$ such that for any $f \in \mathsf{L}^2(\overline{\Gamma}\backslash\overline{G})$ and $a \in \mathsf{G}(\mathbb{Q})$,

$$\|T_a(f) - \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_G(g)\|_2 \le C.\|f\|_2.\xi(a).$$

The proof of this theorem follows an argument similar to the theorem of [5] discussed above.

3.4. Homogeneous varieties. — This is considered in [10]. Let G and U satisfy the assumptions of 3.3, and let $\mathsf{H} \subset \mathsf{G}$ be a $\mathbb{Q}$-subgroup. Let $\overline{H}$ be the image of $\mathsf{H}(\mathbb{R})^0$ in $\overline{G}$. Assume that $\overline{\Gamma} \cap \overline{H}$ is a lattice in $\overline{H}$. Let $\mu_{\overline{H}}$ be the right $\mathsf{H}(\mathbb{R})^0$-invariant measure on $\overline{H}$ which gives $\overline{\Gamma} \cap \overline{H} \backslash \overline{H}$ volume 1. The measures $\mu_{\overline{G}}$ and $\mu_{\overline{H}}$ induce a unique $\mathsf{G}(\mathbb{R})^0$-invariant measure μ on the homogeneous space $\overline{H} \backslash \overline{G} \cong \mathbb{Z}(R)^0 \mathsf{H}(\mathbb{R})^0 \backslash \mathsf{G}(\mathbb{R})^0$. Given an integrable function with compact support on $\overline{H} \backslash \overline{G}$, we define a function F on $\overline{\Gamma} \backslash \overline{G}$ by

$$F(g) = \sum_{\gamma \in (\overline{\Gamma} \cap \overline{H}) \backslash \overline{\Gamma}} f(\gamma g). \tag{3.1}$$

Clearly, F is integrable and we have

$$\int_{\overline{\Gamma} \backslash \overline{G}} F(g)\, d\mu_{\overline{G}}(g) = \int_{\overline{H} \backslash \overline{G}} f(g)\, d\mu(g).$$

It is seen easily that F has compact support if and only if $\overline{\Gamma} \cap \overline{H}$ is cocompact in $\overline{H}$. Here too we have an equidistribution theorem, but only in the weak sense:

Assertion 3.4. — *Let f be an integrable function of compact support on $\overline{H} \backslash \overline{G}$, and let F be as above. Then*

(1) *For any $\psi \in C_c^\infty(\overline{\Gamma} \backslash \overline{G})$,*

$$\langle T_a F, \psi \rangle \to \langle \mu(f), \psi \rangle \quad \text{as } \deg a \to \infty.$$

(2) *For any $\psi \in C_c^\infty(\overline{\Gamma} \backslash \overline{G})$ and $a \in \mathsf{G}(\mathbb{Q})$,*

$$\langle T_a F - \mu(f), \psi \rangle \leq C_f \,.\, C_\psi \,.\, \xi(a^{-1})^\delta$$

with C_f, C_ψ constants depending on f and ψ respectively. Here $0 < \delta \leq 1$ with equality when $\overline{\Gamma} \cap \overline{H}$ is cocompact in $\overline{H}$.

Note that this is not a direct consequence of the equidistribution statement of 3.3 as the function F is not in general smooth of compact support. Statement (1) of the assertion is not hard though. Duality properties of Hecke operators imply that

$$\langle T_a F - \mu(f), \psi \rangle = \langle F, T_{a^{-1}} \psi - \mu_{\overline{G}}(\psi) \rangle.$$

The last integral is equal to

$$\int_{\overline{H} \backslash \overline{G}} f(g) \left(\int_{\overline{\Gamma} \cap \overline{H} \backslash \overline{H}} (T_{a^{-1}}(\psi)(hg) - \mu_{\overline{G}}(\psi))\, d\mu_{\overline{H}}(h) \right) d\mu(g).$$

As $\deg(a^{-1}) = \deg(a)$, we may apply the equidistribution of Hecke operators to ψ and use the dominated convergence theorem to get (1). The pointwise bounds alluded to in 3.3 also give (2) in the cocompact case. In the situation where $\overline{\Gamma} \cap \overline{H}$ is not cocompact in $\overline{H}$ the proof of (2) is much more involved and uses reduction theory and Siegel sets.

4. Applications to rational points

This is considered in [11] and [20]. The basic question is to understand the distribution of rational points on Fano varieties. There are a few conjectures concerning these varieties and their rational points; see [1] for a list. The class of varieties considered in these two papers is the class of *wonderful compactifications* of semi-simple groups of adjoint type ([7]) over number fields. One can in fact verify Manin's conjecture and its generalizations for this class of varieties. For the sake of this exposition we concentrate on a concrete special case of the theorem which is the following. Let G be an arbitrary semi-simple group of adjoint type over a number field F. Let $\varrho : \mathsf{G} \to \mathrm{GL}_N$ be an absolutely irreducible faithful representation of G defined over F. Consider the induced map $\bar{\varrho} : \mathsf{G}(F) \to \mathbb{P}^{N^2-1}(F)$ on rational points. Let H be an arbitrary height function on $\mathbb{P}^{N^2-1}(F)$. Define a counting function H_ϱ on $\mathsf{G}(F)$ by $H \circ \bar{\varrho}$.

Then we are interested in the asymptotic behavior of the following

$$N(T, \varrho, H) = |\{\gamma \in \mathsf{G}(F) \mid H_\varrho(\gamma) \leq T\}| \tag{4.1}$$

as $T \to \infty$. The theorem proved in [11, 20] in this particular case implies that the main term in the formula is of the form $CT^{\mathsf{a}_\varrho}(\log T)^{\mathsf{b}_\varrho - 1}$ with C of arithmetic-geometric nature, and a_ϱ and b_ϱ completely geometric. The approaches of [11] and [20] are different: [11] uses ergodic theoretic methods, while [20] uses height zeta functions and spectral methods. Both of them nonetheless use [16] in a substantial way. Below we sketch the two approaches and try to highlight where exactly Oh's result has been used. While the results of [11, 20] are very general, for this exposition we will explain the arguments in appropriate special cases to avoid technical problems as much as possible.

4.1. Spectral approach of [20]. — Here for the sake of exposition we will assume that G is an F-anisotropic inner form of a split semi-simple group of adjoint type defined over F.

We will now sketch the proof. Using Tauberian theorems one deduces the asymptotic properties of $N(T, \varrho, H)$ from the analytic properties of the *height zeta function*

$$\mathcal{Z}_\varrho(s) = \sum_{\gamma \in \mathsf{G}(F)} H_\varrho(\gamma)^{-s}.$$

Actually, we will use the function

$$\mathcal{Z}_\varrho(s, g) = \sum_{\gamma \in \mathsf{G}(F)} H_\varrho(\gamma g)^{-s}.$$

For $\Re(s) \gg 0$, the right hand side converges (uniformly on compacts) to a function which is holomorphic in s and continuous in g on $\mathbb{C} \times \mathrm{G}(\mathbb{A})$. Since G is F-anisotropic, $\mathsf{G}(F)\backslash\mathrm{G}(\mathbb{A})$ is compact, and if we assume that H_ϱ is right and left invariant under some compact-open subgroup K_0 of $\mathsf{G}(\mathbb{A}_f)$, we get

$$\mathcal{Z} \in \mathsf{L}^2(\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}))^{\mathsf{K}_0}.$$

Since G is anisotropic, we have

$$\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})) = (\widehat{\bigoplus_{\pi}}\mathcal{H}_\pi) \oplus (\bigoplus_{\chi}\mathbb{C}_\chi), \tag{4.2}$$

as a Hilbert direct sum of irreducible subspaces. Here the first direct sum is over infinite-dimensional representations of $\mathsf{G}(\mathbb{A})$ and the second direct sum is a sum over all automorphic characters of $\mathsf{G}(\mathbb{A})$. Consequently,

$$\mathsf{L}^2(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_0} = (\widehat{\bigoplus_{\pi}}\mathcal{H}_\pi^{\mathsf{K}_0}) \oplus (\bigoplus_{\chi}\mathbb{C}_\chi^{\mathsf{K}_0}), \tag{4.3}$$

a sum over representations containing a K_0-fixed vector (in particular, the sum over characters is *finite*). For each infinite-dimensional π occurring in (4.3) we choose an orthonormal basis $\mathcal{B}_\pi = \{\phi_\alpha^\pi\}_\alpha$ for $\mathcal{H}_\pi^{\mathsf{K}_0}$. We have next the following *Automorphic Fourier expansion*

$$\mathcal{Z}_\varrho(s,g) = \sum_{\pi}\sum_{\phi\in\mathcal{B}_\pi}\langle\mathcal{Z}_\varrho(s,.),\phi\rangle\phi(g) + \sum_{\chi}\langle\mathcal{Z}_\varrho(s,.),\chi\rangle\chi(g). \tag{4.4}$$

as an identity of L^2-functions. The first step is to show that the series on the right hand side is normally convergent in g for $\Re s \gg 0$. This follows from the convergence of the spectral zeta function of the Laplace operator. Consequently (4.4) is an identity of continuous functions. Then we can insert $g = e$ to obtain

$$\mathcal{Z}_\varrho(s) = \sum_{\pi}\sum_{\phi\in\mathcal{B}_\pi}\langle\mathcal{Z}_\varrho(s,.),\phi\rangle\phi(e) + \sum_{\chi}\langle\mathcal{Z}_\varrho(s,.),\chi\rangle. \tag{4.5}$$

We will have to establish a meromorphic continuation of the right hand side of (4.5) in order to obtain a proof of the main theorem.

The first step is to find a half plane to which the finite sum $\sum_\chi\langle\mathcal{Z}_\varrho(s,.),\chi\rangle$ has an analytic continuation, plus the determination of the right most pole. This involves a couple of steps. Let $\chi = \otimes_v\chi_v$ be a one-dimensional automorphic representation of G. Let G' be the split group of which G is an inner form. Then by general theory $\mathsf{G}(F_v)$ and $\mathsf{G}'(F_v)$ are isomorphic for almost all v. This gives a local character χ_v' for almost all v. Then one needs to show that there is an automorphic character χ' of G' such that for almost all v the local components of χ' are the χ_v'. This, via the Cartan decomposition, implies a regularization of $\langle\mathcal{Z}_\varrho(s,.),\chi\rangle$ by a product of Hecke L-functions. The Hecke L-functions that appear in this regularization are the compositions $\chi'\circ\check{\alpha}$ for various $\alpha\in\Delta$ (see below for notation).

The second step is to meromorphically continue the inner products $\langle\mathcal{Z}_\varrho,\phi\rangle$ and then show that the sum of the analytically continued functions is holomorphic in an appropriate domain that contains the domain of holomorphy of the sum over characters discussed above. A key ingredient is the computation of the individual inner products $\langle \mathcal{Z}_\varrho, \phi\rangle$. We have

$$\begin{aligned}\langle \mathcal{Z}_\varrho, \phi\rangle &= \int_{\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})} \mathcal{Z}_\varrho(s,g)\overline{\phi(g)}\,dg \\ &= \int_{\mathsf{G}(\mathbb{A})} H_\varrho(g)^{-s}\overline{\phi(g)}\,dg \\ &= \int_{\mathsf{G}(\mathbb{A})} H_\varrho(g)^{-s}\int_{\mathsf{K}_0}\overline{\phi(kg)}\,dk\,dg.\end{aligned}$$

Without loss of generality we can assume that

$$\mathsf{K}_0 = \prod_{v\notin S}\mathsf{K}_v \times \mathsf{K}_0^S,$$

for a finite set of places S. Here for $v \notin S$, K_v is a maximal special open compact subgroup in $\mathsf{G}(F_v)$. After enlarging S to contain all the places where G is not split, we can assume that $\mathsf{K}_v = \mathsf{G}(\mathcal{O}_v)$. In particular, for $v \notin S$ the local representations π_v are spherical. Thus we have a normalized local spherical function φ_v associated to π_v. We have assumed that each ϕ is right K_0-invariant. In conclusion,

$$\begin{aligned}\langle \mathcal{Z}_\varrho, \phi\rangle = \prod_{v\notin S}&\int_{\mathsf{G}(F_v)}\varphi_v(g_v)H_\varrho(g_v)^{-s}\,dg_v \\ &\times\int_{\mathsf{G}(\mathbb{A}_S)} H_\varrho(\eta(g_S)))^{-s}\int_{\mathsf{K}_0^S}\phi(k\eta(g_S))\,dk\,dg_S.\end{aligned}$$

(Here $\eta : \mathsf{G}(\mathbb{A}_S) \to \mathsf{G}(\mathbb{A})$ is the natural inclusion map.) The integral over $\mathsf{G}(\mathbb{A}_S)$ is easy to handle. Our main concern here is the first factor

$$I_\varrho^S(s) = \prod_{v\notin S}\int_{\mathsf{G}(F_v)}\varphi_v(g_v)H_\varrho(g_v)^{-s}\,dg_v. \tag{4.6}$$

Even though there are no non-trivial groups that are both split and anisotropic, we will explain the regularization of this expression in the situation where the group G is split. Let us introduce some notation:

Let G be a split semi-simple group of adjoint type over a number field F. Let T be a split torus in G, and B a Borel subgroup containing T. B then defines an ordering on the set of root, and this gives a set of simple roots $\Delta = \{\alpha_1,\ldots,\alpha_r\}$. Let 2ρ be the sum of all the positive roots, and define numbers κ_i by $2\rho = \sum_i \kappa_i\alpha_i$. As G is adjoint, there is a collection of one-parameter subgroups $\{\check{\alpha}_1,\ldots,\check{\alpha}_r\}$ such that $(\check{\alpha}_i,\alpha_j) = \delta_{ij}$, with δ_{ij} the Kronecker delta. Let S be a large finite set of places of F containing the places at infinity. In particular we assume that S is large enough so that if $v \notin S$, then $\mathsf{G}(\mathcal{O}_v)$ is a maximal compact subgroup of $\mathsf{G}(F_v)$ and satisfies the

Cartan and Iwasawa decompositions. In particular if for each place v, we let $\mathsf{S}(F_v)^+$ be the semi-group generated by $\{\check{\alpha}_1(\varpi_v), \dots, \check{\alpha}_r(\varpi_v)\}$, then

$$G(F_v) = \mathsf{G}(\mathcal{O}_v)\mathsf{S}(F_v)^+\mathsf{G}(\mathcal{O}_v).$$

This way for each $g \in \mathsf{G}(F_v)$ one gets an r-tuple of non-negatives integers $a_v(g) = (a_1(g), \dots, a_r(g)$. Let $\mathsf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$. We set

$$H_v(g, \mathsf{s}) = q_v^{<a_v(g), \mathsf{s}>},$$

where $< a_v(g), \mathsf{s} >= \sum_i a_i(g)s_i$. Note that for any height function H_ϱ, there is a sequence of integers $(u_1, \dots, u_r)$, depending on ϱ, such that if we set $\mathsf{s}_\varrho = (u_1 s, \dots, u_r s)$ then we have

$$H_\varrho(g)^s = H(g, \mathsf{s}_\varrho).$$

Let $\pi = \otimes_v \pi_v$ be an infinite-dimensional irreducible automorphic representation of $\mathsf{G}(\mathbb{A}_F)$ such that for $v \notin S$, π_v has a $\mathsf{G}(\mathcal{O}_v)$ fixed vector. Let φ_v be the normalized spherical function associated to π_v. For $\mathsf{s} \in \mathbb{C}^r$ we set

$$I_v(\mathsf{s}) = \int_{\mathsf{G}(F_v)} \varphi(g) H_v(\mathsf{s}, g)^{-1}\, dg,$$

and

$$I_S(\mathsf{s}) = \prod_{v \notin S} I_v(\mathsf{s}).$$

One of the main technical points of [20] is the proof of the existence of a $\mathsf{w} > 0$ such that $I_S(\mathsf{s})$ is holomorphic as a function of several variables on the open set $\mathcal{T}_{-\mathsf{w}} = \{\mathsf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r \,|\, \Re s_i > \kappa_i + 1 - \mathsf{w}\}$. One then proves, using the Cartan decomposition and some volume estimates, that in order to do this it suffices to consider the (simpler) function of several variables

$$I_S^{\mathtt{simple}}(\mathsf{s}) = \prod_{v \notin S} \left(1 + \sum_{i=1}^{r} q_v^{-s_i + \kappa_i} \varphi_v(\check{\alpha}_i(\varpi_v)) \right),$$

and prove its holomorphy on $\mathcal{T}_{-\mathsf{w}}$. For details please see [20], or [21] for the exposition of a simple case. It is clear that the result would follow if we knew that there is a universal constant C such that

$$|\varphi_v(\check{\alpha}_i(\varpi_v))| \leq C q_v^{-\mathsf{w}}. \tag{4.7}$$

Case 1: semi-simple rank 1. In this case $\mathsf{G} = \mathrm{PGL}(2)$ and any estimate towards the Ramanujan conjecture suffices.

Case 2: semi-simple rank > 1. First we use a strong approximation argument to show that for $v \notin S$, the representation π_v is not one-dimensional, unless π itself is one-dimensional (a similar argument appears in the work of Clozel and Ullmo [6]). Then we apply Oh's result.

Remark 4.1. — When we deal with arbitrary groups we will also need to consider the group U(3). Here we will need to use Rogawski's lifting $U(3) \to GL_3$. Then we will need the bounds on Langlands classes of cuspidal automorphic representations due to Luo, Rudnick and Sarnak, in addition to those mentioned in *Case 1* above.

The meromorphic continuation of the infinite sum over different automorphic representation again follows from analytic properties of the spectral zeta function.

4.2. Ergodic theory approach of [11]. — Here, for simplicity, we will assume that G is a connected split simple group of adjoint type of $\mathbb{Q}$-rank larger than two and $F = \mathbb{Q}$. Also assume that G is equipped with an appropriate $\mathbb{Z}$-structure. For simplicity we will further assume that H_ϱ is invariant under $\mathsf{K}_f = \prod_p \mathsf{G}(\mathbb{Z}_p)$. Set

$$B_T := \{g \in \mathsf{G}(\mathbb{A}) \mid H(g) \leq T\}. \tag{4.8}$$

Note that $N(T, \varrho, H) = |B_T \cap \mathsf{G}(\mathbb{Q})|$. Let τ be the Tamagawa measure of the group G, and set $\tau_{\mathsf{G}} = \tau(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))$. Then we will sketch the proof of the following theorem:

Theorem 4.2. — *We have*

$$|B_T \cap \mathsf{G}(\mathbb{Q})| \sim \frac{1}{\tau_{\mathsf{G}}}.\tau(B_T) \tag{4.9}$$

as $T \to \infty$.

One can in fact get error estimates too, but here we won't worry about that. After this theorem is proved, in order to get an asymptotic formula for $|B_T \cap \mathsf{G}(\mathbb{Q})|$ and consequently for $N(T, \varrho, H)$, one needs to find an asymptotic formula for $\tau(B_T)$. One can use a Tauberian arguments, using a theorem of [20], that

$$\tau(B_T) \sim C T^{\mathsf{a}_\varrho} (\log T)^{\mathsf{b}_\varrho - 1} \tag{4.10}$$

as $T \to \infty$. We will show that Theorem 4.2 follows from the following *mixing theorem*:

Theorem 4.3. — *Let* G *be a connected semi-simple split* $\mathbb{Q}$*-group. Then for any* $f_1, f_2 \in \mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ *we have*

$$\int f_1(h) f_2(hg)\, d\tau(h) \to \frac{1}{\tau_{\mathsf{G}}} \int f_1\, d\tau \,.\int f_2\, d\tau \tag{4.11}$$

as $g \to \infty$.

In the statement of the theorem, $\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ is the collection of L^2-functions that are right invariant under K_f, and $g \to \infty$ means the following. We say a sequence $\{g_i\}$ is going to infinity, if for every compact set Ω there is an N such that $g_i \notin \Omega$ for $i > N$. In this theorem too one can get error estimates.

Let us prove Theorem 4.2 assuming Theorem 4.3. We define a function $F_T(g, h)$ on $\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{A})$ by

$$F_T(g, h) = \sum_{\gamma \in \mathsf{G}(\mathbb{Q})} \chi_{B_T}(g^{-1}\gamma h). \tag{4.12}$$

Then clearly F_T descends to a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ which we will denote again by F_T. It is easily seen that

$$F_T(e,e) = |B_T \cap \mathsf{G}(\mathbb{Q})|. \tag{4.13}$$

Consequently the proof will be finished if we show

$$F_T(e,e) \sim \frac{1}{\tau_{\mathsf{G}}}\tau(B_T) \tag{4.14}$$

as $T \to \infty$. Theorem 4.3 is used to prove the following lemma:

Lemma 4.4. — *For any $\alpha \in C_c(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f\times\mathsf{K}_f}$, we have*

$$\lim_{T\to\infty} \frac{1}{\tau(B_T)} \int F_T.\alpha \, d(\tau\times\tau) = \frac{1}{\tau_{\mathsf{G}}} \cdot \int \alpha \, d(\tau\times\tau). \tag{4.15}$$

It suffices to prove the lemma for α of the form $\alpha_1 \otimes \alpha_2$. To prove the lemma for such functions, we do a straightforward unfolding to obtain

$$\int F_T.\alpha \, d(\tau\times\tau) = \int_{B_T} \langle \alpha_1, g.\alpha_2\rangle \, d\tau(g). \tag{4.16}$$

Since the height function H_ϱ is proper, $g \to \infty$ if an only if $H_\varrho(g) \to \infty$. Hence by the Mixing Theorem for any $\epsilon > 0$ there is $T_0 > 0$ such that

$$\left| \langle \alpha_1, g.\alpha_2\rangle - \frac{1}{\tau_{\mathsf{G}}} \cdot \int \alpha \, d(\tau\times\tau) \right| < \epsilon \tag{4.17}$$

whenever $H(g) > T_0$. This easily implies the lemma.

To continue, we make the observation that the balls B_T are *asymptotically well-rounded* in the following sense: there exist constants $a_\epsilon \geq 1$ and $b_\epsilon \leq 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$ we have

$$b_\epsilon \leq \liminf_T \frac{\tau(B_{(1-\epsilon)T})}{\tau(B_T)} \leq \limsup_T \frac{\tau(B_{(1+\epsilon)T})}{\tau(B_T)} \leq a_\epsilon. \tag{4.18}$$

Fix $\epsilon > 0$. Let Ω_ϵ be a symmetric neighborhood of e in $\mathsf{G}(\mathbb{R})$ such that

$$B_T\Omega_\epsilon \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g\in\Omega_\epsilon} B_T.g \tag{4.19}$$

for all T large. Then if we set $\Omega = \Omega_\epsilon \times \mathsf{K}_f$, we have

$$F_{(1-\epsilon)T}(g,h) \leq F_T(e,e) \leq F_{(1+\epsilon)T}(g,h) \tag{4.20}$$

for all $g, h \in \Omega$. Now let $\psi \in C_c(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))^{\mathsf{K}_f}$ be a non-negative function with support contained in Ω and such that $\int \psi \, d\tau = 1$. Set $\alpha = \psi \otimes \psi$ as a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. Then (4.20) implies

$$\langle F_{(1-\epsilon)T}, \alpha\rangle \leq F_T(e,e) \leq \langle F_{(1+\epsilon)T}, \alpha\rangle. \tag{4.21}$$

Now Lemma 4.4 combined with (4.18) implies that

$$\frac{b_\epsilon}{\tau_{\mathsf{G}}} \leq \liminf_T \frac{F_T(e,e)}{\tau(B_T)} \leq \limsup_T \frac{F_T(e,e)}{\tau(B_T)} \leq \frac{a_\epsilon}{\tau_{\mathsf{G}}}. \tag{4.22}$$

Letting $\epsilon \to 0$ proves (4.14) and consequently Theorem 4.2.

It remains to prove Theorem 4.3. We define a bi-K-invariant function on $\mathsf{G}(\mathbb{A})$ by

$$\xi((g_p)_p) = \prod_p \xi_{\mathcal{S}_p}(g_p)$$

for $(g_p)_p \in \mathsf{G}(\mathbb{A})$. Then $\xi(g) \to 0$ when $g \to \infty$. Then Theorem 4.3 is a consequence of the following theorem which is a consequence of Oh's theorem.

Theorem 4.5. — *Let* G *be as above, and let* π *be an automorphic representation in the orthogonal complement to the one-dimensional representations. Then for any* K*-finite unit vectors* v, w *we have*

$$|\langle \pi(g)v, w\rangle| \leq c_0 \,.\, \xi(g) \,.\, (\dim \mathsf{K}v \,.\, \dim \mathsf{K}w)^{\frac{1}{2}}$$

for a constant c_0 *which depends only on the group* G.

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