Arithmetic aspects of the theta correspondence

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Abstract

We review some recent results on the arithmetic of the theta correspondence for certain symplectic-orthogonal dual pairs and some applications to periods and congruences of modular forms.

Contents

1 Introduction

The theta correspondence provides a very important method to transfer automorphic forms between different reductive groups. Central to the theory is the important notion of a dual reductive pair. This is a pair of reductive subgroups G and G' contained in an ambient symplectic group H that happen to be the centralizers of each other in H. In such a situation, for every choice of additive character ψ of $\mathbb{A}/\mathbb{Q}$ and for automorphic representations π, π' on G, G' respectively, one may define theta lifts $\Theta(\pi, \psi), \Theta(\pi', \psi)$ that (if non-zero) are automorphic representations on G', G respectively ([9].) In the automorphic theory, it is an important and subtle question to characterize when the lift is non-vanishing. For instance, the non-vanishing could depend on both local conditions (compatibility of ε-factors) and global conditions (non-vanishing of an L-value.)

The theta lift has its genesis in the Weil representation of $H(\mathbb{A})$ on a certain Schwartz space $\mathcal{S}(\mathbb{A})$. For any choice of Schartz function $\varphi \in \mathcal{S}(\mathbb{A})$ and vector $f \in \pi$ one may consider the theta lift $\theta(f, \varphi, \psi)$ which is an element of $\Theta(\pi, \psi)$. Now it is often the case that one can define good notions of arithmeticity for elements of π and $\Theta(\pi, \psi)$. Arithmeticity here could mean algebraicity, rationality over a suitable number field or even p-adic integrality. The fundamental problem in the arithmetic theory of the theta correspondence is the following:

Question A: Suppose f is chosen to be arithmetic. For a given canonical choice of φ, is $\theta(f, \varphi, \psi)$ is arithmetic (perhaps up to some canonical transcendental period) ?

This question has been studied in great detail by Shimura ([15], [16], [17], [18]), Harris ([3]), and Harris-Kudla ([4],[5]) in the case of algebraicity and in some cases rationality over suitable number fields. But the study of such questions up to p-adic integrality is much more recent. To the authors knowledge, the only results on p-adic integrality that have appeared in the literature are the following.
(i) $(GU(2), GU(3))$ (Finis [2])
(ii) $(GL(2), GO(B))$ for B a quaternion algebra. (see [11] for the indefinite case with square-free level over $\mathbb{Q}$, work of Emerton [1] for the definite case at prime level over $\mathbb{Q}$ and Hida [8] for the definite case at full level over totally real fields.)
(iii) $(U(n), U(n+1))$ (Harris, Li and Skinner [6], [7])
(iv) $(\widetilde{SL}(2), O(V))$, for V the space of trace 0 elements in an indefinite quaternion algebra over $\mathbb{Q}$. This case and applications are treated in forthcoming work of the author ([12], [13], [14].)

In all these cases, there seem to be intimate connections with Iwasawa theory. For instance, (ii) and (iv) use crucially the main conjecture of Iwasawa theory for imaginary quadratic fields, which is a deep theorem of Rubin. The work of Harris, Li and Skinner has as an application the construction of p-adic L-functions for unitary groups and one divisibility of an associated main conjecture. It is certainly to be expected that other cases of the theta correspondence will yield other applications to Iwasawa theory. In addition to this, one also gets interesting applications to the study of special values of L-functions and integral period relations for modular forms, on which more will be said later.

In the integral theory, one may pose a more refined version of Question A:

Question B: Suppose that the form f is a p-unit. Is $\theta(f, \varphi, \psi)$ a p-unit ? If not, what can be said about the primes p for which $\theta(f, \varphi, \psi)$ has positive p-adic valuation ?

Question B is undoubtedly more difficult than Question A and the answer seems to involve certain kinds of congruences of modular forms and μ-invariants of p-adic L-functions. It is also closely related to the classical question of whether certain spaces of modular forms are (integrally) spanned by theta series.

In this article, we will focus only on cases (ii) and (iv), partly out of the author's lack of knowledge of the other cases. Before we proceed with examples of individual cases, we begin by describing some questions regarding periods of modular forms that motivate the study of arithmeticity of the theta correspondence.

Note and caution: In order to keep the exposition simple, we will ignore many terms in the formulae that appear below. For instance, we ignore powers of π (3.1415...), other explicit constants, abelian L-functions etc. Since we will be interested mostly in p-integrality, we use the symbol $\sim$ instead of $=$ to denote equality up to elements that are units at all places above p. The reader may be rest assured that every formula that occurs below may be worked out precisely, so that $\sim$ may be replaced by $=$ after throwing in the appropriate constants and terms that we have neglected in the present exposition.

2 Periods of modular forms

Let f be a holomorphic newform of weight $2k$ on $\Gamma_0(N)$ and K_f the field generated by its Hecke eigenvalues. We assume N is square-free and that we have picked a factorization $N = N^+N^-$ such that N^- is the product of an even number of primes. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ ramified precisely at the primes dividing N^- and g the Jacquet-Langlands lift of f to the Shimura curve X of level N^+ coming from B. We assume that $p \nmid N$ and normalize g (up to a p-adic unit in K_f) using the integral structure provided by sections of the relative dualizing sheaf on the minimal regular model of X at p. Let F be a field containing K_f and if $2k > 2$ we also assume that B splits over F. It is possible then to attach to f and g certain canonical periods $u_\pm(f), u_\pm(g)$ that are well defined up to p-adic units in F. (See [12] for instance for a definition.) In good situations, namely if certain Hecke modules constructed from cohomology groups are free, one may show that these periods are related to the usual Petersson inner product by

$$\langle f, f\rangle \sim \delta_f \cdot u_+(f) \cdot u_-(f)$$
$$\langle g, g\rangle \sim \delta_g \cdot u_+(g) \cdot u_-(g)$$

Here δ_f is a p-integer that measures congruences between f and other forms on $X_0(N)$. Likewise δ_g is a p-integer that measures congruences between g and other forms on X.

Example 2.1 If $2k = 2$ and $K_f = \mathbb{Q}$, we may pick $F = \mathbb{Q}$. Then f and g correspond to elliptic curves E and E' over $\mathbb{Q}$ that are *strong elliptic curves* for $X_0(N)$ and X respectively i.e. there are modular parametrizations $X_0(N) \to E, X \to E'$ and the dual maps to the induced map $J_0(N) \to E, Jac(X) \to E'$ are injective. In this case, $u_\pm(f), u_\pm(g)$ agree with the usual $\pm$ periods of E, E' respectively.

Suppose p is not an Eisenstein prime for f i.e. the mod p Galois representation associated to E is irreducible. Then, by Faltings' isogeny theorem, one may find an isogeny $E \to E'$ of degree prime to p. It follows that

$$u_+(f) \sim u_+(g)$$
$$u_-(f) \sim u_-(g)$$

and hence

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim \frac{\delta_f}{\delta_g}$$

The number δ_f/δ_g should count congruences between f and forms that do not transfer to the quaternion algebra B. We call such congruences *level-lowering congruences* for f at B.

Our motivation is in proving such results for forms of arbitrary weight. We now make the following assumptions for the rest of this article:

Assumption I: $p > 2k + 1$
Assumption II: $p \nmid \tilde{N} := \prod_{q|N} q(q+1)(q-1)$

It can be shown that assumption II implies in particular that the following condition (*) is satisfied by p.

Condition (*) There exist infinitely many imaginary quadratic fields K that satisfy any prescribed set of splitting conditions at the primes dividing N, are split at p and have class number prime to p.

3 Arithmeticity of theta lifts

3.1 The pair $(GL_2, GO(B))$.

Let B be a quaternion algebra over $\mathbb{Q}$ with discriminant N^- dividing N. We fix isomorphisms

$$\Phi_q : B \otimes \mathbb{Q}_q \simeq M_2(\mathbb{Q}_q)$$

for $q \nmid N^-$. If B is unramified at infinity, we fix an isomorphism

$$\Phi_\infty : B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$$

In the case when B is ramified at infinity, we first pick a model for B,

$$B = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$$

where $a^2 = -N^-$, $b^2 = -l$ and $ab = -ba$. Here $N^- = \mathrm{disc}(B)$ and l is an auxiliary prime chosen such that

$$\begin{aligned} \left(\frac{-l}{q}\right) &= -1 \text{ if } q \mid N^- \text{ and } q \text{ is odd} \\ l &\equiv 3(\mod 8) \end{aligned}$$

Let $\mathcal{O}'$ be the maximal order in B given by

$$\mathcal{O}' = \mathbb{Z} + \mathbb{Z}\frac{1+b}{2} + \mathbb{Z}\frac{a(1+b)}{2} + \mathbb{Z}\frac{(r+a)b}{l}$$

where r is any integer satisfying $r^2 + N^- \equiv 0 \mod l$. We may assume in this case that the isomorphisms Φ_q are chosen such that $\Phi_q(\mathcal{O}') = M_2(\mathbb{Z}_q)$ for $q \nmid N^-$.

Let $\mathbb{H}$ be the division algebra of Hamilton quaternions given by

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

with the relations $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ and fix an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \to \mathbb{H}$ characterized by

$$\Phi_\infty : a \to \sqrt{N^-}j, b \to \sqrt{l}i$$

Note that we can identify the subalgebra of elements of the form $a + bi$ in $\mathbb{H}$ with the field $\mathbb{C}$ of complex numbers. Further note that $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. Let us fix an isomorphism

$$\rho : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$$

characterized by

$$\rho(\gamma + \delta j) = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix}$$

for $\gamma, \delta \in \mathbb{C}$. We denote by the same symbol ρ the composite map $(\rho \otimes 1) \circ \Phi_\infty : B_\infty^\times \to GL_2(\mathbb{C})$. Let F be the subfield of $\mathbb{C}$ generated by $\sqrt{N^-}$ and $\sqrt{-l}$ and $R_0 = \mathcal{O}_{F,(l)}$ the subring of F obtained from $\mathcal{O}_F$ by inverting l. Then one checks immediately that

$$\rho(\mathcal{O}') \subset M_2(R_0)$$

We consider B as a quadratic space over $\mathbb{Q}$, the quadratic form being the reduced norm. Let $\mathrm{GO}(B)$ denote the corresponding orthogonal similitude group. One has a surjection $\rho : B^\times \times B^\times \to \mathrm{GO}(B)^0$ onto the connected component of $\mathrm{GO}(B)$, given by $(\gamma_1, \gamma_2) \mapsto (x \rightsquigarrow \gamma_1 x \gamma_2^{-1})$, the kernel being a copy of $\mathbb{G}_m$ embedded diagonally. Then there are theta lifts

$$\Theta(\cdot, \psi) : \mathcal{A}_0(G) \to \mathcal{A}_0(G')$$
$$\Theta^t(\cdot, \psi) : \mathcal{A}_0(G') \to \mathcal{A}_0(G)$$

for $G = \mathrm{GL}_2, G' = \mathrm{GO}(B)^0$ (see [4], [5], [11] for more details). Note that via ξ, automorphic representations of G' can be identified with pairs (π_1, π_2), the π_i being representations of $B^\times$ such that $\xi_{\pi_1} \cdot \xi_{\pi_2} = 1$. Here ξ_{π_i} denotes the central character of π_i.

Let π' denote an automorphic cuspidal representation of $B^\times$ and set $\pi = \mathrm{JL}(\pi')$.

Theorem 3.1 *(Shimizu)*
1. $\Theta(\pi, \psi) = \pi_B \times \overline{\pi_B}$
2. $\Theta^t(\pi_B \times \overline{\pi_B}, \psi) = \pi$

Suppose now that π corresponds to a holomorphic newform f of weight $2k$ on $\Gamma_0(N)$ with N square-free. We assume that f has first Fourier coefficient equal to 1 and denote by the same symbol f the corresponding adelic automorphic form. On $B^\times$, there is no theory of q-expansions and it is not clear how one might pick a canonical element of π_B analogous to the element f in π. However, the situation can be partially remedied as follows. The representation π_B is a restricted tensor product $\pi_B \simeq \otimes_v \pi_{B,v}$ of local representations. For finite primes $v = q$ such that B is split at q, let g_q be a local new vector in $\pi_{B,q}$ as given by Casselman's theorem i.e. g_q is nonzero and invariant under the action of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_q), c \in N\mathbb{Z}_q \right\}$$

Here we have identified $(B\otimes\mathbb{Q}_q)^\times$ with $GL_2(\mathbb{Q}_q)$ via the isomorphism Φ_q. For finite primes v such that B is ramified at v, the local representation $\pi_{B,v}$ is one dimensional since π_v is Steinberg. In this case, we pick g_v to be any nonzero vector in $\pi_{B,v}$. Finally for $v=\infty$, there are two cases according as B is split or ramified at infinity. In the former case, we pick g_∞ to be the unique nonzero vector up to scaling on which $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by $e^{2ik\theta}$. In the latter case, one has that the representation $\pi_{B,\infty}$ is isomorphic to

$$\rho_k : B_\infty^\times \to GL_{2k-1}(\mathbb{C}), \qquad \rho_k = \mathrm{Sym}^{2k-2}\rho\otimes(\det\rho)^{1-k}$$

Let $V_1=\mathbb{C}^2$ be the representation space associated to ρ and denote by e_1, e_2 the standard basis. Then the set of vectors $e_1^{\otimes r}\otimes e_2^{\otimes 2k-2-r}, 0\le r\le 2k-2$ is a basis for V_k, the representation space of ρ_k. Fixing an isomorphism between $\pi_{B,\infty}$ and V_k, we pick $g_\infty^l = e_1^{\otimes r}\otimes e_2^{\otimes 2k-2-r}$. Notice that g_∞^r spans the unique line in $\pi_{B,\infty}$ on which $e^{i\theta}\in\mathbb{C}^{(1)}$ acts by $e^{2i(r-(k-1))\theta}$. Thus if B is indefinite, $g_B=\otimes_v g_v$ in π_B is well defined up to scaling, while if B is definite, the vector of forms $[g_B^r]$, with $g_B^r = \otimes_{v<\infty} g_v\otimes g_\infty^r$ in π_B is well defined up to scaling. We will see below that for a given prime p, we can pick g (resp. $[g_B^r]$) in such a way that it is well defined up to a p-adic unit in K_f.

We will now pick a Schwartz function φ (resp. φ^r) in $\mathcal{S}(B_\mathbb{A})$ such that $\theta(f,\varphi,\psi)=\beta g_B$ (resp. $\theta(f,\varphi^l,\psi)=\beta_l g_B^r$) in the indefinite (resp. definite) case for some scalars β,β_l. Suppose that $\mathrm{disc}\,B = N^-$, $N=N^+N^-$ and let $\mathcal{O}$ be the unique Eichler order of level N^+ in B such that

$$\Phi_q(\mathcal{O}\otimes\mathbb{Z}_q)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{Z}_q), c\in N^+\mathbb{Z}_q\right\}\ \text{for } q\nmid N^-$$

Note that for $q\mid N^-$, $\mathcal{O}\otimes\mathbb{Z}_q$ is just the unique maximal order in $B\otimes\mathbb{Q}_q$. Now for $v=q$ a finite prime, set $\varphi_q=\mathbb{I}_{\mathcal{O}\otimes\mathbb{Z}_q}$. If $v=\infty$ and B is indefinite, set

$$\varphi_\infty(\beta)=\frac{1}{\pi}Y(\beta)^k e^{-2\pi(|X(\beta)|^2+|Y(\beta)|^2)}$$

where for $\beta=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R})$, $X(\beta)=\frac{1}{2}(a+d)+\frac{i}{2}(b-c)$ and $Y(\beta)=\frac{1}{2}(a-d)+\frac{i}{2}(b+c)$. As usual we have identified $B\otimes\mathbb{R}$ in this case with $M_2(\mathbb{R})$ via Φ_∞. If B is definite, Φ_∞ identifies $B\otimes\mathbb{R}$ with the space of Hamilton quaternions. Set

$$\begin{aligned}\varphi_\infty^r(u+vj) &= \bar{v}^{2l}p_{k-1-|l|}(|u|^2)e^{-2\pi(|u|^2+|v|^2)},\ \text{if } l\ge 0\\ &= v^{2|l|}p_{k-1-|l|}(|u|^2)e^{-2\pi(|u|^2+|v|^2)},\ \text{if } l\le 0\end{aligned}$$

where $l=k-1-r$ and p_m is the Laguerre polynomial of degree m, given by

$$p_m(t)=\sum_{j=0}^m\binom{l}{j}\frac{(-t)^j}{j!}$$

Finally, set $\varphi^r = \otimes_v \varphi_v \otimes \varphi^r_\infty$.

The following proposition follows easily from computations in [23] and [24].

Proposition 3.2 *Suppose that B is indefinite (resp. definite.) Let* $\delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in B^\times_\infty$ *(resp.* $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in B^\times_\infty$*.) Then*

$$\theta(f,\varphi,\psi)(x\cdot\delta) = \overline{\theta(f,\varphi,\psi)(x)}$$

Further, there exist nonzero scalars α,β (resp. α_r,β) such that
(a) $\theta(f,\varphi,\psi) = \beta\cdot(g_B\times g_B)$ (resp. $[\theta(f,\varphi^r,\psi)] = \beta\cdot[g^r_B\times g^r_B]$.)
(b) $\theta^t(g_B\times g_B,\varphi,\psi) = \alpha f$ (resp. $\theta^t(g^r_B\times g^r_B,\varphi^r,\psi) = \alpha_r f$.)

Note that by our assumption that f occurs on $\Gamma_0(N)$, π and π_B both have trivial central character, hence $\overline{\pi_B} = \pi_B$. If B is indefinite (resp. definite) let Ψ (resp. $\tilde\Psi$) be such that $\theta(f,\varphi,\psi) = \Psi\times\Psi$ (resp. $[\theta(f,\varphi^r,\psi)] = \tilde\Psi\times\tilde\Psi$.)

In the following discussion we write $\theta(f)$ instead of $\theta(f,\varphi,\psi)$ for simplicity of notation. There are two important formulae that are very useful in this situation, namely see-saw duality ([9], [5]) and the Rallis inner product formula. In the indefinite case, applying see-saw duality gives

$$\begin{aligned} \langle\theta(f), g_B\times g_B\rangle &= \langle f, \theta^t(g_B\times g_B)\rangle \\ \text{i.e. } \beta\langle g_B, g_B\rangle^2 &= \overline{\alpha}\langle f,f\rangle \end{aligned} \tag{1}$$

Next the Rallis inner product formula gives

$$\begin{aligned} \langle\theta(f),\theta(f)\rangle &\sim L(1,ad^0(\pi))\langle f,f\rangle \\ \text{i.e. } \beta\overline{\beta}\langle g_B,g_B\rangle^2 &\sim \langle f,f\rangle^2 \end{aligned} \tag{2}$$

Combining (1) and (2) yields

$$\overline{\alpha}\overline{\beta} \sim \langle f,f\rangle \tag{3}$$

and

$$\alpha\overline{\alpha} \sim \langle g_B,g_B\rangle^2 \tag{4}$$

Clearly, exactly the same formulas hold also in the definite case, with α,β,g_B being replaced by α_r,β,g^r_B respectively. In particular from (3) we see that $\overline{\alpha_r}\sim\overline{\alpha_{r'}}$ and hence $\alpha_r\sim\alpha_{r'}$. This implies also that $\langle g^r_B,g^r_B\rangle \sim \langle g^{r'}_B, g^{r'}_B\rangle$.

3.1.1 The indefinite case

Suppose that $N = N^+N^-$ with N^- a product of an even number of primes and let B be the unique (up to isomorphism) indefinite quaternion algebra with discriminant N^-. Let $\mathcal{O}$ be an Eichler order of level N^+ in B. The form g_B that we picked in the previous section corresponds in the usual way to a classical modular form on the upper half plane $\mathfrak{H}$ (also denoted g_B) with respect to the group $\mathcal{O}^{(1)}$ consisting of the elements of $\mathcal{O}$ with reduced norm 1. Further we may view $\varsigma = g_B(z)(2\pi i \cdot dz)^{\otimes k}$ as being a section of the line bundle Ω^k on the curve $X = \mathfrak{H}/\mathcal{O}^{(1)}$. One knows from work of Shimura that the curve X admits a canonical model $X_{\mathbb{Q}}$ over $\mathbb{Q}$. Let $\mathcal{X}$ denote the minimal regular model of X over $\mathcal{O}_{K_f}$ and denote by ω the relative dualizing sheaf on $\mathcal{X}/\operatorname{spec}\mathcal{O}_{K_f}$. Since the Hecke eigenvalues of g_B lie in K_f, we may choose g_B such that $\varsigma \in H^0(X_{K_f}, \Omega^k)$ and further such that ς is a p-unit in $H^0(\mathcal{X}, \omega^k)$. Thus g_B is well defined up to a p-unit in K_f. Fixing such a choice of g_B, one has

Theorem 3.3 *(Harris-Kudla [4]) $\beta \in K_f$. Consequently $\langle f, f\rangle/\langle g_B, g_B\rangle \in K_f$.*

Indeed, since K_f is totally real, one gets from (2) that $\beta \cdot \langle g_B, g_B\rangle \sim \langle f, f\rangle$ so that

$$\beta \sim \frac{\langle f, f\rangle}{\langle g_B, g_B\rangle}$$

We are now in a situation where Questions A and B of the introduction make sense, namely we can ask for information about $v_\lambda(\beta)$ for λ a prime in K_f above p. The answer is provided by the following theorem and corollary which constitute the main results of [11].

Theorem 3.4 *(a)*

$$v_\lambda(\beta) = \min_{K,\chi} v_\lambda(\delta(\pi, K, \chi))$$

where

$$\delta(\pi, K, \chi) := \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{\Omega_K^{4k}}$$

Here K ranges over imaginary quadratic field that are split at N^+ and inert at N^-, χ ranges over unramified Hecke characters of K of type $(k, -k)$ at infinity and Ω_K is a suitable CM period i.e. a period of a Neron differential on an elliptic curve that has CM by $\mathcal{O}_K$.

(b)

$$v_\lambda(\delta(\pi, K, \chi)) \geq 0$$

for all K, χ as above. Further if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\delta(\pi, K, \chi)) > 0$.

(c) $v_\lambda(\beta) > 0$. Further, if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\beta) > 0$.

The reader will note that part (c) of the theorem follows immediately from parts (a) and (b). The proof of part (b) uses the Iwasawa *main conjecture* for K (a theorem of Rubin); we refer the reader to [11] for more details. We now indicate briefly some of the ingredients in the proof of part (a). For K, χ as above, pick a *Heegner* embedding $K \hookrightarrow B$ and set

$$L_\chi(g_B) = j(\alpha, i)^{2k} \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B(x\alpha)\chi(x) d^\times x$$

for any $\alpha \in SL_2(\mathbb{R})$ such that $\alpha^{-1} \cdot (K \otimes \mathbb{R}) \cdot \alpha = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$. For a suitable choice of measure on $K_{\mathbb{A}}^\times$, the integral above can be interpreted as a sum of values of g_B at certain CM points associated to K twisted by the values of χ divided by the class number h_K. Since a λ-integral modular form must take λ-integral values at CM points (up to suitable CM periods), one can show roughly that

$$\min_{K, \chi, p \nmid h_K} v_\lambda \left(\frac{L_\chi(g_B)}{\Omega_K^{2k}} \right) = 0 \tag{5}$$

On the other hand, by methods of Waldspurger one can show that

$$\beta L_\chi(g_B)^2 = L_{\chi \times \chi}(\theta(f)) \sim \frac{1}{h_K^2} L(\frac{1}{2}, \pi_K \otimes \chi) \tag{6}$$

Part (a) of the theorem follows now by combining (5) and (6).

3.1.2 The definite case

Now suppose that $N = N^+ N^-$, where N^- is the product of an odd number of primes and suppose that $\operatorname{disc} B = N^-$. Thus B is a definite quaternion algebra. As before, let $\mathcal{O}$ be an Eichler order of level N^+ in B. Recall that for every integer r satisfying $0 \leq r \leq (2k-2)$, we have picked a form g_B^r on $B_{\mathbb{A}}^\times$ such that the vector of forms $[g_B^r]$ is well defined up to a scalar. Set $\tilde{g}_B = [g_B^r]^t$, so that $\tilde{g}_B \in \tilde{S}_k(U)$ where

$$\tilde{S}_k(U) = \{\tilde{g} : B^\times \backslash B_{\mathbb{A}}^\times \to \mathbb{C}^{2k-1} \ \mid \ \tilde{g}(x \cdot u u_\infty) = \rho_k(u_\infty)\tilde{g}(x) \ \forall \ u \in U, u_\infty \in B_\infty^\times\}$$

and $U = \prod_q U_q$ is the open compact subgroup of $B_{\mathbb{A}_f}^\times$ given by $U_q = (\mathcal{O}\otimes\mathbb{Z}_q)^\times$.

Since any element of $\tilde{S}_k(U)$ is determined by its values on $B_{\mathbb{A}_f}^\times$, the space $\tilde{S}_k(U)$ is canonically isomorphic to the space $S_k(U)$ given by

$$S_k(U) = \{g : B_{\mathbb{A}_f}^\times/U \to \mathbb{C}^{2k-1} \ | \ g(\alpha\cdot x) = \rho_k(\alpha^{-1})g(x) \ \forall \alpha \in B^\times\}$$

Denote by g_B the element of $S_k(U)$ corresponding to $\tilde{g}_B$. We now follow [19] in defining a integral (or rather p-integral) structure on $S_k(U)$. For R any ring such that $R_0 \subset R \subset \mathbb{C}$, let $L_k(R)$ be the R-submodule of $\mathbb{C}^{2k-1}$ consisting of vectors all whose components are in R. The group $B_{\mathbb{A}_f}^\times$ acts on R_0-lattices in $L_k(K)$ via the embedding

$$B_{\mathbb{A}_f}^\times \hookrightarrow (B\otimes\mathbb{A}_{K,f})^\times \xrightarrow{\mu_k\otimes 1} GL_{2k-1}(\mathbb{A}_{K,f})$$

Set $L_k(R)\cdot x := L_k(\mathcal{O}_K)\cdot x\otimes R$. We then define $S_k(U;R)$ to be the set of those $h \in S_k(U)$ such that $h(x) \in L_k(R)\cdot x^{-1}$ for all $x \in B_{\mathbb{A}_f}^\times$.

Let $K_{0,f} = K_f F$ be the compositum of K_f and F. We may then normalize g_B by requiring that it be a p-unit in $S_k(U, R_p)$ where R_p is the subring of p-integral elements in $K_{0,f}$. With this normalization, it makes sense to study the arithmetic properties of α_r and β. Note that this case is very different from the indefinite case in that $\alpha_r \in \overline{\mathbb{Q}}$ while $\beta/u_+(f)u_-(f) \in \overline{\mathbb{Q}}$.

Let $(,)$ denote the inner product on $S_k(U)$ defined in [19]. For $g_B \in S_k(U)$ and $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g'_B := \rho_k(\delta)\overline{g} \in S_k(U)$ and it is easy to see that

$$\langle\tilde{g}_B, \tilde{g}_B\rangle = \sum_r \langle g_B^r, g_B^r\rangle = (g_B, g'_B)$$

Note that $\beta\overline{\beta}\langle\tilde{g}_B,\tilde{g}_B\rangle^2 = \langle\tilde{\Psi},\tilde{\Psi}\rangle^2 = (\Psi,\Psi)^2 = \beta^2(g_B,g_B)^2$ since $\overline{\Psi(x)\cdot\delta} = \Psi(x)$.

Set $\delta_g = (g_B, g_B)$. It is known that δ_g divides the (Fitting ideal of) the congruence module that counts congruences modulo λ between g_B and other forms on $B^\times$ and also that δ_g divides δ_f. Since

$$\beta \sim \frac{\langle f, f\rangle}{\delta_g} \sim \frac{\delta_f}{\delta_g}u_+(f)u_-(f)$$

and $\alpha \sim (g_B, g_B)$, we get

Theorem 3.5 *(a)* $v_\lambda(\alpha) \geq 0$.

(b) $v_\lambda\left(\frac{\beta}{u+(f)u_-(f)}\right) \geq 0$.

We now explain the relation between this result and Rankin-Selberg L-values. Let K be an imaginary quadratic field that is split at N^+ and inert at N^-, $i : K \hookrightarrow B$ a Heegner embedding. For any integer r with $0 \leq r \leq 2k-2$, let χ_r be a Hecke character of K of type $(r_0, -r_0)$ at infinity, where $r_0 = r-(k-1)$. With a suitable choice of measure, one defines

$$L_{\chi_r}(g_B^r) = \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B^r(x\gamma)\chi(x)d^\times x$$

for any $\gamma \in (B \otimes \mathbb{R})^\times = \mathbb{H}^\times$ such that $\gamma^{-1}(K \otimes \mathbb{R})\gamma = \mathbb{C} \subset \mathbb{H}$. Again, by methods of Waldspurger one can prove that

$$\beta L_{\chi_r}(g_B^r)^2 = L_{\chi_r \times \chi_r}(\theta(f, \varphi_r)) \sim L(\frac{1}{2}, \pi_K \otimes \chi_r)$$

Combining this with (2) yields

$$|L_{\chi_r}(g_B^r)|^2 \sim L(\frac{1}{2}, \pi_K \otimes \chi_r)\frac{\langle g_B^r, g_B^r\rangle}{\langle f, f\rangle}$$

which is just one form of Gross's special value formula.

The following integrality criterion for forms on $B^\times$ follows quite easily from the equidistribution theorem (Thm. 10) of [10].

Proposition 3.6 *(Integrality criterion for forms on $B^\times$)*
A form $\tilde{\Psi}' = [\Psi_r']$ is p-integral if and only if for some Heegner point $K \hookrightarrow B$ with $p \nmid h_K$ and $h_K >> 0$, and all unramified characters χ_r of $K_{\mathbb{A}}^\times$ of infinity type $(r_0, -r_0)$, $0 \leq r \leq 2k-2$,

$$L_{\chi_r}(\Psi_r') := \int_{K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times} \Psi_r'(x)\chi_r(x)d^\times x \tag{7}$$

is p-integral.

Note that the expression (7) is a finite sum of the values Ψ_r' twisted by the values of the character χ_r. Applying the criterion to the form $\tilde{\Psi}$ constructed earlier and using Thm. 3.5, we see that

Theorem 3.7 *For K, r, χ_r as above, and λ any prime above p,*

$$v_\lambda\left(\frac{L(\frac{1}{2}, \pi_K \times \chi_r)}{u_+(f)u_-(f)}\right) \geq v_\lambda(\frac{\delta_f}{\delta_g})$$

Further, if $k = 2$ (so that $r = 0$), there exists (K, χ_0) such that equality holds.

3.2 The dual pair $(\widetilde{SL_2}, O(V))$, $V = B^0$

3.2.1 The indefinite case

Note that Thm. 3.4 above does not say anything about the periods $u_{\pm}(f), u_{\pm}(g)$ but only about the Petersson norms which are products of periods. It turns out to be much harder to prove results about individual periods. By studying a relevant theta correspondence, we are able to prove the following result about ratios of periods. We denote by $A(f,d)$ the algebraic part of the L-value $L(\frac{1}{2}, \pi_f \otimes \chi_d)$ for d any fundamental quadratic discriminant and χ_d the corresponding character i.e. $A(f,d) = \mathfrak{g}(\chi_d)L(\frac{1}{2}, \pi_f \otimes \chi_d)/u_\tau(f)$ for $\tau = (-1)^k \mathrm{sign}(d)$. It is known that $A(f,d)$ is a p-integer in K_f.

Theorem 3.8 *(a) Suppose $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$. Then*

$$\left(\frac{u_{\pm}(f)}{u_{\pm}(g)}\right)^{\sigma} = \frac{u_{\pm}(f^\sigma)}{u_{\pm}(g^\sigma)}$$

(b)Suppose there exists a quadratic discriminant d such that $p \nmid A(f,d)$. Then

$$v_\lambda\left(\frac{u_\tau(f)}{u_\tau(g)}\right) \geq 0$$

where $\tau = (-1)^k \mathrm{sign}(d)$.

In the case f has weight 2, one can use the rationality of period ratios provided by part (a) to construct directly isogenies between quotients of $J_0(N)$ and $Jac(X)$, completely independent of Faltings' isogeny theorem. Further in the case when $k = 2$, one also has applications to questions about p-divisibility and indivisibility of central values of quadratic twists. (See [12], [13] for more details on these applications.)

The proofs of the above results are based on studying the p-adic properties of the theta lifting for the dual pair $(\widetilde{SL_2}, O(V))$ with V the space of trace zero elements in B. The automorphic theory in this case has been worked out in great detail in three beautiful articles of Waldspurger ([20], [21], [22]). In the arithmetic theory there are three complications that arise. Firstly, there is not one automorphic form on $\widetilde{SL_2}$ but rather a *packet* of forms that corresponds to g_B. Secondly, there is no good theory of newforms for forms of half-integral weight. Lastly, while one can again measure arithmeticity on $PB^\times = SO(V)$ again by means of period integrals on tori, the relevant period integrals are not related to a Rankin-Selberg L-value as in the case of $O(B)$. However, one can still show that for a suitable choice of ψ and $\varphi \in V(\mathbb{A})$ and a suitable form h of weight $k + \frac{1}{2}$ that is p-adically normalized, one has

Theorem 3.9 *(a)* $\theta^t(g,\varphi,\psi) = \alpha u_\pm(g)h$ *for some scalar* α.
(b) $\theta(h,\varphi,\psi) = \beta g$ *for some scalar* β.
(c) $\alpha, \beta \in \overline{\mathbb{Q}}$. *Further* $v_\lambda(\alpha) \geq 0$ *and* $v_\lambda(\beta) \geq 0$.

The proof of the above theorem (especially the p-integrality of β) is rather intricate, so we refer the reader to the article [12] for more details.

3.2.2 The definite case

It is not hard to show in this case that for suitable choices of φ, ψ, h, $\theta^t(g,\varphi,\psi) = \alpha h$ and $\theta(h,\varphi,\psi) = \beta u_\pm(f) g$ for some scalars α, β. Unfortunately, the author does not know how to prove in this case the analog of Thm. 3.5 (b) i.e. the p-integrality of β. One would certainly conjecture that

Conjecture 3.10 $v_\lambda\left(\frac{\beta}{u_\pm(f)}\right) \geq 0$

However the previous methods of proofs break down in that one seems to require rather refined information about Petersson inner products and congruences of half-integral weight forms, that is not presently available. (See [13] for a discussion of this issue.)

3.3 An open question

The reader might have noticed that it is essential that condition (*) be satisfied by p in order to apply the integrality criteria 5 and 3.6. To the author's best knowledge the following is an open (and important) question:

Question 3.11 *Let* p *be a prime not dividing* N. *Does* p *satisfies condition (*) even if it does not satisfy Assumption II ?*

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