A SIMPLE PROOF OF RATIONALITY OF SIEGEL-WEIL EISENSTEIN SERIES

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Introduction

This short article is concerned with the arithmetic properties of the most degenerate holomorphic Eisenstein series on quasi-split $2n$-dimensional unitary similitude groups $GU(n, n)$ attached to totally imaginary quadratic extensions of totally real fields. This topic has been treated in detail in the literature, especially by Shimura. The general principle is that holomorphic Eisenstein series on Shimura varieties are rational over the fields of definition of their constant terms. In the range of absolute convergence this was proved in [H1] by applying a version of the Manin-Drinfeld principle: by a combinatorial argument involving Satake parameters in the range of convergence[1], one shows that the automorphic representations generated by absolutely convergent holomorphic Eisenstein series have multiplicity one in the space of automorphic forms. Hecke operators and the constant term map are rational over the appropriate reflex field, and this suffices to prove that Eisenstein series inherit the rationality of their constant terms, without any further computation.

The holomorphic Eisenstein series considered in this paper are lifted from the point boundary component of the Shimura variety associated to the quasi-split group GH. Rationality of modular forms on such a Shimura variety can be determined by looking at Fourier coefficients relative to the parabolic subgroup stabilizing the point boundary component (the *Siegel parabolic*). In the article [S2] and in the two books [S3, S4], Shimura has obtained almost completely explicit formulas for the nondegenerate Fourier coefficients of the Eisenstein series. This should be enough to cover all relevant cases, and indeed Shimura obtains applications to special values of standard L-functions of unitary groups. However, Shimura makes special hypotheses on the sections defining the Eisenstein series (equivalently, on the constant terms) that are not natural from the point of view of representation

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[1] I later learned this argument is equivalent to Langlands' geometric lemmas in his theory of Eisenstein series.

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theory. In the present article we consider Eisenstein series attached to Siegel-Weil sections. These are defined representation-theoretically in terms of the local theta correspondence between $GU(n,n)$ and the unitary group $U(V)$ of a totally positive-definite hermitian space V. Suppose $\dim V = m \geq n$. The Siegel-Weil Eisenstein series are then holomorphic values of the Eisenstein series in the right-half plane determined by the functional equation. Ichino, following the techniques of Kudla and Rallis, has recently proved the Siegel-Weil formula in this setting: the Siegel-Weil Eisenstein series is the theta lift of the constant function 1 on the adeles of $U(V)$, and a simple formula relates the constant term of the Siegel-Weil Eisenstein series to the Schwartz function defining the theta kernel in the Schrödinger model. For $\dim V < n$, the theta lift of 1 is identified with a residue of an explicit Eisenstein series, and is again a holomorphic automorphic form.

The Siegel-Weil residues obtained when $\dim V < n$ are clearly *singular modular forms*, in that their Fourier coefficients are supported on singular matrices. A special case of a theorem of J.-S. Li shows that the corresponding automorphic representations have multiplicity one. The proof of rationality now fits in a few lines. First apply Li's multiplicity one theorem and the Manin-Drinfeld principle to show that the holomorphic residual Eisenstein series on $GU(n,n)$ when $\dim V = 1$ – call these *rank one theta lifts* – inherit the rationality of their constant terms. Now let $m = \dim V \geq n$ and apply this result to $GU(nm,nm)$. This group contains $GU(n,n) \times U(V)$ (we actually work with a semi-direct product) as a subgroup. The Siegel-Weil Eisenstein series lifted from $U(V)$ to $GU(n,n)$ is obtained by integrating the rank one theta lift on $GU(nm,nm)$ over the adeles (mod principal adeles) of $U(V)$ and restricting to $GU(n,n)$. These operations are rational, and moreover are compatible with a rational map on constant terms, and this completes the proof.

It might be thought that even that sketch is too long. The non-degenerate Fourier coefficients of Siegel-Weil Eisenstein series can be calculated very simply in terms of the moment map. Indeed, the comparison of this calculation with Shimura's calculation for the Eisenstein series underlies Ichino's proofs of the extended Siegel-Weil formula, as it did the Kudla-Rallis extension of the Siegel-Weil formula for orthogonal-symplectic dual reductive pairs. In fact, this calculation is perfectly sufficient for determining rationality up to roots of unity. The problem is that the oscillator representation on finite adèles only becomes rational over the field generated by the additive character used to define it. In the end the choice of additive character doesn't really matter, but it enters into the calculations. In the present approach, the explicit calculations are concealed in the proof of the Siegel-Weil formula.

Although the ideas of the proof fit in a few lines, the paper has stretched to occupy over twenty pages. Most of this consists of notation and references to earlier work. Our use of similitude groups also introduces complications, since only unitary groups are treated in the literature. In fact, Ichino's Siegel-Weil formula does not quite extend to the full adelic similitude group, but only to a subgroup of index two, denoted $GU(n,n)(\mathbf{A})^+$. The values of Eisenstein series on the complement of $GU(n,n)(\mathbf{A})^+$ in the range of interest – to the right of the center of symmetry, but to the left of the half-plane of absolute convergence – are related to values of so-called *incoherent* Eisenstein series, or to an as yet unknown "second-term identity" in the Kudla-Rallis version of the Siegel-Weil formalism. The upshot is that we only obtain rationality of Siegel-Weil Eisenstein series over a specific quadratic extension of $\mathbb{Q}$. Adapting the arguments to $GU(n,n)(\mathbf{A})^+$ involves nothing complicated but adds considerably to the length of the paper.

The paper concludes with the expected applications to special values of L-functions, – up to scalars in the quadratic extension mentioned above – following [H3]. For scalar weights, these are probably all contained in [S97]. Our results are undoubtedly less general in some respects than Shimura's, since the use of Siegel-Weil Eisenstein series removes half the degrees of freedom in the choice of a twisting character. However, the special values we can treat are sufficient for applications to period relations anticipated in [H4]. These matters will be discussed in a subsequent paper.

The reader is advised that the point boundary stratum of the Shimura variety in the applications to Siegel Eisenstein series in [H1,§8] and [H3] is slightly inconsistent with the general formalism for boundary strata described in [H1, §6]. In fact, the latter was treated within the framework of Shimura data, as defined in Deligne's Corvallis article [D1]. However, as Pink realized in [P], this is inadequate for the boundary strata. The problem is that the connected components of the zero-dimensional Shimura variety attached to $\mathbb{G}_m$, with the norm map, are all defined over the maximal totally real abelian extension of $\mathbb{Q}$, whereas the boundary points of the Shimura variety attached to $GU(n,n)$ (or $GL(2)$, for that matter) are in general defined over the full cyclotomic field, as they correspond to level structures on totally degenerate abelian varieties (i.e., on powers of $\mathbb{G}_m$. In fact, the proof of arithmeticity of Eisenstein series in [H86], quoted in [H3], made implicit use of Pink's formalism, although the author did not realize it at the time. For example, the reciprocity law [H3, (3.3.5.4)] is correct, but is consistent with Pink's formalism rather than the framework of [H1,§6]. This reciprocity law is used (implicitly) in the proof of [H3, Lemma 3.5.6], which is the only place the precise determination of arithmetic Eisenstein series is applied. The discussion following Corollary 2.4.4 appeals explicitly to Pink's formalism.

I thank Atsushi Ichino for several helpful exchanges, and for providing me with the manuscript of [I2]. Everything new and substantial in the present article is contained in [I2] and [I1]. I also thank Paul Garrett for providing an updated and expanded version of his calculation of the archimedean zeta integrals, and for allowing me to include his calculation as an appendix.

0. Preliminary notation

Let E be a totally real field of degree d, $\mathcal{K}$ a totally imaginary quadratic extension of E. Let V be an n-dimensional $\mathcal{K}$-vector space, endowed with a non-degenerate hermitian form $< \bullet, \bullet >_V$, relative to the extension $\mathcal{K}/E$. We let Σ_E, resp. $\Sigma_{\mathcal{K}}$, denote the set of complex embeddings of E, resp. $\mathcal{K}$, and choose a CM type $\Sigma \subset \Sigma_{\mathcal{K}}$, i.e. a subset which upon restriction to E is identified with Σ_E. Complex conjugation in $Gal(\mathcal{K}/E)$ is denoted c.

The hermitian pairing $< \bullet, \bullet >_V$ defines an involution $\tilde{c}$ on the algebra $End(V)$ via

$$< a(v), v' >_V = < v, a^{\tilde{c}}(v') >, \tag{0.1}$$

and this involution extends to $End(V \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define $\mathbb{Q}$-algebraic groups $U(V) = U(V, < \bullet, \bullet >_V)$ and $GU(V) = GU(V, < \bullet, \bullet >_V)$ over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra R,

$$U(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = 1\}; \tag{0.2}$$

(0.3) $GU(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = \nu(g) \text{ for some } \nu(g) \in R^{\times}\}.$

Thus $GU(V)$ admits a homomorphism $\nu : GU(V) \to \mathbb{G}_m$ with kernel $U(V)$. There is an algebraic group $U_E(V)$ over E such that $U(V) \xrightarrow{\sim} R_{E/\mathbb{Q}}U_E(V)$, where $R_{E/\mathbb{Q}}$ denotes Weil's restriction of scalars functor. This isomorphism identifies automorphic representations of $U(V)$ and $U_E(V)$.

All constructions relative to hermitian vector spaces carry over without change to skew-hermitian spaces.

The quadratic Hecke character of $\mathbf{A}_E^{\times}$ corresponding to the extension $\mathcal{K}/E$ is denoted

$$\varepsilon_{\mathcal{K}/E} : \mathbf{A}_E^{\times}/E^{\times}N_{\mathcal{K}/E}\mathbf{A}_{\mathcal{K}}^{\times} \xrightarrow{\sim} \pm 1.$$

For any hermitian or skew-hermitian space, let

$$GU(V)(\mathbf{A})^{+} = \ker \varepsilon_{\mathcal{K}/E} \circ \nu \subset GU(V)(\mathbf{A}). \tag{0.4}$$

For any place v of E, we let $GU(V)_v^{+} = GU(V)(E_v) \cap GU(V)(\mathbf{A})^{+}$. If v splits in $\mathcal{K}/E$, then $GU(V)_v^{+} = GU(V)(E_v)$; otherwise $[GU(V)(E_v) : GU(V)_v^{+}] = 2$, and $GU(V)_v^{+}$ is the kernel of the composition of ν with the local norm residue map. We define $GU(V)^{+}(\mathbf{A}) = \prod_v' GU(V)_v^{+}$ (restricted direct product), noting the position of the superscript; we have

$$GU(V)(E) \cdot GU(V)^{+}(\mathbf{A}) = GU(V)(\mathbf{A})^{+}. \tag{0.5}$$

1. Eisenstein series on unitary similitude groups

(1.1) Notation for Eisenstein series. The present section is largely taken from [H3, §3] and [H4, §I.1]. Let E and $\mathcal{K}$ be as in §0. Let $(W, <,>_W)$ be any hermitian space over $\mathcal{K}$ of dimension n. Define $-W$ to be the space W with hermitian form $- <,>_W$, and let $2W = W \oplus (-W)$. Set

$$W^d = \{(v,v) \,|\, v \in W\}, \qquad W_d = \{(v,-v) \,|\, v \in W\}$$

These are totally isotropic subspaces of $2W$. Let P (resp. GP) be the stablizer of W^d in $U(2W)$ (resp. $GU(2W)$). As a Levi component of P we take the subgroup $M \subset U(2W)$ which is stablizer of both W^d and W_d. Then $M \simeq GL(W^d) \xrightarrow{\sim} GL(W)$, and we let $p \mapsto A(p)$ denote the corresponding homomorphism $P \to GL(W)$. Similarly, we let $GM \subset GP$ be the stabilizer of both W^d and W_d. Then $A \times \nu : GM \to GL(W) \times \mathbb{G}_m$, with A defined as above, is an isomorphism. There is an obvious embedding $U(W) \times U(W) = U(W) \times U(-W) \hookrightarrow U(2W)$.

In this section we let $H = U(2W)$, viewed alternatively as an algebraic group over E or, by restriction of scalars, as an algebraic group over $\mathbb{Q}$. The individual groups $U(W)$ will reappear in §4. We choose a maximal compact subgroup $K_\infty = \prod_{v \in \Sigma_E} K_v \subset H(\mathbb{R})$; specific choices will be determined later. We also let $GH = GU(2W)$.

Let v be any place of E, $|\cdot|_v$ the corresponding absolute value on $\mathbb{Q}_v$, and let

$$\delta_v(p) = |N_{\mathcal{K}/E} \circ \det(A(p))|_v^{\frac{n}{2}} |\nu(p)|^{-\frac{1}{2}n^2}, \; p \in GP(E_v). \tag{1.1.1}$$

This is the local modulus character of $GP(E_v)$. The adelic modulus character of $GP(\mathbf{A})$, defined analogously, is denoted $\delta_{\mathbf{A}}$. Let χ be a Hecke character of $\mathcal{K}$. We view χ as a character of $M(\mathbb{A}_E) \xrightarrow{\sim} GL(W^d)$ via composition with det. For any complex number s, define

$$\delta^0_{P,\mathbf{A}}(p,\chi,s) = \chi(\det(A(p))) \cdot |N_{\mathcal{K}/E} \circ \det(A(p))|_v^s |\nu(p)|^{-ns}$$

$$\delta_{\mathbf{A}}(p,\chi,s) = \delta_{\mathbf{A}}(p)\delta^0_{P,\mathbf{A}}(p,\chi,s) = \chi(\det(A(p)))\cdot|N_{\mathcal{K}/E}\circ\det(A(p))|_v^{\frac{n}{2}+s}|\nu(p)|^{-\frac{1}{2}n^2-ns}.$$

The local characters $\delta_{P,v}(\cdot,\chi,s)$ and $\delta^0_{P,v}(\cdot,\chi,s)$ are defined analogously.

Let σ be a real place of E. Then $H(E_\sigma) \xrightarrow{\sim} U(n,n)$, the unitary group of signature (n,n). As in [H3, 3.1], we identify $U(n,n)$, resp. $GU(n,n)$, with the unitary group (resp. the unitary similitude group) of the standard skew-hermitian matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. This identification depends on the choice of extension $\tilde{\sigma}$ of σ to an element of the CM type Σ. We also write $GU(n,n)_\sigma$ to draw attention to the choice of σ. Let $K(n,n)_\sigma = U(n) \times U(n) \subset U(n,n)_\sigma$ in the standard embedding, $GK(n,n)_\sigma = Z \cdot K_{n,n}$ where Z is the diagonal subgroup of $GU(n,n)$, and let $X_{n,n} = X_{n,n,\sigma} = GU(n,n)_\sigma/GK(n,n)_\sigma$, $X^0_{n,n} = U(n,n)/K(n,n)_\sigma$ be the corresponding symmetric spaces. The space $X^0_{n,n}$, which can be realized as a tube domain in the space $M(n,\mathbb{C})$ of complex $n \times n$-matrices, is naturally a connected component of $X_{n,n}$; more precisely, the identity component $GU(n,n)^+$ of elements with positive similitude factor stabilizes $X^0_{n,n}$ and identifies it with $GU(n,n)^+_\sigma/GK(n,n)_\sigma$. Writing $g \in GU(n,n)$ in block matrix form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with respect to bases of W^d_σ and $W_{d,\sigma}$, we identify GP with the set of $g \in GU(n,n)$ for which the block $C = 0$. In the tube domain realization, the canonical automorphy factor associated to GP and $GK(n,n)_\sigma$ is given as follows: if $\tau \in X_{n,n}$ and $g \in GU(n,n)^+$, then the triple

$$J(g,\tau) = C\tau + D,\ J'(g,\tau) = \bar{C}^t\tau + \bar{D}, \nu(g) \tag{1.1.2}$$

defines a canonical automorphy factor with values in $GL(n,\mathbb{C})\times GL(n,\mathbb{C})\times GL(1,\mathbb{R})$ (note the misprint in [H3, 3.3]).

Let $\tau_0 \in X^+_{n,n}$ denote the unique fixed point of the subgroup $GK(n,n)_\sigma$. Given a pair of integers (μ,κ), we define a complex valued function on $GU(n,n)^+$:

$$J_{\mu,\kappa}(g) = \det J(g)^{-\mu} \cdot \det(J'(g))^{-\mu-\kappa} \cdot \nu(g)^{n(\mu+\kappa)} \tag{1.1.3}$$

More generally, let GH^+ denote the identity component of $GH(\mathbb{R})$, and let

$$GK(n,n) = \prod_\sigma GK(n,n)_\sigma, \quad K(n,n) = \prod_\sigma K(n,n)_\sigma.$$

Define $\mathbf{J}_{\mu,\kappa} \to GH^+ \to \mathbb{C}^\times$ by

$$\mathbf{J}_{\mu,\kappa}((g_\sigma)_{\sigma\in\Sigma_E}) = \prod_{\sigma\in\Sigma_E} J_{\mu,\kappa}(g_\sigma) \tag{1.1.4}$$

We can also let μ and κ denote integer valued functions on σ and define analogous automorphy factors. The subsequent theory remains valid provided the value $2\mu(\sigma)+\kappa(\sigma)$ is independent of σ. However, we will only treat the simpler case here.

In what follows, we sometimes write $GU(n,n)$ instead of GH to designate the quasi-split unitary similitude group of degree $2n$ over E or any of its completions. Let $N \subset P \subset GP$ be the unipotent radical, so that $P = M \cdot N$, $GP = GM \cdot N$.

(1.2) Formulas for the Eisenstein series. Consider the induced representation

(1.2.1) $$I_n(s,\chi) = \mathrm{Ind}(\delta^0_{P,\mathbf{A}}(p,\chi,s)) \xrightarrow{\sim} \otimes_v I_{n,v}(\delta^0_{P,v}(p,\chi,s)),$$

the induction being normalized; the local factors I_v, as v runs over places of E, are likewise defined by normalized induction. Explicitly,

(1.2.2)
$$I_n(s,\chi) = \{f : H(\mathbf{A}) \to \mathbb{C} \mid f(pg) = \delta_{P,\mathbf{A}}(p,\chi,s)f(g), p \in P(\mathbf{A}), g \in H(\mathbf{A})\}.$$

With this normalization the maximal E-split torus in the center of GH acts by a unitary character. At archimedean places we assume our sections to be K_∞-finite. For a section $\phi(h,s;\chi) \in I_n(s,\chi)$ (cf. [H4, I.1]) we form the Eisenstein series

(1.2.3) $$E(h,s;\phi,\chi) = \sum_{\gamma \in P(E)\backslash U(2V)(E)} \phi(\gamma h,s;\chi)$$

If χ is unitary, this series is absolutely convergent for $\mathrm{Re}(s) > \frac{n-1}{2}$, and it can be continued to a meromorphic function on the entire plane. Let $m \geq n$ be a positive integer, and assume

(1.2.4) $$\chi|_{\mathbf{A}_E^\times} = \varepsilon_{\mathcal{K}}^m$$

Then the main result of [Tan] states that the possible poles of $E(h,s;\phi,\chi)$ are all simple, and can only occur at the points in the set

(1.2.5) $$\frac{n-\delta-2r}{2},\ r = 0,\ldots,[\frac{n-\delta-1}{2}],$$

where $\delta = 0$ if m is even and $\delta = 1$ if m is odd. We will be concerned with the values of $E(h,s_0;\phi,\chi)$ for s_0 in the set indicated in (1.2.5) when the Eisenstein series is holomorphic at s_0.

We write $I_n(s,\chi) = I_n(s,\chi)_\infty \otimes I_n(s,\chi)_f$, the factorization over the infinite and finite primes, respectively. Define

$$\alpha = \chi \cdot |N_{\mathcal{K}/E}|^{\frac{\kappa}{2}}.$$

We follow [H3, 3.3] and suppose the character χ has the property that

(1.2.6) $$\alpha_\sigma(z) = z^\kappa,\ \alpha_{c\sigma}(z) = 1\ \forall \sigma \in \Sigma$$

Then the function $\mathbf{J}_{\mu,\kappa}$, defined above, belongs to

$$I_n(\mu - \frac{n}{2},\alpha)_\infty = I_n(\mu + \frac{\kappa-n}{2},\chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}.$$

(cf. [H3,(3.3.1)]). More generally, let

$$\mathbf{J}_{\mu,\kappa}(g,s+\mu-\frac{n}{2}) = \mathbf{J}_{\mu,\kappa}(g)|\det(J(g)\cdot J'(g))|^{-s} \in I_n(s,\alpha)_\infty.$$

When $E = \mathbb{Q}$, these formulas just reduce to the formulas in [H3].

(1.3) Holomorphic Eisenstein series.

The homogeneous space $X^d_{n,n}$ can be identified with a $GH(\mathbb{R})$-conjugacy class of homomorphisms of algebraic groups over $\mathbb{R}$:

$$h : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to GH_{\mathbb{R}}.$$

There is a unique conjugacy class with the property that the composition of the map (1.3.1) with any $h \in X_{n,n}$ induces an E-linear Hodge structure of type $(0,-1)+(-1,0)$ on $R_{\mathcal{K}/\mathbb{Q}}2W\otimes_{\mathbb{Q}}\mathbb{R}$. The chosen subgroup $GK(n,n) \subset GH(\mathbb{R})$ is the stabilizer (centralizer) of a unique point $h_0 \in X^0_{n,n}$. Corresponding to h_0 is a Harish-Chandra decomposition

$$Lie(GH)_{\mathbb{C}} = Lie(GK(n,n)) \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are isomorphic respectively to the holomorphic and antiholomorphic tangent spaces of $X^0_{n,n}$ at h_0.

Let π_∞ be an admissible $(Lie(H), K(n,n))$-module. By a *holomorphic vector* in π_∞ we mean a vector annihilated by $\mathfrak{p}^-$. A *holomorphic representation* (called antiholomorphic in [H3]) of $H(\mathbb{R})$ is a $(Lie(H), K(n,n))$-module generated by a holomorphic vector. The same terminology is used for admissible $(Lie(GH), GK(n,n))$-modules. As in [H3, (3.3)], the element

$$\mathbf{J}_{\mu,\kappa} \in I_n(\mu + \frac{\kappa-n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

is a holomorphic vector and generates an irreducible $(Lie(GH), GK(n,n))$-submodule $\mathbb{D}(\mu,\kappa)$, unitarizable up to a twist and necessarily holomorphic, which is a free $U(\mathfrak{p}^+)$-module provided $\mu + \frac{\kappa-n}{2} \geq 0$.

In our normalization, as in [KR] and [I], the value $\mu + \frac{\kappa-n}{2} = 0$ is the center of symmetry of the functional equation of the Eisenstein series. We will be working with an auxiliary definite hermitian space V of dimension m. It will always be assumed that

$$(1.3.2) \qquad m \equiv \kappa \pmod 2, \quad \mu = \frac{m-\kappa}{2}$$

When $m < n$, so that $\mu + \frac{\kappa-n}{2} < 0$ is to the left of the center of symmetry of the functional equation,

$$\mathbb{D}(\mu,\kappa) \subset I_n(\mu + \frac{\kappa-n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

is still unitarizable (up to a twist) but is a torsion $U(\mathfrak{p}^+)$-module (a singular holomorphic representation, cf. [J]). The lowest $GK(n,n)$-type of $\mathbb{D}(\mu,\kappa)$ is in any case given at each archimedean place v of E by

$$(1.3.3) \qquad \Lambda(-\mu,\kappa) = (\mu+\kappa, \mu+\kappa, \ldots, \mu+\kappa; -\mu, \ldots, -\mu; n\kappa)$$

(cf. [H3,(3.3.2)]). When $m = 0$, $\mathbb{D}(\mu,\kappa)$ is the one-dimensional module associated to the character $\det^{\frac{\kappa}{2}}$. We will be most concerned with the case $m = 1$.

Let $\tilde{\mathbb{D}}(\mu,\kappa)$ be the universal holomorphic module with lowest $GK(n,n)$-type $\Lambda(-\mu,-\kappa)$:

$$(1.3.4) \qquad \tilde{\mathbb{D}}(\mu,\kappa) = U(Lie(GH)) \otimes_{U(Lie(GK(n,n))\oplus\mathfrak{p}^-)} \mathbb{C}_{-\mu,-\kappa}$$

where $GK(n,n)$ acts by the character $\Lambda(-\mu,-\kappa)$ on $\mathbb{C}_{-\mu,-\kappa}$. Then $\mathbb{D}(\mu,\kappa)$ is the unique non-trivial irreducible quotient of $\tilde{\mathbb{D}}(\mu,\kappa)$ [ref.**] The same notation is used for the restrictions of these holomorphic modules to $(Lie(H), K(n,n))$.

2. Rank one representations of $GU(n,n)$

(2.1) Local results for unitary groups.

For the time being, G is either GH or H. Unless otherwise indicated, we always assume $n > 1$. Let v be a place of E that does not split in $\mathcal{K}$, E_v the completion. Choose a non-trivial additive character $\psi : E_v \rightarrow \mathbb{C}^\times$. The subgroup $N(E_v) \subset G(E_v)$ is naturally isomorphic with its own Lie algebra, which we identify with the vector group of $n \times n$-hermitian matrices over $\mathcal{K}_v$, and We can then identify $N(E_v)$ with its own Pontryagin dual by the pairing

$$N(E_v) \times N(E_v) \rightarrow \mathbb{C}^\times; (n, n') = \psi_n(n') := \psi(Tr(n \cdot n')). \tag{2.1.1}$$

Formula (2.1.1) defines the character ψ_n, which we also use to denote the space $\mathbb{C}$ on which $N(E_v)$ acts through the character ψ_n. For the remainder of this section we write N, P, M, G, and so on for $N(E_v), P(E_v)$, etc.

Let π be an irreducible admissible representation of G. The N-spectrum of π is the set of $n \in N$ such that $Hom_N(\pi, \psi_n) \neq 0$. The representation π is said to be *of rank one* if its N-spectrum is contained in the subset of matrices of rank one.

The subgroup $GM \subset GP(E_v)$ acts on N by conjugation. For any $\nu \in E_v^\times$, let

$$d(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix} \in GH.$$

The following facts are well-known.

Lemma 2.1.2. *(a) Let $G = GH$ (resp. H). For any π, the N-spectrum of π is a union of GM- (resp. M-) orbits. If π is of rank one, its N-spectrum contains a single GM- (resp. M-) orbit of rank one.*

(b) Let $\alpha \in E_v^\times$ be an element that is not a norm from $\mathcal{K}_v$. The set of rank one matrices in N is the union of two M-orbits $\mathcal{O}_1$ and $\mathcal{O}_\alpha$, represented respectively by the matrices with 1 and α in the upper left-hand corner and zeros elsewhere.

(c) The set of rank one matrices in N is a single GM-orbit; the element d_α of GH exchanges $\mathcal{O}_1$ and $\mathcal{O}_\alpha$.

Thus the irreducible representations π of H of rank one can be classified as of type 1 or type α, as their spectrum contains $\mathcal{O}_1$ or $\mathcal{O}_\alpha$. This classification is relative to the choice of ψ.

Let ε denote the quadratic character of $E_v^\times$ associated to $\mathcal{K}_v$. Let V be any hermitian space over $\mathcal{K}_v$ of dimension m. Let $\chi_v : \mathcal{K}_v^\times \rightarrow \mathbb{C}^\times$ be a character restricting to ε^m on $E_v^\times$. Let $\chi'_v \mathcal{K}_v^\times / E_v^\times \rightarrow \mathbb{C}^\times$ be a second character. The map $x \mapsto x/c(x)$ defines an isomorphism

$$\mathcal{K}_v^\times / E_v^\times \xrightarrow{\sim} U(1)_v = \ker N_{\mathcal{K}_v/E_v} \subset \mathcal{K}^\times;$$

$U(1)_v$ is also the unitary group of any 1-dimensional hermitian space. In particular, any character β of $\mathcal{K}_v^\times / E_v^\times$ defines a character $\tilde{\beta}$ of $U(1)_v$:

$$\tilde{\beta}(x) = \beta(x/c(x)),$$

and the map $\beta \mapsto \tilde{\beta}$ is a bijection. Moreover, χ'_v can be chosen to be the trivial character $triv_v$, and multiplication makes the set of χ_v with given restriction to

$E_v^\times$ into a principal homogeneous space under the group of characters of $U(1)_v$. As in [HKS], the pair (χ_v, χ'_v), together with the additive character ψ, define a Weil representation

$$\omega_{\chi_v,\chi'_v,\psi} : H \times U(V) \to Aut(\mathcal{S}(V^n)),$$

where $\mathcal{S}$ denotes the Schwartz-Bruhat space. The formulas for the action of H are given in [I1]. The group $U(V)$ acts linearly on the argument when χ'_v is trivial, and in general the linear action is twisted by $\chi'_v \circ \det$, cf. (2.2.1), below. Let $R_n(V, \chi_v, \chi'_v) = R_n(V, \chi_v, \chi'_v, \psi)$ denote the maximal quotient of $\mathcal{S}(V^n)$ on which $U(V)$ acts trivially. We abbreviate $R_n(V, \chi_v) = R_n(V, \chi_v, triv_v)$.

More generally, for any irreducible representation β of $U(V)$, let

$$\Theta_{\chi_v,\chi'_v,\psi}(\beta) = \Theta_{\chi_v,\chi'_v,\psi}(U(V) \to U(n,n); \beta)$$

denote the maximal quotient of $\mathcal{S}(V^n) \otimes \beta$ on which $U(V)$ acts trivially; this is a representation of H.

For the remainder of this section $m = \dim V = 1$. Let V^+ denote the 1-dimensional $\mathcal{K}_v$-space with hermitian form $(z, z') = z \cdot \bar{z}'$, where $^-$ denotes Galois conjugation. Let V^- denote the same space with the hermitian form multiplied by α. Up to isomorphism, V^+ and V^- are the only two hermitian spaces over $\mathcal{K}_v$ of dimension 1. We let ψ_α be the character $\psi_\alpha(x) = \psi(\alpha \cdot x)$.

Proposition 2.1.3 [KS]. *The spaces $R_n(V^\pm, \chi_v, \psi)$ are irreducible. More precisely, $R_n(V^+, \chi_v, \psi)$ and $R_n(V^-, \chi_v, \psi)$ are the unique non-trivial irreducible quotients of the induced representation $I(\frac{n-1}{2}, \chi_v)$; alternatively, they are isomorphic to the unique non-trivial irreducible subrepresentations of the induced representation $I(\frac{1-n}{2}, \chi_v)$.*

The first part of the following theorem is equivalent to a special case of a result of Li:

Theorem 2.1.4. *(a) Every rank one representation π of H is isomorphic to a representation of the form $R_n(V^\pm, \chi_v, \chi'_v, \psi)$ for some χ_v, χ'_v, with fixed ψ. The N-spectrum of $R_n(V^+, \chi_v, \chi'_v, \psi)$ (resp. $R_n(V^-, \chi_v, \chi'_v, \psi)$) is contained in the closure of $\mathcal{O}_1$ (resp. $\mathcal{O}_\alpha$).*

(b) For any χ_v, χ'_v,

$$R_n(V^+, \chi_v, \chi'_v, \psi_\alpha) = R_n(V^-, \chi_v, \chi'_v, \psi), \; R_n(V^-, \chi_v, \chi'_v, \psi_\alpha) = R_n(V^+ \chi_v, \chi'_v, \psi).$$

(c) There are natural isomorphisms

$$R_n(V^\pm, \chi_v, \chi'_v, \psi) \xrightarrow{\sim} \Theta_{\chi_v, triv_v, \psi}(U(V^\pm) \to U(n,n); (\chi'_v)^{-1});$$

$$R_n(V^\pm, \chi_v \cdot \beta, \chi'_v, \psi) \xrightarrow{\sim} R_n(V^\pm, \chi_v \cdot \beta, \chi'_v, \psi) \otimes \tilde{\beta} \circ \det$$

if β is trivial on $E_v^\times$

(d) No two representations $R_n(V^\pm, \chi_v, \chi_v^1, \psi)$ and $R_n(V^\pm, \chi_v, \chi_v^2, \psi)$ are isomorphic if $\chi_v^1 \neq \chi_v^2$.

Proof. The first part of (a) follows from [L1, Theorem 4.8], and the second part is a special case of [L1,Lemma 4.4]. Actually, Li's theorem identifies rank one representations as theta lifts only up to character twists, but (c) shows that all such twists are obtained by varying χ'_v. Assertion (b) is standard (cf. [MVW, p. 36 (2)]), while (c) follows from properties of splittings proved in [K] and recalled in [HKS], specifically [HKS, (1.8)]. Finally, (d) follows from the first formula of (c) and Howe duality for the dual reductive pair $(U(1), U(n,n))$.

In accordance with 2.1.4(a), we say V^+ (resp. V^-) *represents* $\mathcal{O}_1$ (resp. $\mathcal{O}_\alpha$). Note that this correspondence between orbits and hermitian spaces depends only on ψ and not on the choice of characters χ_v, χ'_v.

Lemma 2.1.5. *Suppose v is a real place and $\psi(x) = e^{ax}$ with $a > 0$. Then $R_n(V^+, \chi_v, \psi)$ is a holomorphic representation. If $\chi_v(z) = z^\kappa/|z|^\kappa$, then*

$$R_n(V^+, \chi_v, \psi) \xrightarrow{\sim} \mathbb{D}(\mu, \kappa)$$

in the notation of (1.3), with $\mu = \frac{1-\kappa}{2}$.

(2.1.6) Split places. The situation at places v that split in $\mathcal{K}/E$ is simpler in that there is no dichotomy between V^+ and V^-, there is only one orbit $\mathcal{O}_1$, and only one equivalence class of additive characters. Otherwise, Theorem 2.1.4 remains true: every rank one representation is a theta lift from $U(1) = GL(1)$. Proposition 2.1.3 is also valid in this situation: the theta lift can be identified with an explicit constituent of a degenerate principal series representation [KS,Theorem 1.3] with the same parametrization as in the non-split case.

(2.2) Local results for similitude groups.

Let V and $U(V)$ be as in (2.1). Let $GH^+ \subset GH$ be the index two subgroup of elements h such that $\varepsilon \circ \nu(h) = 1$, i.e. the similitude of h is a norm from $\mathcal{K}_v$. Define $GU(V)^+ \subset GU(V)$ analogously. Let $GU(V)$ act on H by the map β:

$$\beta(g)(h) = d(\nu(g))hd(\nu(g))^{-1},$$

with $d(\nu(g)) \in GH$ defined as in (2.1). As in [HK, §3], the Weil representation extends to an action of the group

$$R = R_v = \{(g,h) \in GU(V) \times GH \mid \nu(h) = \nu(g)\}.$$

There is a map $R \to GU(V) \ltimes H$:

$$(g,h) \mapsto (g, d(\nu(h))^{-1}h) = (g, h_1)$$

whose image equals either $GU(V)^+ \ltimes H$ or $GU(V) \ltimes H$. There is a representation of R on $\mathcal{S}(V^n)$ given by

$$\omega_{\chi_v,\chi'_v,\psi}(g,h_1)\Phi(x) = |\nu(g)|^{-\frac{\dim V}{2}}\chi'_v(\det(g))(\omega_{\chi_v,\chi'_v,\psi}(h_1)\Phi)(g^{-1}x). \tag{2.2.1}$$

The power $|\nu(g)|^{-\frac{\dim V}{2}}$ guarantees that the maximal E-split torus in the center of GH acts by a unitary character, as in $I_n(s,\chi)$. This is not what we need for arithmetic applications but is helpful for normalization.

The image $U(V) \subset GU(V)$ in R is a normal subgroup and $R/U(V)$ is isomorphic either to GH^+ or to GH. When $\dim V = 1$, $R \xrightarrow{\sim} GU(V)^+ \ltimes H$ and $R/U(V) \xrightarrow{\sim} GH^+$. If β is a representation of $GU(V)$ that restricts irreducibly to

$U(V)$, then $\Theta_{\chi_v,\chi'_v,\psi}(\beta)$, defined as in (2.1), extends to a representation of GH^+, which we denote $\Theta^+_{\chi_v,\chi'_v,\psi}(\beta)$. We let

$$\Theta_{\chi_v,\chi'_v,\psi}(\beta) = Ind_{GH^+}^{GH}\Theta^+_{\chi_v,\chi'_v,\psi}(\beta).$$

This need not be irreducible, but in the cases we consider it will be. See [H2, §3] for constructions involving similitude groups.

We again restrict attention to $\dim V = 1$, and $\chi'_v = triv_v$. Then $GU(V) = \mathcal{K}^\times$, acting by scalar multiplication on V. For the character β of $GU(V)$ we just take the trivial character; then $\Theta^+_{\chi_v,triv_v,\psi}(triv)$ is an extension of $R_n(V,\chi_v)$ to a representation $R_n^+(V,\chi_v)$ of GH^+, rigged so that the maximal E_v-split torus in the center acts by a unitary character.

Proposition 2.2.2. *The representations $R_n^+(V^+,\chi_v)$ and $R_n^+(V^-,\chi_v)$ of GH^+ are inequivalent and are conjugate under the element $d(\alpha)$ of $GH - GH^+$. In particular,*

$$Ind_{GH^+}^{GH}R_n^+(V^+,\chi_v) = Ind_{GH^+}^{GH}R_n^+(V^-,\chi_v)$$

is an irreducible representation, denoted $R_n(V^\pm,\chi_v)$, of GH. It is the unique non-trivial irreducible quotient of the induced representation $I(\frac{n-1}{2},\chi_v)$. Alternatively, it is the unique non-trivial irreducible subrepresentation of the induced representation $I(\frac{1-n}{2},\chi_v)$.

We have

$$R_n(V^{pm},\chi_v)\ |_H = R_n(V^+,\chi_v)\oplus R_n(V^-,\chi_v).$$

Proof. Theorem 2.1.4 asserts the two irreducible representations $R_n^+(V^+,\chi_v)$ and $R_n^+(V^-,\chi_v)$ have distinct N-spectra upon restriction to H. This implies the first assertion. To show the two representations are conjugate under $d(\alpha)$, we note first that $d(\alpha)$ exchanges their N-spectra (Lemma 2.1.2(c)). On the other hand, the induced representation $I(n-\frac{1}{2},\chi_v)$ of H extends (in more than one way) to a representation of GH; one such extension is defined in (1.1). The claim then follows from Proposition 2.1.3. The remaining assertions are then obvious, given that $I(s,\chi_v)$ has also been rigged to have unitary central character on the maximal E_v-split torus.

(2.2.3) Intertwining with induced representations. In this section $m = \dim V$ is arbitrary. The quotient $R_n(V,\chi)$ of $\mathcal{S}(V^n)$ can be constructed explicitly as a space of functions on GH^+. For any $h\in GH^+$, let $g_0\in GU(V)$ be any element with $\nu(g_0)=\nu(h)$. For $\Phi\in\mathcal{S}(V^n)$, define

$$\varphi_\Phi(h) = (\omega_{\chi_v,triv,\psi}(g_0,h)\Phi)(0). \tag{2.2.3.1}$$

This function does not depend on the choice of g_0 (cf. [HK, *loc. cit.*]) and belongs to the space of restrictions to GH^+ of functions in $I_n(s_0,\chi_v)$ with $s_0=\frac{m-n}{2}$. The action of GH^+ on $I_n(s_0,\chi_v)$ by right translation extends to the natural action of GH, and in this way the function φ_Φ on GH^+ extends canonically to a function on GH. Since $d(\nu)\in GP$ for all $\nu\in E_v^\times$, the formula for this extension is simply

$$\varphi_\Phi(h^+d(\nu)) = \delta_{P,v}(d(\nu),\chi,s_0)\cdot\varphi_\Phi(d(\nu)^{-1}h^+d(\nu)), \tag{2.2.3.2}$$

which is consistent with (2.2.3.1) when $d(\nu)\in GH^+$.

Lemma 2.2.4. *Suppose v is a real place and $\psi(x) = e^{ax}$ with $a > 0$. Then $R_n(V^+, \chi_v, \psi)$ is a holomorphic representation of GH_v^+ . If $\chi_v(z) = z^\kappa/|z|^\kappa$, then*

$$R_n(V^+, \chi_v, \psi) \otimes |\nu|_\infty^{\frac{n\kappa}{2}} \xrightarrow{\sim} \mathbb{D}(\mu, \kappa)$$

in the notation of (1.3), with $\mu = \frac{1-\kappa}{2}$.

Proof. This is an elementary calculation, and the result is in any case a special case of the general results of [LZ], specifically the K-type calculations in §2, Proposition 2.1 and 2.2.

(2.3) Global multiplicity one results.

As above, we always assume $n \geq 2$ unless otherwise indicated. We fix an additive character $\psi : \mathbf{A}_E/E \to \mathbb{C}^\times$ and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), with $m = 1$.

Definition (2.3.1). *An automorphic representation π of GH or H is said to be locally of rank one at the place v of E if π_v is of rank one. The representation π is said to be globally of rank one if π_v is of rank one for all v.*

The main results on rank one representations are summarized in the following theorem. Most of the assertions are special cases of a theorem of J.-S. Li [L2].

Theorem 2.3.2. *(a) An automorphic representation of H is globally of rank one if and only if it is locally of rank one at some place v.*

(b) Let π be an automorphic representation of H of rank one. Then there is a $\mathcal{K}$ hermitian space V of dimension one, a character β of $U(V)(E)\backslash U(V)(\mathbf{A})$, and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), such that

$$\pi = \Theta_{\chi,triv,\psi}(U(V) \to U(n,n); \beta).$$

(c) Any automorphic representation of H of rank one occurs with multiplicity one in the space of automorphic forms on H.

(d) Suppose $\psi_v(x) = e^{a_v x}$ with $a_v > 0$ for all archimedean v. Then an automorphic representation of H of rank one is holomorphic (resp. anti-holomorphic) if and only if $V_v = V^+$ (resp. V^-) for all archimedean v.

(e) Every automorphic representation of H of rank one is contained in the space of square-integrable automorphic forms on H.

(f) Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of H with π_v of rank one for all v. Let the orbit $\mathcal{O}_v$ be the N_v spectrum of π_v, and let V_v be the equivalence class of one-dimensional hermitian spaces over $\mathcal{K}_v$ representing $\mathcal{O}_v$. Suppose there is no global one-dimensional hermitian space V that localizes to V_v at each v. Then π has multiplicity zero in the space of automorphic forms.

Proof. Assertions (a) is due to Howe [Ho]. In view of Li's Theorem 2.1.4 (a), assertions (b), (c), and (e) are contained in Theorem A of [L2]; note that any automorphic form on any $U(1)$ relative to $\mathcal{K}/E$ is necessarily square-integrable! Assertion (d) is a calculation of the local Howe correspondence (cf. [H4, II (3.8)]). Finally, assertion (f) is a formal consequence of (b), and equivalently of the fact that such a π can have no non-constant Fourier coefficients.

Corollary 2.3.3. *(a) An automorphic representation of GH is globally of rank one if and only if it is locally of rank one at some place v.*

(b) Let π be an automorphic representation of GH of rank one. Then there is a $\mathcal{K}$ hermitian space V of dimension one, a character β of $GU(V)(E)\backslash GU(V)(\mathbf{A})$, and a Hecke character χ of $\mathcal{K}^\times$ satisfying (1.2.4), such that $\pi = \Theta_{\chi,triv,\psi}(\beta)$.

(c) Any automorphic representation of GH of rank one occurs with multiplicity one in the space of automorphic forms on GH.

(d) Let π be an automorphic representation of GH of rank one. As representation of $Lie(GH)_\mathbb{C} \times GH(\mathbf{A}_f)$, π is generated by $\{\pm 1\}^{\Sigma_E}$ vectors $v_{(e_\sigma)}$, with each $e_\sigma \in \{\pm 1\}$. The vector $v_{(e_\sigma)}$ is holomorphic (resp. anti-holomorphic) at each place σ with $e_\sigma = +1$ (resp. $e_\sigma = -1$.

(e) Every automorphic representation of GH of rank one is contained in the space of essentially square-integrable automorphic forms on GH.

"Essentially square integrable" in the above corollary means square-integrable modulo the adeles of the center, up to twist by a character.

The global automorphic representation $\Theta_{\chi,triv,\psi}(\beta)$ is isomorphic to the restricted tensor product over v of the local representations $\Theta_{\chi_v,triv,\psi_v}(\beta_v)$. As a space of automorphic forms it is defined as in [H4, §I.4]. The elements of this space are defined below. We are primarily interested in the case of trivial β. Then

$$\Theta_{\chi,triv,\psi}(triv) = R_n(V^\pm,\chi) \stackrel{def}{=} \otimes_v R_n(V^\pm,\chi_v) \xrightarrow{\sim} Ind_{GH^+(\mathbf{A})}^{GH(\mathbf{A})} \otimes_v R_n(V_v,\chi_v), \tag{2.3.4}$$

where V is any one-dimensional hermitian space over $\mathcal{K}$ and V_v is its localization at v. The space $R_n(V^\pm,\chi)$ may be viewed alternatively, as in as the unique non-trivial irreducible $GH(\mathbf{A})$-quotient of the adelic induced representation $I(n-\frac{1}{2},\chi)$ or as the unique non-trivial irreducible $GH(\mathbf{A})$-submodule of $I(\frac{1}{2},\chi)$.

The dimension of V is now arbitrary. Let $R(\mathbf{A}_f) = \prod'_v R_v$, where $R_v \subset GH(E_v)\times GU(V_v)$ is the group defined in (2.2). Let $\Phi \in \mathcal{S}(V^n)$ and define

$$\theta_{\chi,triv,\psi}(\Phi)(g,h) = \sum_{x\in V^n(\mathcal{K})} \omega_{\chi,triv,\psi}(g,h)(\Phi)(x). \tag{2.3.5}$$

For $h \in GH^+(\mathbf{A})$, defined as in the notation section, and for $g_0 \in GU(V)(\mathbf{A})$ such that $\nu(g_0) = \nu(h)$, let

$$I_{\chi,triv,\psi}(\Phi)(h) = \int_{U(V)(E)\backslash U(V)(\mathbf{A})} \theta_{\chi,triv,\psi}(\Phi)(g_0 g,h)dg \tag{2.3.6}$$

The measure dg is Tamagawa measure. This integral does not depend on the choice of g_0 and defines an automorphic form on $GH^+(\mathbf{A})\cap GH(\mathbb{Q})\backslash GH^+(\mathbf{A})$ that extends uniquely to a function on $GH(\mathbb{Q})\backslash GH(\mathbb{Q})\cdot GH^+(\mathbf{A}) = GH(\mathbb{Q})\backslash GH(\mathbf{A})^+$, also denoted $I_{\chi,triv,\psi}(\Phi)$.

In §3 we will realize this function in a certain space of Eisenstein series, and as such it can be extended to an automorphic form on all $GH(\mathbf{A})$. We now assume $\dim V = 1$. As abstract representation, the restriction of $\Theta_{\chi,triv,\psi}(triv)$ to $H(\mathbf{A})$ is calculated by the final assertion of Proposition 2.2.2:

$$\Theta_{\chi,triv,\psi}(triv)\ |_{H(\mathbf{A})} = \bigoplus_S \otimes_{v\notin S} R(V_v^+,\chi_v,\psi_v) \bigotimes \otimes_{v\in S} R(V_v^-,\chi_v,\psi_v), \tag{2.3.7}$$

where S runs over all finite sets of places of E. Let Θ_S denote the summand on the right hand side of (2.3.7) indexed by S. We say Θ_S is *coherent* if there is a global V such that $V_v = V_v^-$ if and only if $v \in S$, incoherent otherwise. By the global reciprocity law for the Hasse invariant of hermitian spaces, S is coherent if and only if $|S|$ is even.

Lemma 2.3.8. *Let F be any extension of $I_{\chi,triv,\psi}(\Phi)$ to an automorphic form on $GH(\mathbf{A})$. Then $F(h) = 0$ for $h \in GH(\mathbf{A}) - GH(\mathbf{A})^+$.*

Proof. An extension of $I_{\chi,triv,\psi}$ to $GH(\mathbf{A})$ corresponds to a homomorphism λ from

$$\Theta_{\chi,triv,\psi}(triv) = Ind_{GH^+(\mathbf{A})}^{GH(\mathbf{A})} \otimes_v R_n(V_v, \chi_v)$$

to the space of automorphic forms on $GH(\mathbf{A})$. Let Θ_S denote the summand on the right hand side of (2.3.7) indexed by S. It follows from (2.3.2)(f) that $\lambda(\Theta_S) = 0$ for any incoherent S. The Lemma now follows easily from Lemma 2.1.2(c).

(2.4) Automorphic forms on the Shimura variety.

Let $\mathcal{A}(n,n)$ denote the space of automorphic forms on GH; i.e., of functions on $GH(E)\backslash GH(\mathbf{A}_E)$ satisfying the axioms of automorphic forms. Let $GH^0(\mathbf{A}_E) = GH^+_\infty \times GH(\mathbf{A}_f)$, where $GH_\infty = GH(E \otimes_{\mathbb{Q}} \mathbb{R})$ and GH^+_∞ is the identity component, which can also be characterized as the subgroup with positive similitude factor at each real place. By "real approximation", $GH(E) \cdot GH^0(\mathbf{A}_E) = GH(\mathbf{A})$, so an automorphic form is determined by its restriction to $GH(E)^0\backslash GH^0(\mathbf{A}_E)$, where $GH(E)^0 = GH(E) \cap GH^0(\mathbf{A})_E$.

The pair $(GH, X^d_{n,n})$ defines a Shimura variety $Sh(n,n)$, or $Sh(n,n)_{\mathcal{K}/E}$, with canonical model over its reflex field, which is easily checked to be $\mathbb{Q}$ (see §(2.6), below). This Shimura variety does not satisfy a hypothesis frequently imposed: the maximal $\mathbb{R}$-split subgroup of the center of G is not split over $\mathbb{Q}$. Let $Z_E \subset Z_{GH}$ be the maximal $\mathbb{Q}$-anisotropic subtorus of the maximal $\mathbb{R}$-split torus; concretely, Z_E is the kernel of the norm from $R_{E/\mathbb{Q}}(\mathbb{G}_m)_E$ to $\mathbb{G}_m$. Automorphic vector bundles over $Sh(n,n)$ are indexed by irreducible representations of $GK(n,n)$ on which Z_E acts trivially.

We only need automorphic line bundles, indexed by the characters $\Lambda(\mu,\kappa)$ of $GK(n,n)$ whose inverses were defined in (1.3.3); note that Z_E acts trivially because the characters μ and κ are constant as functions of real places. Let $\mathcal{E}_{\mu,\kappa}$ be the corresponding line bundle, defined as in [H3, (3.3)]. It is elementary (cf. [H3, (3.3.4)]) that

$$(2.4.1) \qquad Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa), \mathcal{A}(n,n)) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}).$$

(In [*loc. cit.*] the right-hand side is replaced by the space of sections of the canonical extension of $\mathcal{E}_{\mu,\kappa}$) over some toroidal compactification $\widetilde{Sh(n,n)}$ of $Sh(n,n)$. Since $n > 1$, Koecher's principle guarantees these spaces are canonically isomorphic.)

Theorem 2.4.2. *In fact, every homomorphism on the left-hand side of (2.4.1) factors through $\mathbb{D}(\mu,\kappa)$.*

Proof. This theorem is essentially due to Resnikoff [R]; the current formulation is from [HJ].

Corollary 2.3.3 and Lemma 2.1.5 then imply:

Corollary 2.4.3. *(a) The space $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ is generated by the images with respect to (2.4.1) of forms in*

$$\Theta^{hol}_{\chi,triv,\psi}(\beta)\otimes|\nu|_\infty^{\frac{n\kappa}{2}}$$

where β runs through characters of $GU(V)(\mathbf{A})/GU(V)(E)$ with $\beta_\infty = 1$, and where $\chi_v(z) = z^\kappa/|z|^\kappa$ for all real v.

(b) The representation of $GH(\mathbf{A}_f)$ on $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ is completely reducible and multiplicity free.

Proof. The condition $\beta_\infty = 1$ implies, in view of Lemma 2.2.4, that the archimedean components of $\Theta^{hol}_{\chi,triv,\psi}(\beta)|\nu|_\infty^{\frac{n\kappa}{2}}$ are all isomorphic to $\mathbb{D}(\mu,\kappa)$. This is not possible for $\beta_\infty \neq 1$ by Howe duality, thus only β with trivial archimedean component contribute to the left-hand side of (2.4.1). Thus (a) follows from Corollary 2.3.3 (a) and (b) and Lemma 2.1.5. Assertion (b) is a consequence of (2.3.3) (e) and (c): the L^2 pairing defines a hermitian (Petersson) pairing on $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ with respect to which the action of $GH(\mathbf{A}_f)$ is self-adjoint.

It follows from the theory of canonical models of automorphic vector bundles [H85,Mi] that the automorphic line bundle $\mathcal{E}_{\mu,\kappa}$ is defined over the field of definition $E(\mu,\kappa)$ of the conjugacy class under the normalizer of $GK(n,n)$ in GH of the character $\Lambda(\mu,\kappa)$, which is contained in the reflex field $E(\Sigma)$ of the CM type Σ. If $\kappa \neq 0$, as will always be the case in applications, then $E(\mu,\kappa) = E(\Sigma)$. Improvement on this field of definition requires a more careful analysis of the interchange of holomorphic and anti-holomorphic forms on $Sh(n,n)$.

The isomorphism (2.4.1) implicitly depends on the choice of a basis for the fiber of $\mathcal{E}_{\mu,\kappa}$ at the fixed point of $GK(n,n)$ in $X^+_{n,n}$, called a *canonical trivialization* in [H3]. This choice enters into explicit calculations with automorphic forms but is irrelevant for the present purposes. All that matters is that, with respect to this choice, which we fix once and for all, the $E(\mu,\kappa)$-rational structure on the right-hand side of (2.4.1) defines an $E(\mu,\kappa)$-rational structure on the left-hand side.

The next corollary is obvious from what has been presented thus far.

Corollary 2.4.4. *Let π_f be an irreducible representation of $GH(\mathbf{A}_f)$ that occurs in $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$, and let $M[\pi_f] \subset H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ denote the π_f-isotypic subspace. Then π_f is defined over a finite extension $E(\pi_f)$ of $E(\mu,\kappa)$, $M[\pi_f]$ is isomorphic as $GH(\mathbf{A}_f)$-module to π_f, and $\gamma(M[\pi_f])) = M([\gamma(\pi_f)])$ for any $\gamma \in Gal(\overline{\mathbb{Q}}/E(\mu,\kappa))$.*

As in [H3,3.3.5], we denote by $Sh(n,n)_{GP}$ the point boundary stratum of the minimal (Baily-Borel-Satake) compactification of $Sh(n,n)$. Our description of this stratum follows the formalism of Pink [P]. Let $(G_{h,P}, h^{\pm}_P)$ be the Shimura datum associated to this boundary stratum. The group $G_{h,P}$ is a torus contained in GP, isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$, as in [H3] (where there is however a misprint):

$$G_{h,P} = \{g \in GP \ \mid A = aI_n, D = dI_n\} \subset GP$$

and $h^{\pm}_P$ is a homogeneous space for $G_{h,P}(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times$, consisting of two points on which (a,d) acts trivially if and only if $a > 0$; this covers (in Pink's sense) the $\mathbb{R}$-homomorphism $h_P : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to G_{h,P}$ defined by [H3,(3.3.5.2)]. More precisely, in [H3] we have $E = \mathbb{Q}$; in general, h_P is given by [H3,(3.3.5.2)] at each real place of E. The same is true for the formula defining the limit bundle $\mathcal{E}_{\mu,\kappa,GP}$: the line bundle $\mathcal{E}_{\mu,\kappa}$ extends to a line bundle on the minimal compactification, whose restriction to $Sh(n,n)_{GP}$ is denoted $\mathcal{E}_{\mu,\kappa,GP}$, associated to a character denoted $\lambda_{\mu,\kappa,n}$ on p. 143 of [H3]. The exact formula does not matter.

Let $\mathfrak{r}_P : H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa}) \to H^0(Sh(n,n)_{GP},\mathcal{E}_{\mu,\kappa,GP})$ denote the Siegel Φ-operator (constant term map) for holomorphic forms. In classical language, $\mathfrak{r}_P$ takes a holomorphic form of weight (μ,κ) to the constant term of its Fourier expansion relative to the tube domain realization of the universal cover of $Sh(n,n)$. It follows from the general theory [H1, §6] that $\mathfrak{r}_P$ is rational over $E(\mu,\kappa) = E(\Sigma)$ and intertwines the $H(\mathbf{A}_f)$ actions. Thus:

Lemma 2.4.5. *Let π_f be an irreducible admissible representation of $GH(\mathbf{A}_f)$ occurring in $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ and let $\mathfrak{r}_P[\pi_f]$ denote the restriction of r_P to $M[\pi_f] \subset H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$. Then $\mathfrak{r}_P[\pi_f]$ is either zero or an isomorphism onto its image. For any $\gamma \in Gal(\overline{\mathbb{Q}}/E(\mu,\kappa))$, $\mathfrak{r}_P[\gamma(\pi_f)] = \gamma \circ \mathfrak{r}_P[\pi_f]$.*

Lemma 2.4.5 is expressed in terms of sections of automorphic vector bundles. With respect to the isomorphism (2.4.1), r_P can be identified as the constant term in the theory of automorphic forms. More precisely, there is a commutative diagram:

$$\begin{array}{ccc} Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa),\mathcal{A}(n,n)) & \xrightarrow[\sim]{} & H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa}) \\ \Big\downarrow r_P & & \mathfrak{r}_P \Big\downarrow \\ Hom_{GP(\mathbb{R})}(\lambda^{-1}_{\mu,\kappa,n}, I^{GH(\mathbf{A}_f)}_{GP(\mathbf{A}_f)}\mathcal{A}(GM(\mathbf{A}))) & \xrightarrow[\sim]{} & H^0(Sh(n,n)_{GP},\mathcal{E}_{\mu,\kappa,GP}) \end{array} \tag{2.4.6}$$

Here $\lambda^{-1}_{\mu,\kappa,n}$ is the inverse of the character of $GP(\mathbb{R})$ which defines the line bundle $\mathcal{E}_{\mu,\kappa,GP}$, and $\mathcal{A}(GM(\mathbf{A}))$ is the space of automorphic forms on $GM(\mathbf{A})$. Strong approximation for $SL(n)$ implies that any irreducible automorphic representation of $GM(\mathbf{A})$ whose archimedean component is a character, in this case $\lambda^{-1}_{\mu,\kappa,n}$, is necessarily one-dimensional. The effect of $I^{GH(\mathbf{A}_f)}_{GP(\mathbf{A}_f)}$ is to induce at finite places but leave the archimedean place alone, cf. [HZ, Cor. 3.2.9]. The left-hand vertical arrow r_P is defined by the usual constant term map on $\mathcal{A}(n,n)$. The lower horizontal isomorphism also depends on the choice of a basis (canonical trivialization) of $D_{\mu,\kappa,P}$, which is determined by the remaining three arrows. As above, the right half of the diagram determines rational structures on the left half which makes r_P $E(\mu,\kappa)$-rational.

(2.5) Twisting by characters on the Shimura variety.

The map $\det : GH \to T_{\mathcal{K}} \overset{def}{=} R_{\mathcal{K}/\mathbb{Q}}(\mathbb{G}_m)_{\mathcal{K}}$ defines a morphism of Shimura data

$$(GH, X^d_{n,n}) \to (T_{\mathcal{K}}, h_{n,n}). \tag{2.5.1}$$

The homomorphism $h_{n,n} : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to T_{\mathcal{K},\mathbb{R}}$ is defined by (2.5.1) and is given explicitly by the condition that, for every embedding $\sigma : \mathcal{K} \to \mathbb{C}$, the composition $\sigma \circ h_{n,n}(z) = (z\bar{z})^n$ (cf. [H3, (2.9.1)]).

In particular, the image of $h_{n,n}$ is contained in the subtorus $T_E \overset{def}{\to} R_{E/\mathbb{Q}}(\mathbb{G}_m)_E$ of $T_{\mathcal{K}}$. By the yoga of automorphic vector bundles, this means that, if ρ is an algebraic character of $T_{\mathcal{K}}$ trivial on T_E, the corresponding automorphic line bundle $\mathcal{L}_\rho$ on

$Sh(T_{\mathcal{K}}, h_{n,n})$ is $T_{\mathcal{K}}(\mathbf{A}_f)$-equivariantly isomorphic to the trivial line bundle. Thus, let β be a Hecke character of $\mathcal{K}$ that defines a section $[\beta] \in H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_\rho)$; equivalently, $\beta_\infty = \rho^{-1}$ (cf. [H3, (2.9.2)]. Then $[\beta]$ is a motivic Hecke character (character of type A_0), so the field $E[\beta]$ generated by the values of β on finite idèles is a finite extension of $\mathbb{Q}$.

Lemma 2.5.2. *Let E_ρ be the field of definition of the character ρ. The section $[\beta]$ is rational over the field $E[\beta]$; moreover, for any $\sigma \in Gal(\overline{\mathbb{Q}}/E_\rho)$, $\sigma[\beta] = [\sigma(\beta)]$, where σ acts on β by acting on its values on finite idèles.*

Let $\mathcal{E}_\rho$ denote the pullback of $\mathcal{L}_\rho$ to $Sh(n,n)$. Then twisting by $[\beta]$ defines an $E[\beta]$-rational isomorphism

$$i_\beta : H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa} \otimes \mathcal{E}_\rho).$$

Lemma 2.5.2 implies that $\sigma \circ i_\beta = i_{\sigma(\beta)}$, where the notation has the obvious interpretation.

Complex conjugation c defines an automorphism, also denoted c, of the torus $T_{\mathcal{K}}$. Every integer k defines a character ρ_k of $T_{\mathcal{K}}$ trivial on T_E:

$$\rho_k(z) = (z/c(z))^{-k}.$$

Let $\alpha_k = \rho_k \otimes N_{\mathcal{K}/\mathbb{Q}}^{-k}$. One can also twist by sections $[\beta]$ of $H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_{\alpha_k})$. The function β no longer defines a rational section, but we have:

Corollary 2.5.3. *Suppose β is a Hecke character with $\beta_\infty = \alpha_k^{-1}$, and let $[\beta] \in H^0(Sh(T_{\mathcal{K}}, h_{n,n}), \mathcal{L}_{\alpha_k})$ be the section defined by $(2\pi i)^{nk} \cdot \beta$. Then multiplication by $[\beta]$ defines an $E[\beta]$-rational isomorphism*

$$i_\beta : H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa}) \xrightarrow{\sim} H^0(Sh(n,n), \mathcal{E}_{\mu,\kappa} \otimes \mathcal{E}_{\alpha_k}).$$

Moreover, for any $\sigma \in Gal(\overline{\mathbb{Q}}/E_\rho)$, $\sigma \circ i_\beta = i_{\sigma(\beta)}$

2.6. The subvariety $Sh(n,n)^+ \subset Sh(n,n)$.

Let $\underline{S}$ denote the real algebraic group $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. In Deligne's formalism, the space $X_{n,n}$ is a conjugacy class of homomorphisms $h : \underline{S} \to GH_{\mathbb{R}}$. The group $GK(n,n)$, introduced in §1, is the centralizer of one such h, namely $h_0 = (h_{0,\sigma})$, with $h_{0,\sigma}$, the projection of h on the factor $GU(n,n)_\sigma$ of $GH(\mathbb{R})$, given on $z = x + iy \in \underline{S}(\mathbb{R}) \simeq \mathbb{C}^\times$ by

$$h_{0,\sigma}(x+iy) = \begin{pmatrix} xI_n & yI_n \\ -yI_n & xI_n \end{pmatrix} \tag{2.6.1}$$

in the block matrix form of (1.1). Over $\mathbb{C}$ there is an isomorphism

$$(\mu, \mu') : \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\sim} \underline{S}$$

and the Cayley transform conjugates $h_{0,\mathbb{C}}$ to

$$r_0 = (r_{0,\sigma}) : \mathbb{G}_m \times \mathbb{G}_m \to \prod_\sigma GU(n,n)_\sigma,$$

where

$$r_{0,\sigma}(t,t') = \begin{pmatrix} tI_n & 0_n \\ 0_n & t'I_n \end{pmatrix}. \tag{2.6.2}$$

The cocharacter $\mu_{h_0} = h_{0,\mathbb{C}} \circ \mu : \mathbb{G}_m \to GH_{\mathbb{C}}$ is thus conjugate to the cocharacter μ_{r_0}, defined by

$$t \mapsto \begin{pmatrix} tI_n & 0_n \\ 0_n & I_n \end{pmatrix}. \tag{2.6.3}$$

The reflex field $E(GH, X^d_{n,n})$ is the field of definition of the conjugacy class of μ_{h_0}, or equivalently of μ_{r_0}; since μ_{r_0} is defined over $\mathbb{Q}$, it follows that $E(GH, X^d_{n,n}) = \mathbb{Q}$.

In particular, Shimura's reciprocity law for the connected components of $Sh(n,n)$ shows that these are defined over $\mathbb{Q}^{ab}$. We recall Shimura's reciprocity law in the version of Deligne [D1], §2.6, which is correct up to a sign (depending on normalizations) that does not matter. Define a $\mathbb{Q}$-subgroup of $T_{\mathcal{K}} \times \mathbb{G}_m$ by

$$T = \{(u,t) \in T_{\mathcal{K}} \times \mathbb{G}_m \ \mid \ N_{\mathcal{K}/E}(t^{-n}u) = 1\} \tag{2.6.4}$$

The map $d = \det \times \nu : GH \to T_{\mathcal{K}} \times \mathbb{G}_m$ takes values in T, and the simply-connected semisimple group $\ker d = SU(n,n)$ is the derived subgroup GH^{der} of GH. It follows from strong approximation for simply-connected semisimple groups, as in [D1,§2.1], that the set $\pi_0(Sh(n,n))$ of geometrically connected components of $Sh(n,n)$ is a principal homogenous space under

$$\bar{\pi}_0\pi(G) = d(GH(\mathbf{A}_f))/\overline{d(GH(\mathbb{Q})_+)}.$$

Here $GH(\mathbb{Q})_+ = GH(\mathbb{Q}) \cap GH^+$ and the bar over $d(GH(\mathbb{Q})_+$ denotes topological closure.

Let $GH(\mathbf{A})^+ = \ker \varepsilon_{\mathcal{K}/E} \circ \nu$ as in (0.4), $GH(\mathbf{A}_f)^+ = GH(\mathbf{A})^+ \cap GH(\mathbf{A}_f)$, and let $Sh(n,n)^+$ denote the image of $(X^+_{n,n})^d \times GH(\mathbf{A}_f)^+$ in $Sh(n,n)$. This is a union of connected components of $Sh(n,n)$, defined over the subfield of $\mathbb{Q}^{ab}$ determined by the reciprocity law [D1, 2.6.3]. In [D1, *loc. cit*], the reflex field E is just $\mathbb{Q}$, and for the map $q_M : \pi(\mathbb{G}_m) \to \pi(G)$, in Deligne's notation, we can just take the map μ_{r_0} of (2.6.3). The reciprocity map [D1,(2.6.2.1)] is just deduced from the composite

$$d_{r_0} = d \circ \mu_{r_0} : \mathbb{G}_m \to T; t \mapsto (t^n, t) \tag{2.6.5}$$

It follows from the reciprocity law that $Sh(n,n)^+$ is defined over the field $L_{\mathcal{K}/E}$ defined by the kernel in $\mathbf{A}^\times/\mathbb{Q}^\times \cdot \mathbb{R}^\times_+ \xrightarrow{\sim} Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ of

$$\varepsilon(n,n) = \varepsilon_{\mathcal{K}/E} \circ \nu \circ d_{r_0}.$$

This is a quadratic character, so $[L : \mathbb{Q}] \leq 2$. If $E = \mathbb{Q}$, it is easy to see that $L_{\mathcal{K}/E} = \mathcal{K}$; if $\mathcal{K}$ contains no quadratic extension of $\mathbb{Q}$, then $L_{\mathcal{K}/E} = \mathbb{Q}$.

The theory of automorphic vector bundles is valid over $Sh(n,n)^+$, provided $L_{\mathcal{K}/E}$ is taken as the base field. The Siegel-Weil formula only determines Eisenstein series over $GH(\mathbf{A})^+$, hence only determines the corresponding section of automorphic vector bundles on $Sh(n,n)^+$. Let $Sh(n,n)^+_{GP}$ be the point boundary stratum of the minimal compactification of $Sh(n,n)^+$; i.e., it is the intersection of $Sh(n,n)_{GP}$ with the closure of $Sh(n,n)^+$ in the minimal compactification. Then $Sh(n,n)^+_{GP}$ is also defined over $L_{\mathcal{K}/E}$, and Lemma 2.4.5 remains true with the superscript $^+$. We note the following:

Lemma 2.6.6. *Let $\mathcal{E}$ be any automorphic vector bundle over $Sh(n,n)$, $f \in H^0(Sh(n,n),\mathcal{E})$, and define $f^+ \in H^0(Sh(n,n),\mathcal{E})$ to equal f on $Sh(n,n)^+$ and zero on the complement. If $\mathcal{E}$ and f are rational over the field L, then f^+ is rational over $L_{\mathcal{K}/E} \cdot L$.*

Proof. Let 1 denote the constant section of $H^0(Sh(n,n),\mathcal{O}_{Sh(n,n)})$ identically equal to 1. Since $f^+ = f \cdot 1^+$, it suffices to verify the lemma for $f = 1$, but this follows immediately from the reciprocity law.

3. The Siegel-Weil formula and arithmetic Eisenstein series

(3.1) The case of rank one. The constant term of the theta lift is easy to calculate. We first consider the theta lift from $U(V)$ with $\dim V = 1$, as above. The standard calculation of the constant term (cf. [KR, (1.3)]) gives

$$r_P(I_{\chi,triv,\psi}(\Phi))(h) = \tau(U(V)) \cdot \phi_\Phi(h),\ h \in GH^+(\mathbf{A}). \tag{3.1.1}$$

The constant $\tau(U(V))$ is Tamagawa measure, equal to 2; all that matters to us is that it is a non-zero rational number. The map r_P is equivariant with respect to the action of $GH(\mathbf{A}_f)$, and we can extend the left-hand side to a function of $GH(\mathbf{A})$ so that (3.1.1) remains valid for all $h \in GH(\mathbf{A}_f)$.

Write $GH_\infty = GH(E \times_{\mathbb{Q}} \mathbb{R})$, and define $GU(V)_\infty$, $U(V)_\infty$, and H_∞ likewise. Howe duality for the pair $(H, U(V))$ implies that $\mathcal{S}(V^n)_\infty = \mathcal{S}(V^n)(E \times_{\mathbb{Q}} \mathbb{R})$ decomposes, as representation of $(U(Lie(H_\infty)), K(n,n))) \times U(Lie(U(V)_\infty)$, as an infinite direct sum of irreducible representations, indexed by a certain subset of the set of characters of the torus $U(V)_\infty$. Assuming $V_v = V^+$ and $\psi_v(x) = e^{a_v x}$ with $a_v > 0$ for all real places v of E, each summand is a holomorphic representation of H_∞. The summand corresponding to the trivial character of $U(V)_\infty$ is just the tensor product of d copies of $\mathbb{D}(\mu,\kappa)$, indexed by real places of E. Let $\Phi^0_\infty \in \mathcal{S}(V^n)_\infty$ be a non-zero $U(V)_\infty$-invariant function in the (one-dimensional) holomorphic subspace of $\mathbb{D}(\mu,\kappa)^{\otimes^d}$. With this choice of Φ^0_∞, unique up to scalar multiples, $I_{\chi,triv,\psi}(\Phi) \otimes |\nu^{\frac{n\kappa}{2}}|$ defines an element of $Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(\mu,\kappa),\mathcal{A}(n,n))$, and thus of $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$. The function Φ^0_∞ can be written explicitly as a Gaussian, but any choice will do.

Recall that (2.4.6) has been normalized so that the map (3.1.1) is rational over $E(\kappa,\mu)$. One can characterize the rational structure on the functions on the right-hand side of (3.1.1) explicitly, and some choices of Φ^0_∞ are more natural than others for this purpose, but this is unnecessary. The following proposition is an immediate consequence of Lemma 2.4.5.

Proposition 3.1.2. *Let L be an extension of $E(\kappa,\mu)$. Let $\Phi = \Phi^0_\infty \otimes \Phi_f \in \mathcal{S}(V(\mathbf{A})^n)$ be any function such that*

$$\phi_\Phi \otimes |\nu^{\frac{n\kappa}{2}}| \in H^0(Sh(n,n)_{GP},\mathcal{E}_{\mu,\kappa,GP})(L) \tag{3.1.3}$$

(i.e., is rational over L) in terms of the bottom isomorphism in (2.4.6). Then $I_{\chi,triv,\psi}(\Phi) \otimes |\nu^{\frac{n\kappa}{2}}|$ defines an L-rational element of $H^0(Sh(n,n),\mathcal{E}_{\mu,\kappa})$ in terms of the top isomorphism in (2.4.6).

More generally, let $\gamma \in Gal(\overline{\mathbb{Q}}/E(\kappa,\mu))$ and suppose

$$\Phi_1 = \Phi^0_\infty \otimes \Phi_{1,f}, \Phi_2 = \Phi^0_\infty \otimes \Phi_{2,f} \in \mathcal{S}(V(\mathbf{A})^n)$$

have the property that ϕ_{Φ_i} is an L-rational element of $H^0(Sh(n,n)_{GP}, \mathcal{E}_{\mu,\kappa,GP})$, $i = 1, 2$, for some $L \subset \overline{\mathbb{Q}}$ containing $E(\kappa,\mu)$, and such that $\gamma(\phi_{\Phi_1}) = \phi_{\Phi_2}$. Then

$$\gamma(I_{\chi,triv,\psi}(\Phi_1)) = I_{\chi,triv,\psi}(\Phi_2).$$

This can be improved by taking $\Phi_{i,f}$ to be $\overline{\mathbb{Q}}$-valued functions. The twist by a power of $|\nu|$ implicitly introduces a power of $(2\pi i)$ in the normalization, as in Corollary 2.5.3. This can be absorbed into Φ^0_∞ and is invisible in the above statement, because it is present in the boundary value as well as in the theta integral. However, the Galois group acts on the additive character ψ as well as on the values of $\Phi_{i,f}$, so the result is not completely straightforward. Note that the statement of Proposition 3.1.2 depends only on the image of Φ_f in $R_n(V,\chi)_f$.

(3.2) Siegel-Weil theta series in general.

Now let V be of arbitrary dimension m, but always assume V to be positive-definite at all real places of E. Let A_V be the matrix of the hermitian form of V, in some basis. Recall that H was defined to be $U(2W)$. For the theta correspondence it is best to view $2W$ as the skew-hermitian space $\mathcal{K}^{2n}$, with skew-hermitian matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Define $\mathbb{W} = \mathcal{W} \otimes V$. This is naturally a skew-hermitian space, with matrix $\begin{pmatrix} 0 & I_n \otimes A_V \\ -I_n \otimes A_V & 0 \end{pmatrix}$. However, all even maximally isotropic skewhermitian spaces are isomorphic, so $\mathbb{W}$ is equivalent to the space with matrix $\begin{pmatrix} 0 & I_{nm} \\ -I_{nm} & 0 \end{pmatrix}$.

The subgroups H and $U(V)$ of $U(\mathbb{W})$ form a dual reductive pair. In particular, any automorphic form on $U(\mathbb{W})$ restricts to an automorphic form on $H \times U(V)$ whose integral over $U(V)(E)\backslash U(V)(\mathbf{A}_E)$ defines an automorphic form on H. More generally, the integral over $U(V)(E)\backslash U(V)(\mathbf{A}_E)$ of an automorphic form on $GU(\mathbb{W})$ defines an automorphic form on GH.

Let V_1 be the one-dimensional hermitian space $\mathcal{K}$ with the norm form. We fix $\mu = \frac{1-\kappa}{2}$, as is appropriate for the theta lift from $U(V_1)$ to $U(\mathbb{W})$. Attached to $GU(\mathbb{W})$ we have the Shimura variety $Sh(nm,nm)$. We let $(GU(V), \{point\})$ be the trivial Shimura datum attached to $GU(V)$: $\{point\}$ is the conjugacy class of the trivial homomorphism $\mathbb{C}^\times \to GU(V)(E \times_{\mathbb{Q}} \mathbb{R})$ over $\mathbb{R}$. Let $Sh(V)$ be the corresponding (zero-dimensional) Shimura variety. Tensor product defines a natural homomorphism $GU(n,n) \times GU(V) \to GU(\mathbb{W})$ which induces a natural map of Shimura data:

$$(GU(n,n) \times GU(V), X_{n,n} \times \{point\}) \to (GU(\mathbb{W}), X_{nm,nm})$$

and hence a morphism of Shimura varieties

$$Sh(n,n) \times Sh(V) \to Sh(nm,nm) \tag{3.2.1}$$

defined over the reflex field, which is $\mathbb{Q}$. The pullback of $\mathcal{E}_{\mu,\kappa}$ from $Sh(nm,nm)$ to $Sh(n,n) \times Sh(V)$ is just the pullback from $Sh(n,n)$ of $\mathcal{E}_{m\mu,m\kappa}$. This is because the restriction to $K(n,n) = U(n) \times U(n)$ of the determinant character

$$K(nm,nm) = U(nm) \times U(nm) \to U(1) \times U(1),$$

defined by (two copies of) the diagonal homomorphism

$$U(n) \to U(nm) = U(V^n),$$

is the m-power of the determinant character on $K(n,n)$.

Lemma 3.2.2. *Let $\tau = \tau(U(V)) = 2$. There is a commutative diagram*

$$\begin{array}{ccc} Hom_{(GU(\mathbb{W}),GK(nm,nm))}(\tilde{\mathbb{D}}(\mu,\kappa),\mathcal{A}(nm,nm)) & \xrightarrow[\sim]{} & H^0(Sh(nm,nm),\mathcal{E}_{\mu,\kappa}) \\ \Big\downarrow \tau(U(V))^{-1}\int_{U(V)(E)\backslash U(V)(\mathbf{A}_E)} & & I_V \Big\downarrow \\ Hom_{(Lie(GH),GK(n,n))}(\tilde{\mathbb{D}}(m\mu,m\kappa),\mathcal{A}(n,n)) & \xrightarrow[\sim]{} & H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa}) \end{array}$$

where the right vertical arrow is $E(\mu,\kappa) = E(m\mu,m\kappa)$-rational.

Proof. The fibers of $\mathcal{E}_{\mu,\kappa}$ on $Sh(nm,nm)$ and of $\mathcal{E}_{m\mu,m\kappa}$ on a point in the image of $Sh(n,n)$ in $Sh(nm,nm)$ are identical, so the trivialization (horizontal) maps can be made compatible. The right hand arrow I_V is then just given by projection on $U(V)(\mathbf{A}_f)$-invariants – which form a direct summand, since $U(V)$ is anisotropic – followed by restriction to $Sh(n,n)$. The action of $GU(\mathbb{W})(\mathbf{A}_f)$ is $E(\mu,\kappa)$-rational, so every step in the above description of the right hand arrow is $E(\mu,\kappa)$-rational.

We let P_{nm} and P_n denote the Siegel parabolic subgroups of $U(\mathbb{W})$ and H, respectively, and define GP_{nm} and GP_n likewise.

Lemma 3.2.3. *There is a commutative diagram*

$$\begin{array}{ccc} H^0(Sh(nm,nm),\mathcal{E}_{\mu,\kappa}) & \xrightarrow{\mathfrak{f}_{P_{nm}}} & H^0(Sh(nm,nm)_{GP_{nm}},\mathcal{E}_{\mu,\kappa,GP_{nm}}) \\ \Big\downarrow I_V & & \Big\downarrow \\ H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa}) & \xrightarrow{\mathfrak{f}_{P_n}} & H^0(Sh(n,n)_{GP_n},\mathcal{E}_{m\mu,m\kappa,GP_n}). \end{array}$$

The right-hand vertical arrow is given by restriction of functions from $GU(\mathbb{W})(\mathbf{A}_f)$ to $GH(\mathbf{A}_f)$.

Proof.

The two dual reductive pairs $(U(2W),U(V))$ and $(U(\mathbb{W}),U(V_1))$ form a seesaw:

$$\begin{array}{cc} U(\mathbb{W}) & U(V) \\ | & | \\ U(2W) & U(V_1). \end{array}$$

Here $U(V_1)$ is just the center of $U(V)$. The splittings of the metaplectic cover for the two pairs are compatible if one takes the character χ for the pair $(U(\mathbb{W}),U(V_1))$, with $\chi\mid_{\mathbf{A}_E^\times} = \varepsilon$, and $\chi^{\dim V} = \chi^m$ for the pair $(U(2W),U(V))$ (cf. [H1, §3] for the rules governing compatible splittings.

Corollary 3.2.4. *Let χ be a Hecke character of $\mathcal{K}^\times$ satisfying $\chi\mid_{\mathbf{A}_E^\times} = \varepsilon$. Let L be an extension of $E(\kappa,\mu)$. Let $\Phi = \Phi_\infty^0 \otimes \Phi_f \in \mathcal{S}(V(\mathbf{A})^n)$ be any function such that $\phi_\Phi \otimes |\nu^{\frac{nm\kappa}{2}}|$ is an L-rational element of $H^0(Sh(n,n)_{GP},\mathcal{E}_{m\mu,m\kappa,GP})$ in terms of the bottom isomorphism in (2.4.6). Then $I_{\chi^m,triv,\psi}(\Phi) \otimes |\nu^{\frac{nm\kappa}{2}}|$ defines an L-rational element of $H^0(Sh(n,n),\mathcal{E}_{m\mu,m\kappa})$ in terms of the top isomorphism in (2.4.6).*

More generally, let $\gamma \in Gal(\overline{\mathbb{Q}}/E(\kappa,\mu))$ and suppose

$$\Phi_1 = \Phi_\infty^0 \otimes \Phi_{1,f}, \Phi_2 = \Phi_\infty^0 \otimes \Phi_{2,f} \in \mathcal{S}(V(\mathbf{A})^n)$$

have the property that ϕ_{Φ_i} is an L-rational element of $H^0(Sh(n,n)_{GP}, \mathcal{E}_{m\mu,m\kappa,GP})$, $i = 1,2$, for some $L \subset \overline{\mathbb{Q}}$ containing $E(m\kappa, m\mu)$, and such that $\gamma(\phi_{\Phi_1}) = \phi_{\Phi_2}$. Then

$$\gamma(I_{\chi^m,triv,\psi}(\Phi_1)) = I_{\chi^m,triv,\psi}(\Phi_2).$$

Proof. We consider the first claim, the proof of the second claim being similar. When n is replaced by nm, V by V_1, and $(m\mu, m\kappa)$ by (μ,κ), this is just Proposition 3.1.2. Now by construction $\mathbb{W}\otimes V_1 = 2W\otimes V$ so $V_1(\mathbf{A})^{nm}$ is canonically isomorphic to $V(\mathbf{A})^n$. We apply Proposition 3.1.2 to the top line in the commutative diagram in Lemma 3.3.3 and obtain the conclusion under the stronger hypothesis that ϕ_Φ is an L-rational element of $H^0(Sh(nm,nm)_{GP}, \mathcal{E}_{\mu,\kappa,GP})$.

Now consider the commutative diagram

$$\begin{array}{ccc} \mathcal{S}(V_1(\mathbf{A}_f)^{nm}) & \xrightarrow{\phi_{\bullet,nm}} & H^0(Sh(nm,nm)_{GP_{nm}}, \mathcal{E}_{\mu,\kappa,GP_{nm}}) \\ \downarrow = & & \downarrow \\ \mathcal{S}(V(\mathbf{A}_f)^n) & \xrightarrow{\phi_{\bullet,nm}} & H^0(Sh(n,n)_{GP_n}, \mathcal{E}_{m\mu,m\kappa,GP_n}) \end{array} \tag{3.2.5}$$

Here $\phi_{\bullet,nm}$, resp. $\phi_{\bullet,n}$, is the map $\Phi \mapsto \phi_\Phi$ for the pair $(U(\mathbb{W}), U(1))$, resp. $(U(2n), U(V))$. By (3.1.1), we have

$$\phi_{\bullet,nm} = 2\cdot r_P \cdot I_{\chi,triv,\psi}; \quad \phi_{\bullet,n} = 2\cdot r_P \cdot I_{\chi^m,triv,\psi}.$$

Let $B_{nm} = Im(\phi_{\bullet,nm})$, $B_n = Im(\phi_{\bullet,n})$. It follows from Corollary 2.4.3 that B_{nm} is an $E(\mu,\kappa)$-rational subspace of $H^0(Sh(nm,nm)_{GP_{nm}}, \mathcal{E}_{\mu,\kappa,GP_{nm}})$. But the equality on the left of (3.2.5) implies that B_n is the image of B_{nm} under the right-hand vertical map. Thus B_n is also $E(\mu,\kappa)$-rational. It thus follows that, with $I_{\chi^m,triv,\psi}(\Phi)$ as in the statement of the lemma, we can assume $\phi_{\Phi,nm}$ as well as $\phi_{\Phi,n}$ is L-rational. The Corollary then follows from Proposition 3.1.2 and Lemma 3.2.2.

Corollary 3.2.4 is stated in terms of the character χ, but could just as well be stated in terms of the character χ^m; the condition is that the splitting character used to define the theta lift has to be an m-th power, where $m = \dim V$. However, this is unnecessary:

Corollary 3.2.5. *The assertions of Corollary 3.2.4 remain valid when $I_{\chi^m,triv,\psi}(\Phi)$ is replaced by $I_{\xi,triv,\psi}(\Phi)$ where ξ is any Hecke character of $\mathcal{K}^\times$ satisfying $\xi\mid_{\mathbf{A}_E^\times} = \varepsilon^m$.*

Proof. Indeed, by Theorem 2.1.4(c),

$$I_{\xi,triv,\psi}(\Phi) = I_{\chi^m,triv,\psi}(\Phi)\otimes(\xi/\chi^m)\circ\det.$$

Let $\beta = \xi/\chi^m$. This is a character trivial on the idèles of E, hence of the type considered in §2.5. Bearing in mind the various implicit normalizations and character twists, Corollary 3.2.5 follows from 2.5.3 and the previous corollary.

(3.3) Applications of the Siegel-Weil formula.

We now assume V is a positive-definite hermitian space over $\mathcal{K}$ of dimension $m \geq n$. Let $s_0 = \frac{m-n}{2}$. The Eisenstein series is normalized as in §1. Let $\Phi \in \mathcal{S}(V^n)(\mathbf{A})$, and let χ satisfy (1.2.4). Ichino has proved the following analogue of results of Kudla and Rallis:

Theorem 3.3.1 [I2]. *The extended Siegel-Weil formula is valid for Φ: the Eisenstein series $E(h,s,\phi_\Phi,\chi)$ has no pole at $s=s_0$, and*

$$I_{\chi,triv,\psi}(\Phi)(h) = c\cdot E(h,s_0,\phi_\Phi,\chi)$$

for $h\in H(\mathbf{A})$, where $c=1$ if $m=n$ and $c=\frac{1}{2}$ otherwise.

For applications to special values, we need the analogue of Ichino's theorem for $h\in GH(\mathbf{A})^+$.

Corollary 3.3.2. *The extended Siegel-Weil formula is valid on $GH(\mathbf{A})^+$:*

$$I_{\chi,triv,\psi}(\Phi)(h) = c\cdot E(h,s_0,\phi_\Phi,\chi)$$

for $h\in GH(\mathbf{A})^+$, where $c=1$ if $m=n$ and $c=\frac{1}{2}$ otherwise.

Proof. Since both sides are left-invariant under $GH(\mathbb{Q})$, it suffices to establish the identity for $h\in GH^+(\mathbf{A})$. The extension from $H(\mathbf{A})$ to $GH^+(\mathbf{A})$ is carried out as in §4 of [HK].

Corollary 3.2.5 then immediately has the following consequence:

Corollary 3.3.3. *Let χ be a Hecke character of $\mathcal{K}^\times$ satisfying $\chi\mid_{\mathbf{A}_E^\times}=\varepsilon$. Let L be an algebraic extension of $L_{\mathcal{K}/E}\cdot E(\kappa,\mu)$. Let $\Phi=\Phi_\infty^0\otimes\Phi_f\in\mathcal{S}(V(\mathbf{A})^n)$ be any function such that $f=\phi_\Phi$ is an L-rational element of $H^0(Sh(n,n)^+_{GP},\mathcal{E}_{m\mu,\kappa,GP})$ in terms of the bottom isomorphism in (2.4.6). Then $E(h,s_0,f,\chi)$ defines an L-rational element of $H^0(Sh(n,n)^+,\mathcal{E}_{m\mu,m\kappa})$ in terms of the top isomorphism in (2.4.6).*

More generally, if $f\in H^0(Sh(n,n)^+_{GP},\mathcal{E}_{m\mu,\kappa,GP})$ is an L-rational Siegel-Weil section for the pair $(U(2W),U(V))$ then for all $\gamma\in Gal(\overline{\mathbb{Q}}/L_{\mathcal{K}/E}\cdot E(\mu,\kappa))$

$$\gamma(E(h,s_0,f,\chi)) = E(h,s_0,\gamma(f),\gamma(\chi))$$

where γ acts on the finite part of χ.

Earlier work of Ichino [I1] considered the case of $m<n$. The results of [I1] are valid whether or not V is a definite hermitian space, and identify certain residues of Eisenstein series with explicit theta functions. In this way one can apply Corollary 3.2.5 to residues in certain cases. Since this is unnecessary for applications to special values of L-functions, we omit the details.

4 Special values of L-functions

We provide the expected application to special values of L-functions, extending Theorem 3.5.13 of [H3] to the middle of the critical strip for characters α satisfying (1.2.6). As explained in the introduction, this is a somewhat restrictive hypothesis, about which more will be said later. For the sake of simplicity, we assume $E=\mathbb{Q}$, so that $\mathcal{K}$ is imaginary quadratic, and the field $L_{\mathcal{K}/E}=\mathcal{K}$. The techniques can be applied without much difficulty to general CM fields.

Recall that in [H3], whose notation we use without further explanation, we have chosen a cuspidal automorphic representation π of $G=GU(W)$ with $\dim W=n$, $G(\mathbb{R})\xrightarrow{\sim}GU(r,s)$, and two algebraic Hecke characters χ and α of fixed weights. It is assumed that π, or rather its finite part π_f, occurs non-trivially in the middle-dimensional cohomology $\bar{H}^{rs}(W_\mu^\nabla)$ of the Shimura variety naturally associated to G wit coefficients in the local system denoted W_μ^∇ ($\bar{H}$ denotes the image of cohomology with compact support in cohomology). In [H3] the following condition, which should hold automatically, was inadvertently omitted:

Condition (4.1). *The representation π contributes to the* antiholomorphic *component of $\bar{H}^{rs}(W_{\mu}^{\nabla})$.*

The antiholomorphy of π on G is equivalent to the holomorphy of π viewed as an automorphic representation of the isomorphic group $GU(-W)$, where $-W$ is the space W with its hermitian form multiplied by -1; this amounts in the notation of [H3] to replacing the Shimura datum $(G, X_{r,s})$ by the (complex conjugate) Shimura datum $(G, X_{s,r})$. In fact this holomorphy (or antiholomorphy) is the only property used in the proof of Theorem 3.5.13, and was assumed explicitly on p. 151 of [H3] but omitted in the statement of the theorem. The hypothesis of belonging to middle-dimensional cohomology is only made in keeping with the overall motivic theme of [H3] and is in fact irrelevant to the proof, which works just as well for holomorphic forms contributing to cohomology in other degrees. Let $G(W,-W) \subset G \times G = GU(W) \times GU(-W)$ be the subgroup of pairs (g, g'), $g, g' \in G$, with equal similitude factors. We let $Sh(W) = Sh(G, X_{r,s})$, $Sh(-W) = Sh(G, X_{s,r})$, $Sh(W,-W)$ the Shimura variety corresponding to $(G(W,-W), X_{r,s} \times X_{s,r})$.

Let μ be the highest weight of a finite-dimensional representation of G (i.e., of $G(\mathbb{C})$). Then μ can be represented, as in [H3,2.1], by an $n+1$-tuple

$$(a_1, \ldots, a_r; a_{r+1}, \ldots, a_n; c)$$

of integers with $c \equiv \sum_i a_j \pmod 2$ and $a_1 \geq a_2 \cdots \geq a_n$. To μ we can also associate an automorphic vector bundle E_μ on $Sh(W)$. If $K_\infty \subset G(\mathbb{R})$ is the stabilizer of a base point in $X_{r,s}$ – then K_∞ is a maximal connected subgroup of $G(\mathbb{R})$, compact modulo the center – then μ is also the highest weight of an irreducible representation of K_∞ and E_μ is obtained from the corresponding hermitian equivariant vector bundle on $X_{r,s}$, cf. [H3,2.2]. If $k \in \mathbb{Z}$ we let $\eta_k(z) = z^{-k}$ for $z \in \mathbb{C}^\times$ and say the algebraic Hecke character χ of $\mathcal{K}^\times$ is *of type* η_k if $\chi_\infty = \eta_k$.

Let S be a finite set of finite places of $\mathbb{Q}$. We define the motivically normalized standard L-function, with factors at S (and archimedean factors) removed, to be

$$L^{mot,S}(s, \pi \otimes \chi, St, \alpha) = L^S(s - \frac{n-1}{2}, \pi, St, \alpha) \tag{4.2}$$

The motivically normalized standard zeta integrals are defined by the corresponding shift in the integrals [H3,(3.2.5)]. With these conventions, Theorem 3.5.13 of [H3] is the special case of the following theorem in which $m > n - \frac{\kappa}{2}$. Unexplained notation is as in [H3]:

Theorem 4.3. *Let $G = GU(W)$, a unitary group with signature (r, s) at infinity, and let π be a cuspidal automorphic representation of G. We assume $\pi \otimes \chi$ occurs in anti-holomorphic cohomology $\bar{H}^{rs}(Sh(W), E_\mu)$ where μ is the highest weight of a finite-dimensional representation of G. Let χ, α be algebraic Hecke characters of $\mathcal{K}^\times$ of type η_k and η_κ^{-1}, respectively. Let s_0 be an integer which is critical for the L-function $L^{mot,S}(s, \pi \otimes \chi, St, \alpha)$; i.e. s_0 satisfies the inequalities (3.3.8.1) of [H3]:*

$$\frac{n-\kappa}{2} \leq s_0 \leq min(q_{s+1}(\mu) + k - \kappa - \mathcal{Q}(\mu), p_s(\mu - k - \mathcal{P}(\mu)), \tag{4.3.1}$$

where notation is as in [loc. cit]. Define $m = 2s_0 - \kappa$. Let α^* denote the unitary character $\alpha/|\alpha|$ and assume

$$\alpha^* \mid_{\mathbf{A}_{\mathbb{Q}}^\times} = \varepsilon_{\mathcal{K}}^m. \tag{4.3.2}$$

Suppose there is a positive-definite hermitian space V of dimension m, a factorizable section $\phi_f(h,s,\alpha^*) \in I_n(s,\alpha^*)_f$, and factorizable vectors $\varphi \in \pi \otimes \chi$, $\varphi' \in \alpha^* \cdot (\pi \otimes \chi)^\vee$, such that

(a) *For every finite v, $\phi_v \in R_n(V_v, \alpha^*)$;*

(b) *For every finite v, the normalized local zeta integrals $\tilde{Z}_v^{mot}(s,\varphi_v,\varphi'_v,\phi_v,\alpha_v^*)$ do not vanish at $s = s_0$.*

Then

(i) One can find ϕ_f, φ, φ' satisfying (a) and (b) such that ϕ_f takes values in $(2\pi i)^{(s_0+\kappa)n} L \cdot \mathbb{Q}^{ab}$, and such that φ, φ' are arithmetic over the field of definition $E(\pi)$ of π_f.

(ii) Suppose φ is as in (i). Then

$$L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$$

where $P(s_0,k,\kappa,\pi,\varphi,\chi,\alpha)$ is the period

$$(2\pi i)^{s_0 n - \frac{nw}{2} + k(r-s) + \kappa s} g(\varepsilon_{\mathcal{K}}^{[\frac{n}{2}]} \cdot \pi^c P^{(s)}(\pi, *, \varphi) g(\alpha_0)^s p((\chi^{(2)} \cdot \alpha)^\vee, 1)^{r-s}$$

appearing in Theorem 3.5.13 of [H3].

(4.4) Remarks. (i) Condition (b) can be reinterpreted as the condition that the theta lift from $\pi_v \otimes \chi_v$ to $U(V)(E_v)$ be non-trivial, cf. [HKS,top of p. 975]. This will be discussed in more detail in [H5]. There it will be shown that, as long as π is locally tempered at all v, as will generally be the case for the π of arithmetic interest, and as long as $m-n \geq 2$, one can always find V and functions ϕ_v satisfying (a) and (b). When $m = n$ this is still possible unless the L-function vanishes at s_0, in which case the conclusion is vacuously true. When $m = n+1$ the conditions are necessary.

(ii) In Theorem 3.15.3 of [H3] π_f was assumed tempered. The hypothesis is used at no point in the proof and was only included because of the motivic context. However, the hypothesis does allow certain simplifications, as already indicated in the previous remark. If one assumes π admits a (weak) base change to a cuspidal automorphic representation of $GL(n,\mathcal{K})$ as in [HL, 3.1.3], then the local Euler factors at bad primes are the standard (Godement-Jacquet) local Euler factors of the base change; moreover, π_v is locally tempered everywhere [HL, 3.1.5], hence these Euler factors have no poles to the right of the center of symmetry. In this case, or more generally if π_v is assumed tempered everywhere, one can replace "normalized zeta integrals" by "zeta integrals" in condition (b).

(iii) Garrett's calculation of the archimedean local factor, cited without proof in [H3] as Lemma 3.5.3, has now appeared as Theorem [2.1] of [G]. Garrett's formulation is slightly different from that assumed in [H3]; whereas the latter asserted that the archimedean local factor is an element of $\mathcal{K}^\times$, Garrett's integral is a $\mathcal{K}$-multiple of π^{rs}. The following remark explains why one can conclude that Garrett's integral does not vanish. The power of π in Garrett's integral is compensated by our choice of measure (see the formula on p. 83 of [H3]).

There is a more subtle difference. Garrett calculates an integral over G as a an operator on the representation π, whereas the archimedean zeta integral in [H3] is an integral over G and gives a number as a result. One obtains a number by taking $g = 1$ in Garrett's Theorem [2.1]. However, this process implicitly depends on the choice of base point (choice of maximal compact subgroup) and one needs to compare Garrett's implicit normalizations with those considered in [H3]. This will be addressed in [H5].

(iv) The meaning of the inequalities (4.3.1), is that the anti-holomorphic representation $[\pi\otimes\chi]\otimes[\pi^\vee\otimes(\chi\cdot\alpha)^{-1}]$ pairs non-trivially with the holomorphic subspace $I_n(s_0,\alpha^*)^{hol}$ of the degenerate principal series representation $I_n(s_0,\alpha^*)$; this is the content of Lemma 3.3.7 and Corollary 3.3.8 of [H3]. It is well-known, and follows from the results of [LZ], particularly Proposition 5.8, that

$$I_n(s_0,\alpha^*)^{hol} = R_n(V(m,0),\alpha^*) \tag{4.4.1}$$

where $V(m,0)$ is the positive definite hermitian space over $\mathbb{C}$ of dimension m. As in [HKS], Proposition 3.1, the hypothesis that s_0 satisfies (4.3.1) then implies that (the contragredient of) $[\pi\otimes\chi]\otimes[\pi^\vee\otimes(\chi\cdot\alpha)^{-1}]$ has a non-trivial theta lift locally to $U(V(m,0))\times U(V(m,0))$. By Lemma 2.3.13 of [H5], it then follows that the (analytic continuation of the) archimedean local zeta integral calculated up to rational factors by Garrett does not vanish at $s=s_0$. Thus the archimedean counterpart of condition (b) is an automatic consequence of (4.3.1).

There is a global proof of the non-vanishing of the archimedean zeta integral that is simpler but requires more notation from [H3], as well as some notation for totally real fields I prefer to leave to the reader's imagination. The holomorphic Eisenstein series denoted $E(g,\alpha,s,\phi)$ in Corollary 3.3.10 of [H3] is non-zero at $s=0$ (because its constant term is non-zero). Thus $\Delta(m,\kappa,\Lambda)E(g,\alpha,0,\phi)\neq 0$ (because the archimedean component (4.4.1) is irreducible). Now if we replace the base field $\mathbb{Q}$ by a real quadratic extension L, with real places and $\mathcal{K}$ by the CM field $\mathcal{K}\cdot L$ then the analogous fact remains true. Let σ_1,σ_2 be the real places of L and suppose W is a hermitian space over $\mathcal{K}\dot{L}$ with signatures (r,s) at σ_1 and $(n,0)$ at σ_2, and let $\Delta_{\sigma_1}(m,\kappa,\Lambda)$ be the differential operator defined locally as in [H3] at the place σ_1. (Note that in [H3] the hermitian space is denoted V rather than W.) Then the holomorphic automorphic form $\Delta_{\sigma_1}(m,\kappa,\Lambda)E(g,\alpha,0,\phi)$ on $Sh(W,-W)$ is still non-trivial. But $U(W)$ is now anisotropic, so $\Delta_{\sigma_1}(m,\kappa,\Lambda)E(g,\alpha,0,\phi)$ is cuspidal. It follows that the analogue for $Sh(W)$ of the cup product map in Corollary 3.3.10 is non-trivial. In other words, the integral of $\Delta_{\sigma_1}(m,\kappa,\Lambda)E(g,\alpha,0,\phi)$ against some factorizable anti-holomorphic cusp form on $Sh(W,-W)$ of the right infinity type is non-zero. Thus all the local zeta integrals in the factorization [H3, (3.2.4)] are non-vanishing at $s=s_0$. But the local integral at σ_1 is the one of interest to us.

(4.5) $GH(\mathbf{A})$ vs. $GH(\mathbf{A})^+$. The proof of Theorem 4.3 presented below follows the treatment of the absolutely convergent case in [H3], with the difference that we have only proved arithmeticity of the Siegel-Weil Eisenstein series on the variety $Sh(n,n)^+$. This makes no difference to the final result, where the special value is only specified up to the equivalence relation $\sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}}$, but we need to modify the argument in two points. The notation of [H3] is used without comment.

(4.5.1). The basic identity of Piatetski-Shapiro and Rallis is stated in [H3], (3.2.4) for unitary similitude groups. Let $G=G(U(V)\times U(-V))\subset GH$, as in [loc. cit.], and let $G(\mathbf{A})^+ = G(\mathbf{A})\cap GH(\mathbf{A})^+$. Define the modified zeta integral

$$Z^+(s,f,f',\alpha,\phi) = \int_{Z(\mathbf{A})\cdot G(\mathbb{Q})\backslash G(\mathbf{A})^+} E(i_V(g,g'),\alpha,s,\phi)f(g)f'(g')dgdg'$$

where notation is as in [H3,(3.2.3)]. Then [H3, (3.2.4)] is replaced by

$$d_n^S(\alpha, s)Z^+(s, f, f', \alpha, \phi) = (f^+, f')_{V,\alpha} Z_S L^{mot,S}(s + \frac{1}{2}n, \pi, St, \alpha), \tag{4.5.2}$$

where Z_S is the product of the local zeta integrals at places in S, and the automorphic form f^+ is defined as in Lemma 2.6 to equal f on $G(\mathbf{A})^+$ and zero on the complement. The point is that the Euler factors only see the unitary group, and the similitudes are incorporated into the automorphic period factor $(f^+, f')_{V,\alpha}$.

(4.5.3). The motivic period factor $P^{(s)}(\pi, *, \varphi)$, also written $P^{(s)}(\pi, V, \varphi)$, is related to the automorphic period $(f, f')_{V,\alpha}$ by formulas (3.5.10), (3.5.8.2), and (3.5.12.1) of [H3]:

$$(2\pi)^c (f, f')_{V,\alpha} \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} P^{(s)}(\pi, V, \beta)^{-1} \cdot X(k, r, s, \chi, \alpha), \tag{4.5.4}$$

where $X(k, r, s, \chi, \alpha)$ is an explicit abelian period and $f = \beta \otimes \chi$, $f' = \beta' \otimes (\chi\alpha)^{-1}$ in the notation of [H3, §3.5]. Given (4.5.2), it now suffices to show that

$$(f, f')_{V,\alpha} \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} (f^+, f')_{V,\alpha} \tag{4.5.5}$$

when $f = \beta \otimes \chi$ and f' are chosen to correspond to arithmetic antiholomorphic forms. But this is an easy consequence of Lemma 2.6.6 (and Serre duality, which translates Lemma 2.6.6 into a dual assertion concerning antiholomorphic forms.

Proof of Theorem 4.3. We first observe that the subspaces $R_n(V_v, \alpha_v^*) \subset I_n(s_0, \alpha_v^*)$, for v finite, are rational over the field of definition of the finite part α_f of α. Indeed, the induced representation $I_n(s_0, \alpha_v)$ has a natural model as a space of functions transforming with respect to a certain character of the maximal parabolic P. This model is visibly defined over the field of definition of α_v. Indeed, the modulus character (1.1.1) is in general only defined over $\mathbb{Q}(\sqrt{p_v})$, where p_v is the residue characteristic of v, but one verifies that the half-integral shift in the motivic normalization ensures that no odd powers of the square root of the norm occur for critical values of s.[2] It is obvious (by considering restriction to $GU(2W, \mathcal{O}_v)$, for instance) that $\sigma(R_n(V_v, \alpha_v)) = R_n(V_v, \sigma(\alpha_v))$ for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$. Finally, the pairing (2.1.7.1) – with π replaced by $\pi \otimes \chi$ – is defined over the field $E(\pi, \chi, \alpha) \cdot \mathbb{Q}^{ab}$. This completes the verification of (i).

With (i) in hand, (ii) is proved exactly as in the absolutely convergent case in [H3], bearing in mind the modifications (4.5.1) and (4.5.3). First suppose $L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \neq 0$. As already mentioned in [H3,3.8], in order to treat the general case of critical s_0 to the right of the center of symmetry, it suffices to show that the holomorphic Eisenstein series that enter into the proof remain arithmetic at the point corresponding to s_0, and that one can find Eisenstein series that pair non-trivially with arithmetic vectors in $\pi \otimes \alpha \cdot \pi^\vee$. Corollary 3.3.2 asserts that the holomorphic Eisenstein series are arithmetic provided they are attached to arithmetic sections ϕ_f that satisfy (a). Given the basic identity of Piatetski-Shapiro and Rallis ([H3,3.2.4]) the non-triviality of the pairing is guaranteed by (b), Remark 4.4 (iv), and our hypothesis on non-vanishing of the L-value, and the proof is complete.

[2] in [H3] the rationality of the induced representation at critical points is implicitly derived from the rationality of the boundary data lifted in the definition of the Eisenstein series.

On the other hand, if $L^{mot,S}(s_0, \pi\otimes\chi, St, \alpha) = 0$, we need to prove that $L^{mot,S}(s_0, \pi^\sigma\otimes\chi^\sigma, St, \alpha^\sigma) = 0$ for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$. But this is standard (cf. [Ro]): since we can choose arithmetic data for which for the local zeta integrals at primes dividing S do not vanish at s_0, it suffices to observe that the global pairing between Eisenstein series and (anti-holomorphic) cusp forms is rational over the reflex field of $Sh(W, -W)$, which is either $\mathbb{Q}$ or $\mathcal{K}$.

§5. Normalizations: Comparison with [G]

Garrett's calculation in [G] of archimedean zeta integrals, up to algebraic factors, is based on a choice of abstract rational structure on the enveloping algebra and its holomorphic highest weight modules. The zeta integrals considered in §4 and in [H3, Lemma 3.5.3] involve explicit choices of data, especially an explicit choice of automorphy factor. The purpose of the present section is to verify that these two rational structures are compatible. In what follows $E = \mathbb{Q}$ and $\mathcal{K}$ is an imaginary quadratic field, but it should be routine to modify these remarks to apply to the general case.

The rationality invoked in [H3] is that inherited from [H1]. The group G, here $GU(W)$, is rational over $\mathbb{Q}$. The compact dual symmetric space $\hat{M} = \hat{M}(G, X)$ is endowed with a natural structure over the reflex field $E(G, X)$, which in our case is contained in $\mathcal{K}$. This natural $\mathcal{K}$-rational structure is compatible with the action of G, and all G-equivariant vector bundles on $\hat{M}$, along with the corresponding automorphic vector bundles on $Sh(G, X)$, are naturally defined over $\mathcal{K}$. All these remarks apply equally to the group $H = GU(2W)$. Though the corresponding Shimura variety, which we denote $Sh(H, X_{n,n})$ (cf. [H3] for $X_{n,n}$), is naturally defined over $\mathbb{Q}$, we will only need a $\mathcal{K}$-rational structure.

Let $\hat{M}_H = \hat{M}(H, X_{n,n})$. Both $\hat{M}$ and $\hat{M}_H$ have $\mathcal{K}$-rational points h and h_H, respectively, and the stabilizers K_h and $K_{h,H}$ are defined over $\mathcal{K}$, as are the Harish-Chandra decompositions, for example

$$Lie(H)_{\mathcal{K}} = Lie(K_{h,H}) \oplus \mathfrak{p}_H^+ \oplus \mathfrak{p}_H^- \tag{5.1}$$

(cf. [H1, 5.2] or §(1.3), above). For example, if W has a $\mathcal{K}$-basis in terms of which the hermitian form is given by the standard matrix $\begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$, then K_h is the group $K_\infty(r, s)$ defined in [H3, §2.1]. For general W, we can take K_h to be the stabilizer of the diagonal hermitian form $diag(a_1, \ldots, a_n)$ where $a_i \in \mathbb{Q}$, $a_i > 0, 1 \leq i \leq r$, $a_i < 0, r+1 \leq i \leq n$. The exact choice of hermitian form has no bearing on rationality, though it may be relevant to integrality questions. With respect to the canonical embedding of Shimura data

$$(G(U(W) \times U(-W)), X_{r,s} \times X_{s,r}) \hookrightarrow (H, X_{n,n})$$

we may assume $K_H \supset (K_h \times K_h) \cap H$. In other words, identifying $X_{r,s}$ with $X_{s,r}$ antiholomorphically, as in [H3,(2.5.1)], we may assume that the point (h, h) maps to h_H. This antiholomorphic map is the restriction to $X_{r,s}$ of a $\mathcal{K}$-rational isomorphism

$$\phi_{r,s} : R_{\mathcal{K}/\mathbb{Q}}(X_{r,s}) \xrightarrow{\sim} R_{\mathcal{K}/\mathbb{Q}}(X_{s,r}).$$

Let K be either K_h or $K_{h,H}$, $\mathcal{G}$ correspondingly either G or H, and let (τ, V_τ) be a finite-dimensional representation of K, which can be taken rational over $\mathcal{K}$. Since the decomposition (5.1) and its analogue for G are $\mathcal{K}$-rational, the holomorphic highest weight module

$$\mathbb{D}_\tau = U(Lie(\mathcal{G})) \otimes_{U(Lie(K)\oplus\mathfrak{p}^-)} V_\tau \tag{5.2}$$

has a natural $\mathcal{K}$-rational structure; here $\mathfrak{p}^-$ is the anti-holomorphic summand in the Harish-Chandra decomposition for $\mathcal{G}$. When $\mathcal{G} = H$ we only need to consider one-dimensional τ, generated by a holomorphic automorphy factor of the form $J_{\mu,\kappa}$ of (1.1.3). The main point of the comparison is that $J_{\mu,\kappa}$ is a rational function on the algebraic group H, defined over $\mathcal{K}$. The $\mathcal{K}$-rational form of $\mathbb{D}_\tau = \mathbb{D}(\mu,\kappa)$ generated by $U(\mathfrak{p}_H^+)(\mathcal{K}) \otimes J_{\mu,\kappa}$ is a $\mathcal{K}$-subspace of the space of rational functions on H.[3]

Since the rational structures on $\hat{M}$ and $\hat{M}_H$ and the corresponding Harish-Chandra decompositions are compatible, we are now in the situation considered by Garrett. Note that the representations of K_h considered in [G] arise in practice as the irreducible $U(Lie(GU(W) \times U(-W))$-summands of $\mathbb{D}(\mu,\kappa)$, as in [H1, 7.4, 7.11]. It remains to show that the rational structure considered above is also the one used to define archimedean zeta integrals in [H3]. This comes down to four points:

(1) Rationality for holomorphic Eisenstein series on H is defined in terms of rationality of the constant term, i.e. in terms of functions in the induced representation, cf. (3.1.3) and Corollary 3.2.4, as well as [H3, 3.3.5.3]. By the results of [H1,§5], the archimedean condition for rationality of the constant term is compatible up to $\mathcal{K}$ with rationality of $J_{\mu,\kappa}$.
(2) The $\mathcal{K}$-rational differential operators of [H3,Lemma 3.3.7] are defined in terms of the $\mathcal{K}$-rational basis of $\mathfrak{p}_H^+$. This is because the *canonical trivializations* discussed in [H3, §2.5] are defined in terms of the $\mathcal{K}$ rational structure of homogeneous vector bundles on $\hat{M}$ and $\hat{M}_H$. Applying this remark to the (homogeneous) normal bundle of $\hat{M} \times \hat{M}$ in $\hat{M}_H$ (cf. [H1, 7.11.7]) we see that the dual basis to the $\mathcal{K}$-rational basis of $\mathfrak{p}_H^+$ chosen above can serve to define a canonical trivialization.
(3) Garrett's Theorem 2.1 calculates the local zeta integral as an E-rational multiple of π^{pq}, multiplied by the value $f(1)$, where f is a discrete-series matrix coefficient. As in [H3,§3.2], the Euler product factorization of the global zeta integral is normalized in such a way as to allow us to assume that $f(1) = 1$.
(4) Finally, as already explained in (4.4)(iii), the π^{pq} in Garrett's final result is compensated by our choice of measure, so that in our normalization the zeta integral is in fact in $\mathcal{K}^\times$.

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[3] This is not quite right as stated, because the automorphy factor has only been defined on a connected component of the real points of H. However, the explicit definition of $J_{\mu,\kappa}$ in terms of matrix entries shows that it extends to a rational function on all of H.

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