

EQUIVARIANT BIRATIONAL TYPES AND BURNSIDE VOLUME

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ABSTRACT. We introduce equivariant Burnside groups, new invariants in equivariant birational geometry, generalizing birational symbols groups for actions of finite abelian groups, due to Kontsevich, Pestun, and the second author, and study their properties. We establish a specialization map for the equivariant birational type of a smooth algebraic variety with an action of a finite group.

1. INTRODUCTION

Let G be a finite group, acting on algebraic varieties X and X' , defined over a field k . A classical problem in higher-dimensional algebraic geometry is to determine whether or not there exists a G -equivariant birational morphism between X and X' . This is a formidable challenge already when G is the trivial group and $X' = \mathbb{P}^n$: the *rationality problem* has occupied generations of mathematicians and inspired the introduction of many important ideas and techniques. Recently, there has been a resurgence of interest and activity in rationality questions, with the introduction of specialization techniques in [32], [9], [23], [17], and their applications to various classes of varieties, e.g., in [30], [11], [12], [13], [27], [28], [22].

There is also an extensive literature on G -equivariant birationality. In dimension two, the problem reduces to the study of the G -module structure of the Picard lattice $\text{Pic}(X)$, coupled with the detailed study of the geometry of Del Pezzo surfaces (see, e.g., [20], [10], [7]). The situation is much less clear in higher dimensions, where the main tools come from the Minimal Model Program and birational rigidity.

In this paper, we introduce and study a new invariant in G -equivariant birational geometry. Informally, it is defined as follows. Let $X \curvearrowright G$ be a smooth projective variety of dimension n , together with a generically free action of G . Then:

- After a sequence of blow-ups in smooth G -invariant subvarieties, we can assume that the G -action on X has only *abelian* stabilizers.

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- To each (orbit of a) subvariety with *generic* abelian stabilizer $H \subseteq G$ we attach a symbol

$$(H, N_G(H)/H \curvearrowright K, \beta), \quad (1.1)$$

where K is the function field of the stratum with action of the normalizer $N_G(H)$ of H in G , such that the restriction to H is the trivial action, and β encodes the generic representation of H on the normal bundle.

- The invariant $[X \curvearrowright G]$ of $X \curvearrowright G$ is the sum of all such symbols.

The invariant takes values in the *equivariant Burnside group*,

$$\text{Burn}_n(G) = \text{Burn}_{n,k}(G),$$

a quotient of the \mathbb{Z} -span of symbols as above by explicit relations, imposed so that

$$[X \curvearrowright G] = [X' \curvearrowright G]$$

whenever X and X' are G -equivariantly birational. In the definition of $[X \curvearrowright G]$ (Definition 4.4), we rely on

- the divisorialification algorithm of Bergh and Bergh-Ryd, applied to the quotient stack $[X/G]$, to obtain abelian stabilizers,
- the G -equivariant form of weak factorization.

Throughout, we make the assumption that k has characteristic zero, and we make an additional, simplifying assumption concerning roots of unity in k . (These are stated in Section 2.)

When G is trivial, $\text{Burn}_n(G)$ is the Burnside group of fields

$$\text{Burn}_n = \text{Burn}_{n,k}$$

defined in [17]. When G is abelian, $\text{Burn}_n(G)$ admits a surjective homomorphism to the group $\mathcal{B}_n(G, k)$ introduced in [16]; the comparison is explained in Section 8. For general G , we construct a homomorphism

$$\text{Burn}_n(G) \rightarrow \overline{\text{Burn}}_n,$$

to the Burnside group of orbifolds of [18]. When X is a smooth projective variety with generically free G -action, this sends $[X \curvearrowright G]$ to the class in $\overline{\text{Burn}}_n$ of the quotient stack $[X/G]$.

One of our main constructions is a homomorphism of abelian groups, the *equivariant Burnside volume* (Definition 6.4)

$$\rho_\pi^G: \text{Burn}_{n,K}(G) \rightarrow \text{Burn}_{n,k}(G),$$

when K is the function field of a complete DVR \mathfrak{o} with residue field k and uniformizer π . This is an equivariant version of the map of [17, §5].

As a consequence we obtain the equivariant specialization of birationality (Corollary 6.8): *If*

$$\mathcal{X} \rightarrow \text{Spec}(\mathfrak{o}) \quad \text{and} \quad \mathcal{X}' \rightarrow \text{Spec}(\mathfrak{o})$$

are smooth and projective, with generically free actions of a finite group G and G -equivariantly birational generic fibers, then the special fibers are G -equivariantly birational.

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2. PRELIMINARIES: ALGEBRA

In this section we outline an algebraic condition, in terms of group cohomology, and some of its consequences, that apply to the function fields in a symbol (1.1).

Let G be a finite group; write $N_G(H)$ for the normalizer of a subgroup $H \subseteq G$. Fix a choice of representatives

$$H_1, \dots, H_r$$

of the conjugacy classes of *abelian* subgroups of G and let e be the least common multiple of the exponents of H_1, \dots, H_r . Throughout, we work over a field k of characteristic zero; for convenience, we assume that k contains primitive e -th roots of unity. The character group of an abelian subgroup $H \subseteq G$ will be denoted by

$$H^\vee := \text{Hom}(H, k^\times).$$

We will use standard facts about Galois algebras K/K_0 for a finite group Γ , where K_0 is a field. For instance (see, e.g., [14, §4.3]):

- Writing $K \cong K^1 \times \dots \times K^\ell$ as product of fields, for the subgroup Γ^1 that sends the factor K^1 to itself we have K equivariantly isomorphic to $\text{Ind}_{\Gamma^1}^{\Gamma}(K^1)$.
- (Hilbert's Theorem 90) $H^1(\Gamma, K^\times) = 0$.

Assumption 1. If $H \subseteq G$ is an abelian subgroup, K_0 a field containing k , and K/K_0 a Galois algebra for the group $N_G(H)/H$, then the composite homomorphism

$$H^1(N_G(H), K^\times) \rightarrow H^1(H, K^\times)^{N_G(H)/H} \rightarrow H^\vee \tag{2.1}$$

is surjective, where the first map comes from the Hochschild-Serre spectral sequence and the second is obtained by writing

$$K \cong K^1 \times \cdots \times K^\ell,$$

where each K^i is a field, and projecting to a factor

$$H^1(H, (K^i)^\times) = \text{Hom}(H, (K^i)^\times) = H^\vee.$$

Remark 2.1. That K_0 is a field in Assumption 1 implies:

- (i) The choice of a different factor $K^{i'}$ in Assumption 1 modifies the composite homomorphism (2.1) by the automorphism of H^\vee , induced by conjugation of H by a suitable element of $N_G(H)$.
- (ii) Assumption 1 is therefore independent of the choice of factor K^i .
- (iii) The composite homomorphism (2.1) is always injective, since by the Hochschild-Serre spectral sequence, the kernel is identified with $H^1(N_G(H)/H, K^\times)$, which is zero by Hilbert's Theorem 90. Thus, an equivalent form of Assumption 1 is that the composite homomorphism (2.1) is an isomorphism.

Proposition 2.2. *Let H be an abelian normal subgroup of a finite group N , K_0 a field containing primitive e -th roots of unity, where e is the exponent of H , and K/K_0 a Galois algebra for the group N/H . Write*

$$K \cong K^1 \times \cdots \times K^\ell,$$

where K^1, \dots, K^ℓ are fields, choose some $i \in \{1, \dots, \ell\}$, and suppose that the composite homomorphism

$$H^1(N, K^\times) \rightarrow H^1(H, K^\times)^{N/H} \rightarrow H^1(H, (K^i)^\times)$$

is an isomorphism. Then, for every $m \in \mathbb{N}$ there is a unique map of non-abelian cohomology sets

$$H^1(N, GL_m(K)) \rightarrow H^1(H, GL_m(K_0)) \tag{2.2}$$

that is compatible with extension of scalars

$$H^1(H, GL_m(K_0)) \rightarrow H^1(H, GL_m(K^i))$$

and restriction

$$H^1(N, GL_m(K)) \rightarrow H^1(H, GL_m(K^i)),$$

and this map (2.2) is bijective.

Proof. The extension of scalars map is injective (by standard representation theory). It suffices to exhibit a bijective map (2.2) that is compatible with the maps to $H^1(H, GL_m(K^i))$.

Let $\zeta \in K_0$ be a primitive e -th root of unity. We write, according to the structure theorem of finite abelian groups,

$$H \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z}, \quad n_1, \dots, n_r \geq 2, \quad n_i \mid n_{i+1} \text{ for all } i;$$

with $n_r = e$. Sending the i -th generator of H to ζ^{e/n_i} and all other generators to 1, we have an element of $H^1(H, K_0^\times)$, which by hypothesis comes from a 1-cocycle $(u_{i,g})_{g \in N}$ with values in K^\times . Furthermore, $(u_{i,g}^{n_i})_{g \in N}$ is a 1-coboundary, i.e., for some $v_i \in K^\times$ we have

$$u_{i,g}^{n_i} = {}^g v_i / v_i, \quad \text{for all } g \in N.$$

The data of $(u_{i,g}^{n_i})_{i,g}$ and $(v_i)_i$ give us a way to assign, functorially, a Galois algebra for the group H over an étale K_0 -algebra L_0 to every Galois algebra L/L_0 for the group N with N -equivariant K_0 -algebra homomorphism $K \rightarrow L$. Specifically, given L/L_0 and $\iota: K \rightarrow L$, for every i we apply Hilbert's Theorem 90 to obtain $w_i \in L^\times$ such that

$$\iota(u_{i,g}) = {}^g w_i / w_i, \quad \text{for all } g \in N.$$

For every i , then, $\iota(v_i)w_i^{-n_i}$ is Galois-invariant, i.e., lies in L_0 , and is unique up to multiplication by an element of $(L_0^\times)^{n_i}$. This observation lets us assign, functorially, the Galois algebra

$$L_0[t_1, \dots, t_r]/(t_1^{n_1} - \iota(v_1)w_1^{-n_1}, \dots, t_r^{n_r} - \iota(v_r)w_r^{-n_r})$$

for the group H over L_0 . The functorial association is fully faithful. Essential surjectivity follows easily from the fact that any two Galois algebras L'_0/L_0 and L''_0/L_0 for the group H become equivariantly isomorphic after passage to a suitable étale algebra over L_0 .

The previous paragraph gives us an equivalence of categories. This in turn induces an equivalence of categories of the functorial associations of a free L_0 -module of rank m to every Galois algebra L/L_0 for the group N with N -equivariant K_0 -algebra homomorphism $K \rightarrow L$, respectively, to Galois algebras L'_0/L_0 for the group H . The isomorphism classes of objects are given by $H^1(N, GL_m(K))$, respectively, $H^1(H, GL_m(K_0))$, so the equivalence of categories gives us a bijective map (2.2). The compatibility with the maps to $H^1(H, GL_m(K^i))$ is immediate from the construction and the fact that m -dimensional representations of H over K_0 , whose equivalence classes are parametrized by $H^1(H, GL_m(K_0))$, decompose as direct sums of 1-dimensional representations. \square

Action construction (A): Let

$$H \subseteq G, \quad K_0, \quad N_G(H)/H \curvearrowright K,$$

satisfy Assumption 1, let

$$a_1, \dots, a_c \in A := H^\vee$$

be characters, and define

$$\overline{H} := \bigcap_{i=1}^c \ker(a_i),$$

with character group

$$\overline{A} := A/\langle a_1, \dots, a_c \rangle.$$

We take

$$u_1, \dots, u_c$$

to be 1-cocycles, $u_i = (u_{i,g})_{g \in N_G(H)}$, with values in K^\times , corresponding by the isomorphism that we have, thanks to Remark 2.1(iii), to a_1, \dots, a_c ; these determine an action

$$N_G(H) \curvearrowright K(t_1, \dots, t_c).$$

Writing, as in Assumption 1,

$$K \cong K^1 \times \dots \times K^\ell,$$

we define K' to be the product of all K^i such that the action

$$H \curvearrowright K^i(t_1, \dots, t_c)$$

restricts to the *trivial* action of \overline{H} . Then the action of

$$\overline{H}' := N_{N_G(H)}(\overline{H})$$

on $K'(t_1, \dots, t_c)$ restricts to the trivial action of \overline{H} , thus we have

$$\overline{H}'/\overline{H} \curvearrowright K'(t_1, \dots, t_c).$$

We define

$$\overline{K} := \text{Ind}_{\overline{H}'}^{N_G(\overline{H})} K'(t_1, \dots, t_c).$$

3. PRELIMINARIES: GEOMETRY

Let G be a finite group and X a reduced quasiprojective scheme over k , equipped with a generically free action of G . By convention, we suppose that the generic point of any component has dense orbit in X . Then we will say that X is a quasiprojective variety with generically free G -action. We suppose that X is smooth and remind the reader that this can be achieved with equivariant resolution of singularities. (Resolution of singularities is available in a functorial form, see, e.g., [31], [6].) We will also make use of functorial weak factorization, established in [1].

By a *smooth blow-up* of X we mean the blow-up of X along a smooth subscheme. We say that a smooth blow-up is *equivariant* when it is the blow-up of X along a smooth G -invariant subscheme.

Remark 3.1. Geometry naturally leads to situations where Assumption 1 is relevant. Suppose that G acts on X with abelian stabilizer groups. Let $V \subset X$ be a reduced subscheme, such that if we express the decomposition into irreducible components as

$$V = V^1 \cup \cdots \cup V^q,$$

then G acts transitively on $\{V^1, \dots, V^q\}$. The generic stabilizer groups of the V^i comprise a single conjugacy class of abelian subgroups of G , say, the one represented by $H \in \{H_1, \dots, H_r\}$. We write

$$\{Y^1, \dots, Y^\ell\} := \{V^i \mid \text{the stabilizer of the generic point of } V^i \text{ is } H\},$$

$$Y := Y^1 \cup \cdots \cup Y^\ell,$$

and note that $N_G(H)$ acts transitively on $\{Y^1, \dots, Y^\ell\}$. We choose a trivialization of the fiber of the normal bundle

$$\mathcal{N}_{Y/X}$$

at the generic point of each Y^i . Letting m denote the codimension of Y , the $N_G(H)$ -action gives rise to a 1-cocycle, representing a class in

$$H^1(N_G(H), GL_m(k(Y))),$$

where $k(Y)$ denotes $k(Y^1) \times \cdots \times k(Y^\ell)$. We have a Galois algebra $k(Y)/k(Z)$ for the group $N_G(H)/H$, where Z denotes the quotient variety

$$Y/N_G(H) \cong V/G,$$

and when $m = 1$ (respectively $m > 1$) the class may be studied via the map in Assumption 1 (respectively in Proposition 2.2).

For $H \in \{H_1, \dots, H_r\}$, the H -fixed locus X^H is smooth and stable under the action of $N_G(H)$. We consider an $N_G(H)$ -orbit of components of X^H where the stabilizer at the generic points of the components is equal to H . Denoting one such by Y , we have a generically free action of $N_G(H)/H$ on Y with an irreducible quotient variety Z .

Assumption 2. The G -action on X satisfies:

- All stabilizers of the action of G on X are abelian.
- For every $H \in \{H_1, \dots, H_r\}$, with character group H^\vee , and every Y as above, the composite homomorphism

$$\text{Pic}^G(X) \rightarrow H^1(N_G(H), k(Y)^\times) \rightarrow H^1(H, k(Y)^\times)^{N_G(H)/H} \rightarrow H^\vee$$

is surjective.

Remark 3.2. We make the following observations:

- (i) Assumption 2 implies Assumption 1, for all abelian $H \subseteq G$ and Galois algebras as above $k(Y)/k(Z)$ for the group $N_G(H)/H$.

- (ii) Assumption 2 implies that there exists a finite collection of G -linearized line bundles on X such that the images of the classes in $\text{Pic}^G(X)$ generate H^\vee , for every H and Y as above.
- (iii) Assumption 2 is equivalent to the existence of a finite collection of G -linearized line bundles on X such that the associated morphism of stacks

$$[X/G] \rightarrow B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m$$

(where the number of factors $B\mathbb{G}_m$ is the number of linearized line bundles) is representable.

In (iii), $[X/G]$ is the quotient stack associated with the G -action on X and $B\mathbb{G}_m$ is the classifying stack of \mathbb{G}_m , so that each G -linearized line bundle on X determines a morphism $[X/G] \rightarrow B\mathbb{G}_m$. For a given finite collection of G -linearized line bundles on X the equivalence of the conditions stated in (ii) and (iii) is given in [4, Rem. 7.14].

Proposition 3.3. *Let X be a smooth quasiprojective variety with a generically free action of G . There exists a sequence of equivariant smooth blow-ups*

$$X' = X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X,$$

such that X' , with its G -action, satisfies Assumption 2.

Proof. This is a consequence of the *divisorialification* algorithm of Bergh [4] and Bergh-Ryd [5], applied to the quotient stack $[X/G]$. For the convenience of the reader we explain how this works in the language of equivariant geometry. Let $D = D_1 \cup \cdots \cup D_\ell$ be a simple normal crossing divisor on X , where each D_i is G -invariant. The divisorial index is a quantity attached to a point x (closed or not) of X and the divisor D . Let H denote the stabilizer of x (so H acts trivially on $\overline{\{x\}}$), and let \mathcal{N} denote the normal bundle $\mathcal{N}_{\overline{\{x\}}/X}|_{\{x\}}$. Each D_i that contains x determines a one-dimensional linear representation of H . Denoting the intersection of the kernels of these representations by H' , the divisorial index at x is the dimension of the nontrivial part of \mathcal{N} , as a representation of H' .

We regard X as equipped with a simple normal crossing divisor $D = D_1 \cup \cdots \cup D_\ell$ on X , where each D_i is G -invariant: initially D may be taken to be empty, and with every blow-up we replace each D_i by its proper transform and adjoin the exceptional divisor as $D_{\ell+1}$. Let m denote the maximal value of the divisorial index, over all the points of X . If $m > 0$, then by [4, §7–8] and [5, §6] the points with divisorial index m are the points of a smooth G -invariant closed subscheme $W \subset X$ which has normal crossing with D_1, \dots, D_ℓ , and after blowing up X along W , every point of the blow-up has divisorial index less than m . Iterating, we achieve $m = 0$. \square

Remark 3.4. In the proof of Proposition 3.3 we may start with any simple normal crossing divisor $D = D_1 \cup \dots \cup D_\ell$ on X , where each D_i is G -invariant. This transforms in the indicated manner with each blow-up to yield, finally, a simple normal crossing divisor $D' = D'_1 \cup \dots \cup D'_{\ell'}$ on X' . A variant, which does not change the variety X' that we get at the end (up to isomorphism), is to write at every step

$$W = W_1 \cup \dots \cup W_t,$$

a disjoint union, where each W_i is a G -orbit of components, and to index the exceptional divisor over each W_i separately. Denoting by $\varphi: X' \rightarrow X$ the composite map, with this variant the support of $\varphi^{-1}(D)$ is necessarily of the form $\bigcup_{i \in S} D'_i$ for suitable $S \subseteq \{1, \dots, \ell'\}$.

Example 3.5. Even if k contains all roots of unity (e.g., is algebraically closed), Assumption 1 is nontrivial. For instance, let $G = \mathfrak{D}_8$ be the dihedral group of order 8, generated by ρ of order 4 and σ of order 2. For $H = \langle \rho \rangle$ we consider the Galois algebra $\mathbb{C}(z)/\mathbb{C}(z^2)$ for the group G/H . Assumption 1 is not satisfied, since $|H^1(G, \mathbb{C}(z)^\times)| = 2$. When this arises geometrically, e.g., from G acting on $\mathbb{A}_{\mathbb{C}}^3$ with

$$\rho \cdot (x, y, z) = (-y, x, z) \quad \text{and} \quad \sigma \cdot (x, y, z) = (x, -y, -z),$$

divisorialification in the form of Proposition 3.3 leads to a situation where Assumption 2 is satisfied, hence as well Assumption 1.

Proposition 3.6. *Let X and X' be smooth projective varieties, each equipped with a generically free G -action, satisfying Assumption 2. Given a G -equivariant birational map*

$$\varphi: X' \dashrightarrow X,$$

restricting to an isomorphism over an open $U \subset X$, there exists a weak factorization of φ , where each map is, or is inverse to, a equivariant smooth blow-up along a center disjoint from U , and the intermediate varieties in the weak factorization also satisfy Assumption 2.

Proof. It suffices, by equivariant resolution of singularities, to treat the case that φ is a morphism. Then we apply the functorial weak factorization of [1] and notice that, since the intermediate varieties admit a G -equivariant morphism to X , by Remark 3.2(iii) they all satisfy Assumption 2. \square

Remark 3.7. The proof of Proposition 3.6 starts by applying equivariant resolution of singularities to the closure in $X' \times X$ of the graph of the restriction of φ to U , to reduce to the case that φ is a morphism. In the presence of boundary divisors $D \subset X$, $D = D_1 \cup \dots \cup D_\ell$, and $D' \subset X'$, $D' = D'_1 \cup \dots \cup D'_{\ell'}$, simple normal crossing divisors with respective

complement U , where each D_i is G -invariant, equivariant resolution of singularities with boundary divisor lets us reduce to the case that φ is a morphism and $\varphi^{-1}(D)$ is a simple normal crossing divisor. The equivariant weak factorization of [1] is applicable to $\varphi: X' \rightarrow X$ with pair of boundary divisors $(\varphi^{-1}(D), D)$.

Remark 3.8. The correct formulation of Assumptions 1 and 2, when k does not contain enough roots of unity, is that a factor K^i of K contains primitive e -th roots of unity, where e is the exponent of H , and the composite map to $H^1(H, (K^i)^\times)$ is surjective. (This does not imply that K_0 contains primitive e -th roots of unity, as we may see by taking G and H as in Example 3.5 and $G/H \curvearrowright \mathbb{C}$ over $K_0 := \mathbb{R}$.)

4. EQUIVARIANT BURNSIDE GROUP

In this section we define the equivariant Burnside group and study its first properties. It receives G -equivariant birational invariants of G -varieties over a field k , where G is a finite group. We maintain the assumption that k has characteristic zero and contains primitive e -th roots of unity, where e is the least common multiple of the exponents of abelian subgroups of G .

It will be convenient for the following to proceed in two steps.

Definition 4.1. The *symbols group*

$$\text{Burn}_n^0(G)$$

is the \mathbb{Z} -module with

Generators:

$$(H, N_G(H)/H \curvearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup,
- K/K_0 is a Galois algebra for the group $N_G(H)/H$, where K_0 is a finitely generated field of transcendence degree $d \leq n$ over k , up to isomorphism, satisfying Assumption 1, and
- β is a faithful $(n-d)$ -dimensional linear representation of H over k , with trivial space of invariants, up to equivalence; by Assumption 1, β decomposes as a sum of one-dimensional representations, hence we may write β as a sequence of characters, up to order:

$$\beta = (a_1, \dots, a_{n-d}), \quad a_i \in A := H^\vee.$$

These are subject to the following **conjugation relations**:

(C1): Triples with same subgroup and algebra extension are identified,

$$(H, N_G(H)/H \curvearrowright K, \beta) = (H, N_G(H)/H \curvearrowright K, \beta'),$$

when β and β' are related by conjugation by an element of $N_G(H)$.

(C2): Triples with subgroups and algebra extensions related by conjugation by $g \in G$ are identified,

$$(H, N_G(H)/H \curvearrowright K, \beta) = (H', N_G(H')/H' \curvearrowright K, \beta'), \quad H' = gHg^{-1},$$

when β and β' are related by conjugation by g .

Notice, if $H' = gHg^{-1} = g'Hg'^{-1}$, then the two symbols, identified with $(H, N_G(H)/H \curvearrowright K, \beta)$ by applying **(C2)** to g and g' , will be related by **(C1)**. By **(C2)**, the class of any triple in $\text{Burn}_n^0(G)$ may be expressed as $(H, N_G(H)/H \curvearrowright K, \beta)$ with $H \in \{H_1, \dots, H_r\}$.

Definition 4.2. The *equivariant Burnside group*

$$\text{Burn}_n(G) = \text{Burn}_{n,k}(G)$$

is the \mathbb{Z} -module with

Generators:

$$(H, N_G(H)/H \curvearrowright K, \beta),$$

as above, satisfying conjugation relations **(C1)**–**(C2)** and the following **blow-up relations** ($n - d \geq 2$):

(B1): For all H , K , and $a_1, a_3, \dots, a_{n-d} \in A$ we have

$$(H, N_G(H)/H \curvearrowright K, (a_1, -a_1, a_3, \dots, a_{n-d})) = 0.$$

(B2): For all H , K , and $\beta = (a_1, \dots, a_{n-d})$, $a_i \in A$,

$$(H, N_G(H)/H \curvearrowright K, \beta) = \Theta_1 + \Theta_2,$$

where

$$\Theta_1 = \begin{cases} 0, & \text{if } a_1 = a_2, \\ (H, N_G(H)/H \curvearrowright K, \beta_1) + (H, N_G(H)/H \curvearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (a_1, a_2 - a_1, a_3, \dots, a_{n-d}), \quad \beta_2 := (a_2, a_1 - a_2, a_3, \dots, a_{n-d}),$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } a_i \in \langle a_1 - a_2 \rangle \text{ for some } i, \\ (\overline{H}, N_G(\overline{H})/\overline{H} \curvearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{A} := A/\langle a_1 - a_2 \rangle, \quad \overline{H}^\vee = \overline{A}, \quad \bar{\beta} := (\bar{a}_2, \bar{a}_3, \dots, \bar{a}_{n-d}), \quad \bar{a}_i \in \overline{A},$$

and \overline{K} , with the action, described in Construction **(A)** in Section 2, applied to the character $a_1 - a_2$ (with $c = 1$).

Example 4.3. Let $G = \mathfrak{D}_8$ be as in Example 3.5, and $H := \langle \rho^2, \sigma \rangle \subset G$. In $\text{Burn}_2(G)$ we have

$$(H, G/H \curvearrowright \mathbb{C} \times \mathbb{C}, (a_1, a_2)),$$

with generators a_1, a_2 of the character group A of H , equal to

$$\begin{aligned} (H, G/H \curvearrowright \mathbb{C} \times \mathbb{C}, (a_1, a_1 + a_2)) + (H, G/H \curvearrowright \mathbb{C} \times \mathbb{C}, (a_2, a_1 + a_2)) \\ + (\langle \sigma \rangle, \langle \rho^2, \sigma \rangle / \langle \sigma \rangle \curvearrowright \mathbb{C}(t), a), \end{aligned}$$

where Θ_1 from **(B2)** appears on the first line, and Θ_2 , on the second; in a character group of order 2 we write a for the non-identity element. Here, Θ_2 is obtained from the action of G on $\mathbb{C}(t) \times \mathbb{C}(u)$ given by

$$\begin{aligned} \rho \cdot t &= -u, & \sigma \cdot t &= -t, \\ \rho \cdot u &= t, & \sigma \cdot u &= u, \end{aligned}$$

and the nontrivial character in $\overline{A} = A / \langle a_1 + a_2 \rangle$. Now $\overline{H} = \langle \rho^2 \sigma \rangle$ acts trivially only on the first factor $\mathbb{C}(t)$. So Θ_2 is

$$(\langle \rho^2 \sigma \rangle, \langle \rho^2, \sigma \rangle / \langle \rho^2 \sigma \rangle \curvearrowright \mathbb{C}(t), a),$$

written above in an equivalent form by applying **(C2)**.

Definition 4.4. Let X be a smooth projective variety over k with generically free G -action, satisfying Assumption 2. Put

$$[X \curvearrowright G] := \sum_{i=1}^r \sum_{\substack{Y \subset X \text{ with} \\ \text{generic stabilizer } H_i}} (H_i, N_G(H_i)/H_i \curvearrowright k(Y), \beta_Y(X)). \quad (4.1)$$

In the inner sum, Y is an $N_G(H_i)$ -orbit of components where the generic stabilizer is H_i . We understand $k(Y)$ to be the product of the function fields of the components, and we let the generic normal bundle representation along Y be recorded as $\beta_Y(X)$.

Example 4.5. There exist projective equivariant compactifications of faithful representations of abelian groups, of the same dimension, which are not birationally equivalent. Indeed, by [26, Theorem 7.1] (see also [29, Section 1]), if V and W are d -dimensional faithful representations of an abelian group G of rank $r \leq d$, and χ_1, \dots, χ_d , respectively $\eta_1, \dots, \eta_d \in G^\vee$, are the characters appearing in V , respectively W , then these are G -equivariantly birational if and only if

$$\chi_1 \wedge \dots \wedge \chi_d = \pm \eta_1 \wedge \dots \wedge \eta_d \in \bigwedge^d (G^\vee).$$

(This condition is nontrivial only when $r = d$.)

Remark 4.6. In motivic integration, one has considered an equivariant version $K_0^G(\text{Var}_k)$ of the Grothendieck group of algebraic varieties over a field k (see, e.g., [19, Lemma 5.1]). Note that in the definition of $K_0^G(\text{Var}_k)$, one *trivializes* the actions of finite groups on \mathbb{A}^n , which is a significant coarsening, from the birational perspective, as can be seen from Example 4.5.

Proposition 4.7. *In $\text{Burn}_n(G)$ we have the following relations:*

(i) *If $a_1 + \dots + a_j = 0$, for some $j \in \{2, \dots, n-d\}$, then*

$$(H, N_G(H)/H \curvearrowright K, (a_1, \dots, a_{n-d})) = 0.$$

(ii) *For any $2 \leq j \leq n-d$,*

$$(H, N_G(H)/H \curvearrowright K, (a_1, \dots, a_{n-d})) = \sum_{(I, C_I)} (H_I, N_G(H_I)/H_I \curvearrowright K_I, \beta_I)$$

where the sum is over pairs (I, C_I) such that

- $I = \{i_0, \dots, i_c\} \subseteq \{1, \dots, j\}$ is nonempty,
- C_I is a nontrivial coset of the subgroup

$$\langle a_{i_1} - a_{i_0}, \dots, a_{i_c} - a_{i_0} \rangle \subset A,$$

- $I = \{1 \leq i \leq j \mid a_i \in C_I\}$,
- the elements

$$\bar{a}_{j+1}, \dots, \bar{a}_{n-d} \in A_I := A / \langle a_{i_1} - a_{i_0}, \dots, a_{i_c} - a_{i_0} \rangle$$

are nonzero,

and Construction (A) in Section 2 is applied to the characters

$$a_{i_1} - a_{i_0}, \dots, a_{i_c} - a_{i_0}$$

to obtain

$$(H_I, N_G(H_I)/H_I \curvearrowright K_I, \beta_I),$$

with $H_I := \overline{H}$, $K_I := \overline{K}$, and

$$\beta_I := (\bar{a}_{i_0}, \bar{a}_{i'_1} - \bar{a}_{i_0}, \dots, \bar{a}_{i'_{j-c-1}} - \bar{a}_{i_0}, \bar{a}_{j+1}, \dots, \bar{a}_{n-d}),$$

where $\{i'_1, \dots, i'_{j-c-1}\} \cup I = \{1, \dots, j\}$.

Proof. We prove (i) and (ii) by induction on j , where the base case $j = 2$ is (B1), respectively (B2). For the inductive step of (i), we set

$$\beta := (a_1, a_1 + a_2, a_3, \dots, a_{n-d}).$$

If $a_1 + \dots + a_{j+1} = 0$, then (B2) and the induction hypothesis yield

$$(H, N_G(H)/H, (a_1, a_2, \dots, a_{n-d})) = 0.$$

We carry out the inductive step of (ii). Apply the induction hypothesis:

$$(H, N_G(H)/H \supseteq K, (a_1, \dots, a_{n-d})) = \sum_{(I, C_I)} (H_I, N_G(H_I)/H_I \supseteq K_I, \beta_I),$$

and, for each (I, C_I) , apply **(B2)** to weights in β_I corresponding to \bar{a}_{i_0} and \bar{a}_{j+1} :

$$(H, N_G(H)/H \supseteq K, (a_1, \dots, a_{n-d})) = \sum_{(I, C_I)} \Theta_1 + \Theta_2.$$

For instance, the contribution of Θ_1 is

$$\begin{aligned} \sum_{(I, C_I)} \Theta_1 = & \sum_{(I, C_I)} (H_I, N_G(H_I)/H_I \supseteq K_I, \beta'_I) \\ & + \sum_{(I, C_I)} (H_I, N_G(H_I)/H_I \supseteq K_I, \beta''_I), \end{aligned} \quad (4.2)$$

$$+ \sum_{(I, C_I)} (H_I, N_G(H_I)/H_I \supseteq K_I, \beta''_I), \quad (4.3)$$

where

$$\begin{aligned} \beta'_I &= (\bar{a}_{i_0}, \bar{a}_{i'_1} - \bar{a}_{i_0}, \dots, \bar{a}_{i'_{j-c-1}} - \bar{a}_{i_0}, \bar{a}_{j+1} - \bar{a}_{i_0}, \bar{a}_{j+2}, \dots, \bar{a}_{n-d}), \\ \beta''_I &= (\bar{a}_{j+1}, \bar{a}_{i'_1} - \bar{a}_{i_0}, \dots, \bar{a}_{i'_{j-c-1}} - \bar{a}_{i_0}, \bar{a}_{i_0} - \bar{a}_{j+1}, \bar{a}_{j+2}, \dots, \bar{a}_{n-d}). \end{aligned}$$

Now there are two cases. First we treat the case $a_{j+1} \notin \{a_1, \dots, a_j\}$. Then the induction hypothesis identifies

$$(H, N_G(H)/H \supseteq K, (a_1 - a_{j+1}, \dots, a_j - a_{j+1}, a_{j+1}, a_{j+2}, \dots, a_{n-d})) \quad (4.4)$$

with (4.3). The sum of (4.2), (4.4), and

$$\sum_{(I, C_I)} \Theta_2$$

is equal to the required sum of triples over (I, C_I) , $I \subseteq \{1, \dots, j+1\}$. In the remaining case $a_{j+1} \in \{a_1, \dots, a_j\}$, we use **(B1)** to see that (4.3) vanishes. What remains gives the required sum of triples. \square

5. EQUIVARIANT BIRATIONAL INVARIANTS

Our first task is to show that the class introduced in Definition 4.4 is an equivariant birational invariant, i.e.,

$$[X \hookrightarrow G] \in \text{Burn}_n(G).$$

Then we extend the definition to include the case of quasiprojective G -varieties. Finally, we extend the definitions to remove the dependence on Assumption 2.

Theorem 5.1. *Let X and X' be smooth projective varieties of dimension n over k , each with a generically free action of a finite group G , satisfying Assumption 2. If X and X' are G -equivariantly birationally equivalent, then*

$$[X \curvearrowright G] = [X' \curvearrowright G]$$

in $\text{Burn}_n(G)$.

Proof. By Proposition 3.6, it suffices to treat the case that X' is obtained from X by an equivariant smooth blow-up. So, let W be a G -invariant smooth subscheme of X , of pure dimension $\dim(W) \leq n - 2$. Now we split $[X \curvearrowright G]$ into two sums:

$$\begin{aligned} & \sum_{i=1}^r \sum_{\substack{Y \not\subset W \\ \text{generic stabilizer } H_i}} (H_i, N_G(H_i)/H_i \curvearrowright k(Y), \beta_Y(X)) \\ & + \sum_{i=1}^r \sum_{\substack{Y \subset W \\ \text{generic stabilizer } H_i}} (H_i, N_G(H_i)/H_i \curvearrowright k(Y), \beta_Y(X)). \end{aligned}$$

Letting E denote the exceptional divisor of the blow-up, we similarly split $[X' \curvearrowright G]$ into two sums:

$$\begin{aligned} & \sum_{i'=1}^r \sum_{\substack{Y' \not\subset E \\ \text{generic stabilizer } H_{i'}}} (H_{i'}, N_G(H_{i'})/H_{i'} \curvearrowright k(Y'), \beta_{Y'}(X')) \\ & + \sum_{i'=1}^r \sum_{\substack{Y' \subset E \\ \text{generic stabilizer } H_{i'}}} (H_{i'}, N_G(H_{i'})/H_{i'} \curvearrowright k(Y'), \beta_{Y'}(X')). \end{aligned}$$

The two first sums are equal. It remains to verify that the two second sums are equal in $\text{Burn}_n(G)$. We consider some $Y' \subset E$ in the second sum for $[X' \curvearrowright G]$. The G -orbit of its image in X determines, by the generic stabilizer, a conjugacy class representative H_i , a union of components Z with generic stabilizer H_i , and Y containing Z , appearing in the first or second sum for $[X \curvearrowright G]$. Then

$$j := n - \dim(Y) - \dim(W) + \dim(Z)$$

is positive. Let $\text{codim}(Y)$ be the codimension of Y in X ; then $\text{codim}(Y) = n - \dim(Z)$ if and only if $Y \subset W$. In this case, the corresponding summand from the second sum for $[X \curvearrowright G]$ is equal in $\text{Burn}_n(G)$ to the corresponding terms from the second sum for $[X' \curvearrowright G]$, by Proposition 4.7 (ii). Otherwise, we have $\text{codim}(Y) < n - \dim(Z)$, and we have the vanishing of the terms of the second sum for $[X' \curvearrowright G]$ by (B1). \square

We proceed to define $[U \curvearrowleft G] \in \text{Burn}_n(G)$ for a smooth quasiprojective variety U with a faithful action of G , satisfying Assumption 2. Imitating (4.1) in the most naive way, we define

$$[U \curvearrowleft G]^{\text{naive}} := \sum_{i=1}^r \sum_{\substack{V \subset U \text{ with} \\ \text{generic stabilizer } H_i}} (H_i, N_G(H_i)/H_i \curvearrowleft k(V), \beta_V(U)). \quad (5.1)$$

This suffers from an important drawback: A key property of classes in the non-equivariant Burnside group, which fails for $[U \curvearrowleft \{1\}]^{\text{naive}}$, is that when $U \subset X$ is open, $[X] - [U] \in \text{Burn}_n$ carries information about the boundary $X \setminus U$, e.g., it is essentially what we get by application of the specialization map on Burnside groups [17] when X fibers over a smooth curve C and U is the pre-image of $C \setminus \{c\}$ for $c \in C(k)$. However, the following is an immediate consequence of the definition of $[U \curvearrowleft G]^{\text{naive}}$.

Lemma 5.2. *Let $U' \subset U$ be a G -invariant open subvariety, with the property that every $V \subset U$ in (5.1) has nontrivial intersection with U' . Then in $\text{Burn}_n(G)$ we have*

$$[U \curvearrowleft G]^{\text{naive}} = [U' \curvearrowleft G]^{\text{naive}}.$$

The correct definition of $[U \curvearrowleft G]$ (Definition 5.4, below) involves an alternating sum over boundary components, as in [18, (1.1)], and the normal bundles

$$\pi_I: \mathcal{N}_{D_I/X} \rightarrow D_I,$$

in X of intersections of divisors

$$D_I := \bigcap_{i \in I} D_i, \quad I \subseteq \mathcal{I} := \{1, \dots, \ell\};$$

we agree by convention that $D_\emptyset = X$.

The class $[U \curvearrowleft G]^{\text{naive}}$ is at least an equivariant birational invariant of U ; we recall (see, e.g., [18]), that, by definition, U and U' are equivariantly birationally equivalent if there exist a quasiprojective G -variety V and equivariant birational projective morphisms $V \rightarrow U$ and $V \rightarrow U'$.

Lemma 5.3. *Let U and U' be smooth quasiprojective varieties of dimension n over k , each with a generically free action of G , satisfying Assumption 2. If U and U' are equivariantly birationally equivalent, then*

$$[U \curvearrowleft G]^{\text{naive}} = [U' \curvearrowleft G]^{\text{naive}}.$$

Proof. As in the proof of Theorem 5.1, it suffices to treat the case that U' is obtained from U by an equivariant smooth blow-up. The proof of Theorem 5.1 carries over without change to establish the lemma. \square

Definition 5.4. Let X be a smooth projective variety over k with a generically free G -action satisfying Assumption 2. For smooth quasiprojective

$$U = X \setminus D, \quad D = \bigcup_{i \in \mathcal{I}} D_i, \quad \mathcal{I} := \{1, \dots, \ell\},$$

with a generically free G -action, and a simple normal crossing divisor D , with a compatible G -action, where each D_i is G -invariant, we define

$$[U \curvearrowleft G] := [X \curvearrowleft G] + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}_{D_I/X} \curvearrowleft G]^{\text{naive}}. \quad (5.2)$$

Remark 5.5. In Definition 5.4, G is not allowed to permute the divisors D_1, \dots, D_ℓ nontrivially. For instance, the compactification of

$$U := \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}),$$

with action of $G := \mathbb{Z}/2\mathbb{Z}$ by

$$(x, y) \mapsto (xy^{-1}, y^{-1}),$$

by \mathbb{P}^2 is not permitted; instead we may work with the compactification by the blow-up of \mathbb{P}^2 at the point where the line $y = 0$ meets the line at infinity.

It is possible to obtain an alternative formula by recognizing the cancellation of many terms in (5.2). We may work in the symbols group

$$\text{Burn}_n^0(G).$$

Definition 4.4 gives a well-defined class of $\text{Burn}_n^0(G)$ attached to $X \curvearrowleft G$. We may analogously view $[U \curvearrowleft G]^{\text{naive}}$ as an element of $\text{Burn}_n^0(G)$ and hence, as well, $[U \curvearrowleft G]$ by the formula (5.2) (whose *a priori* dependence on the presentation of U as $X \setminus D$ will be removed in Proposition 5.8, below). Lemma 5.2 is valid in $\text{Burn}_n^0(G)$. The canonical homomorphism

$$\text{Burn}_n^0(G) \rightarrow \text{Burn}_n(G)$$

relates the classes in $\text{Burn}_n^0(G)$ to the ones defined in $\text{Burn}_n(G)$.

Definition 5.6. We adopt the notation of Definition 5.4. Recognizing, $I \subseteq \mathcal{I}$, that $\mathcal{N}_{D_I/X}$ may be identified with the direct sum over $i \in I$ of the restrictions $\mathcal{N}_{D_i/X}|_{D_I}$, we define the *punctured normal bundles*

$$\mathcal{N}_{D_I/X}^\circ := \mathcal{N}_{D_I/X} \setminus \bigcup_{i \in \mathcal{I}} \begin{cases} \bigoplus_{j \in I \setminus \{i\}} \mathcal{N}_{D_j/X}|_{D_I}, & \text{when } i \in I, \\ \pi_I^{-1}(D_{I \cup \{i\}}), & \text{when } i \notin I, \end{cases}$$

with projections

$$\pi_I^\circ: \mathcal{N}_{D_I/X}^\circ \rightarrow D_I^\circ := D_I \setminus \bigcup_{i \in \mathcal{I} \setminus I} D_i.$$

Lemma 5.7. *We adopt the notation of Definitions 5.4 and 5.6. Then in $\text{Burn}_n^0(G)$, and hence in $\text{Burn}_n(G)$, we have*

$$[U \curvearrowleft G] = [U \curvearrowleft G]^{\text{naive}} + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}_{D_I/X}^\circ \curvearrowleft G]^{\text{naive}}.$$

Proof. There is a group homomorphism to $\text{Burn}_n^0(G)$ from

$$\bigoplus_{H \in \{H_1, \dots, H_r\}} \bigoplus_{M \subseteq \mathcal{I}} \bigoplus_{\substack{W \subset D_M \text{ with} \\ \text{generic stabilizer } H \\ \text{and } \{i \in \mathcal{I} \mid W \subset D_i\} = M}} \bigoplus_{\substack{J \subseteq M \text{ satisfying} \\ \text{(i)-(ii) below}}} \mathbb{Z}[J, W], \quad (5.3)$$

where $A := H^\vee$ and $a_i \in A$ is determined by the divisor D_i for $i \in M$, so that A is generated by a_i for $i \in M$ and $\beta_W(D_M)$ (characters of the generic normal bundle representation along W in D_M), and $J \subseteq M$ in the final sum is required to satisfy:

- (i) $\{j \in M \mid a_j \in \langle a_i \rangle_{i \in M \setminus J}\} = M \setminus J$;
- (ii) no character in $\beta_W(D_M)$ lies in $\langle a_i \rangle_{i \in M \setminus J}$.

Letting $H_J \subseteq H$ be the subgroup with $H_J^\vee = A/\langle a_i \rangle_{i \in M \setminus J}$, then, $[J, W]$ is mapped to the triple

$$(H_J, N_G(H_J)/H_J \curvearrowright K_J, \beta_J),$$

where $N_G(H_J)/H_J \curvearrowright K_J$ arises by application of Construction (A) in Section 2 to $N_G(H)/H \curvearrowright k(W)$ and $(a_i)_{i \in M \setminus J}$, and

$$\beta_J = \gamma_J \oplus \beta_W(D_M)|_{H_J},$$

with γ_J given by \bar{a}_i for $i \in J$.

We interpret (5.2) as taking values in the group (5.3) as follows. Given $I \subseteq \mathcal{I}$ and V appearing in the definition of $[\mathcal{N}_{D_I/X}^\circ \curvearrowleft G]^{\text{naive}}$, we associate $[J, W]$ for $W = V \cap D_I$, where with $M = \{i \in \mathcal{I} \mid W \subset D_i\}$ there is a unique subset J such that $I \cup J = M$ and, for the vector bundle

$$\varphi_{J,M}: \bigoplus_{i \in M \setminus J} \mathcal{N}_{D_i/X}|_{D_M} \rightarrow D_M,$$

we have $V = \varphi_{J,M}^{-1}(W)$. We observe that $[\mathcal{N}_{D_I/X}^\circ \curvearrowleft G]^{\text{naive}}$ contributes just the terms with $J = \emptyset$.

The proposition is thus reduced to the observation that in the group (5.3) the sum of terms from (5.2) with $J \neq \emptyset$ is zero. For given W (which determines M) and $J \neq \emptyset$, each $I \subseteq M$ with $I \cup J = M$ appears with coefficient $(-1)^{|I|}$. These coefficients sum to zero. \square

Proposition 5.8. *Let $U \curvearrowleft G$ be a smooth quasiprojective variety of dimension n over k with a generically free G -action. Then its associated class in $\text{Burn}_n^0(G)$, and hence in $\text{Burn}_n(G)$, is independent of the choice*

of presentation as $U = X \setminus D$, i.e., if X' is another smooth projective variety with a generically free G -action, with an equivariant embedding $U \rightarrow X'$ with complement

$$D' = \bigcup_{i \in \mathcal{I}'} D'_i, \quad \mathcal{I}' := \{1, \dots, \ell'\},$$

a simple normal crossing divisor, where each D'_i is G -invariant, such that X' satisfies Assumption 2, then formula (5.2) agrees with the analogous formula for X' and D' .

The proof of Proposition 5.8, along with the birational invariance of $[U \hookrightarrow G]$ stated below as Proposition 5.13, will use Lemmas 5.3 and 5.7, along with a careful cancellation argument much like that used in the proof of Lemma 5.7. Because of the intricate combinatorics, we present first an example in a simple setting to illustrate the cancellation scheme.

Example 5.9. Let X be a smooth projective variety of dimension n with a generically free G -action, satisfying Assumption 2. Let $U = X \setminus D$, where D is a smooth invariant divisor, let $d \geq 2$, and let Z be a smooth purely $(n - d)$ -dimensional subscheme of D :

$$Z \subset D \subset X.$$

Let \tilde{X} be the blow-up of Z in X with exceptional divisor E and proper transform \tilde{D} of D . Then an instance of Proposition 5.8 is the following equality:

$$\begin{aligned} [X \hookrightarrow G] - [\mathcal{N}_{D/X} \hookrightarrow G]^{\text{naive}} = \\ [\tilde{X} \hookrightarrow G] - [\mathcal{N}_{\tilde{D}/\tilde{X}} \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{E/\tilde{X}} \hookrightarrow G]^{\text{naive}} + [\mathcal{N}_{\tilde{D} \cap E/\tilde{X}} \hookrightarrow G]^{\text{naive}}. \end{aligned}$$

As a first step, we apply Lemma 5.7 to both sides. Then the terms $[U \hookrightarrow G]^{\text{naive}}$ on each side cancel, leaving us to verify

$$\begin{aligned} [\mathcal{N}_{D/X}^{\circ} \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{\tilde{D}/\tilde{X}}^{\circ} \hookrightarrow G]^{\text{naive}} = \\ [\mathcal{N}_{E/\tilde{X}}^{\circ} \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{\tilde{D} \cap E/\tilde{X}}^{\circ} \hookrightarrow G]^{\text{naive}}. \end{aligned} \tag{5.4}$$

The left-hand side of (5.4) is just the contribution to $[\mathcal{N}_{D/X}^{\circ} \hookrightarrow G]^{\text{naive}}$ from loci V with various generic stabilizer groups that are contained in the pre-image under $\mathcal{N}_{D/X}^{\circ} \rightarrow D$ of Z . In the spirit of the proof of Lemma 5.7, these may be labeled by $W \subset Z$ with some generic stabilizer $H \in \{H_1, \dots, H_r\}$, where the divisor D determines an element $a \in A := H^{\vee}$, such that no character of $\beta_W(D)$ lies in $\langle a \rangle$. The left-hand side of (5.4) is a sum over such W of elements of $\text{Burn}_n(G)$ arising by Construction **(A)** in Section 2 from $N_G(H)/H \curvearrowright k(W)$ and a , with the representation

obtained by restriction from $\beta_W(D)$. On the right-hand side of (5.4), we have

$$\mathcal{N}_{E/\tilde{X}}^\circ \cong \mathcal{N}_{Z/X} \setminus \mathcal{N}_{Z/D}, \quad (5.5)$$

while $\mathcal{N}_{\tilde{D} \cap E/\tilde{X}}$ is a direct sum of the line bundles

$$\mathcal{N}_{\tilde{D} \cap E/\tilde{D}} \cong \mathcal{O}_{\mathbb{P}(\mathcal{N}_{Z/D})}(-1) \quad (5.6)$$

and

$$\mathcal{N}_{\tilde{D} \cap E/E} \cong \mathcal{N}_{D/X}|_{\mathbb{P}(\mathcal{N}_{Z/D})} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}_{Z/D})}(1). \quad (5.7)$$

Here, for (5.6)–(5.7) we use $E \cong \mathbb{P}(\mathcal{N}_{Z/X})$ and $\tilde{D} \cap E \cong \mathbb{P}(\mathcal{N}_{Z/D})$. So,

$$\mathcal{N}_{\tilde{D} \cap E/\tilde{X}}^\circ \cong (\mathcal{N}_{Z/D} \setminus Z) \times_D (\mathcal{N}_{D/X} \setminus D). \quad (5.8)$$

We relate the two punctured normal bundles on the right-hand side of (5.4) by choosing, Zariski locally on Z , a G -equivariant splitting of the short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{N}_{Z/D} \longrightarrow \mathcal{N}_{Z/X} \xrightarrow{s} \mathcal{N}_{D/X}|_Z \longrightarrow 0. \quad (5.9)$$

This is possible, since splittings exist Zariski locally and may be averaged with their translates to yield equivariant splittings, or in fancier language, since the invariant local section functor on coherent sheaves is exact [2, Lemma 2.3.4]. Using (5.5) and (5.8), we obtain from (5.9) isomorphisms

$$\mathcal{N}_{E/\tilde{X}}^\circ \setminus s(\mathcal{N}_{D/X}|_Z) \cong \mathcal{N}_{\tilde{D} \cap E/\tilde{X}}^\circ.$$

over G -invariant Zariski open subsets of Z . A splitting does not necessarily exist globally but does exist and is unique upon restricting to the locus Z' where no character of $\beta_Z(D)$ is equal to a , by Schur's lemma. This observation, together with the fact that every W as above is contained in Z' , lets us identify (see Lemma 5.10) the right-hand side of (5.4) with a sum over $W \subset Z$ as before and thereby establish (5.4).

Lemma 5.10. *Let $d \leq n$, let Z be a smooth purely $(n-d)$ -dimensional quasiprojective G -scheme over k , and let U and U' be quasiprojective purely n -dimensional schemes over k , with generically free G -actions and G -equivariant smooth morphisms*

$$U \rightarrow Z \quad \text{and} \quad U' \rightarrow Z.$$

Suppose that for every $z \in Z$ there exist a G -invariant Zariski neighborhood $Y \subset Z$ of z and a G -equivariant open immersion

$$U' \times_Z Y \rightarrow U \times_Z Y$$

which commutes with the projection maps to Y and satisfies the condition in Lemma 5.2. Then in $\text{Burn}_n^0(G)$ we have

$$[U \odot G]^{\text{naive}} = [U' \odot G]^{\text{naive}}.$$

Proof. There exists a finite collection Y_1, \dots, Y_m of G -invariant Zariski neighborhoods as in the statement of the lemma, whose union is Z . Over each Y_j we choose an open immersion as in the statement. In the definition of $[U' \odot G]^{\text{naive}}$, to each $1 \leq i \leq r$ and $V' \subset U'$ with generic stabilizer H_i we associate $j \in \{1, \dots, m\}$, taken to be minimal with the property that the image of V' in Z has nontrivial intersection with Y_j . Then we use the chosen G -equivariant open immersion over Y_j to identify a corresponding $V \subset U$ with generic stabilizer H_i , appearing in the definition of $[U \odot G]^{\text{naive}}$. The hypotheses guarantee that every $V \subset U$ appearing in the definition of $[U \odot G]^{\text{naive}}$ is accounted for. \square

Proof of Proposition 5.8. By equivariant weak factorization (see Remark 3.7) it suffices to compare the presentations of U as $X \setminus D$ and $\tilde{X} \setminus \tilde{D}$, where \tilde{X} is obtained from X by equivariant smooth blow-up, with a center of blow-up Z that is disjoint from U and has normal crossing with the divisors D_i , and

$$\tilde{D} = \tilde{D}_1 \cup \dots \cup \tilde{D}_\ell \cup E,$$

where \tilde{D}_i denotes the proper transform of D_i for $i = 1, \dots, \ell$, and E , the exceptional divisor.

We introduce the notation

$$\mathcal{I}' := \{i \in \mathcal{I} \mid Z \subset D_i\} \quad \text{and} \quad \mathcal{I}'' := \{i \in \mathcal{I} \mid Z \not\subset D_i\}$$

and, for $I \subseteq \mathcal{I}$,

$$I' := \mathcal{I}' \cap I \quad \text{and} \quad I'' := \mathcal{I}'' \cap I.$$

Additionally, we define

$$Z_I := D_{\mathcal{I}' \cup I}^\circ \cap Z.$$

Application of Lemma 5.7 to the expressions for $[U \odot G]$ from $U \subset X$ and $U \subset X'$ reduces the proposition to the verification of

$$\begin{aligned} \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} ([\mathcal{N}_{D_I/X}^\circ \odot G]^{\text{naive}} - [\mathcal{N}_{\tilde{D}_I/\tilde{X}}^\circ \odot G]^{\text{naive}}) = \\ \sum_{I \subseteq \mathcal{I}} (-1)^{|I|+1} [\mathcal{N}_{\tilde{D}_I \cap E/\tilde{X}}^\circ \odot G]^{\text{naive}}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} ([\mathcal{N}_{D_I/X}^\circ \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{\tilde{D}_I/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}}) = \\ \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} ([\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{\tilde{D}_I \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}}). \end{aligned} \quad (5.10)$$

Indeed, taking $i_0 \in \mathcal{I}'$, the summand with $I = \{i_0\}$ has first term $-[\mathcal{N}_{E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}}$, while the first terms from the remaining summands cancel, as we see by pairing the terms indexed by I and $I \cup \{i_0\}$ for $\emptyset \neq I \subseteq \mathcal{I} \setminus \{i_0\}$.

For any $I \subseteq \mathcal{I}$ and $i \in \mathcal{I}'$ we have $D_I \setminus D_i \cong \tilde{D}_I \setminus (\tilde{D}_i \cup E)$. It follows easily that the left-hand side of (5.10) is 0 whenever $\mathcal{I}' \not\subseteq I$. When $\mathcal{I}' \subseteq I$, the left-hand side of (5.10) is equal to the contribution to $[\mathcal{N}_{D_I/X}^\circ \hookrightarrow G]^{\text{naive}}$ from loci V with various stabilizer groups, contained in

$$(\pi_I^\circ)^{-1}(Z_I).$$

Any such V is equal to $(\pi_I^\circ)^{-1}(W)$, where W is the image of V in Z_I . Let H be the generic stabilizer of (a union of components of) W . Then the contribution, specifically, is gotten by pairing the outcome of Construction **(A)** in Section 2, applied to $N_G(H)/H \supset k(W)$ and $(a_i)_{i \in I}$, with the characters of $\beta_W(D_I^\circ)$. We have

$$W \subset Z',$$

where

$$Z' \subset Z_I$$

is defined to be the locus where, for all $i \in \mathcal{I}'$, the character a_i in the corresponding character group does not appear in $\beta_{Z_I}(D_I^\circ)$.

We now turn to an analysis of the right-hand side of (5.10). Since Z meets $D_{I''}$ transversally we have

$$\mathcal{N}_{E/\tilde{X}}|_{\tilde{D}_{I''} \cap E} \cong \mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{D}_{I''}},$$

hence as well

$$\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}} \cong \mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{D}_{I''}} \oplus \left(\bigoplus_{j \in I''} \mathcal{N}_{D_j/X}|_{\tilde{D}_{I''} \cap E} \right).$$

The complement of the zero-section in the line bundle $\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{D}_{I''}}$ may be identified with the complement of the zero-section in the vector bundle $\mathcal{N}_{D_I \cap Z/D_{I''}}$. To obtain $\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ$, we remove the exceptional divisors of

the blow-up of $D_{I'' \cup \{i\}}$ along $D_I \cap Z$ for every $i \in \mathcal{I}'$ as well as everything over divisors indexed by $\mathcal{I}'' \setminus I$:

$$\mathcal{N}_{\tilde{D}_{I''} \cap E / \tilde{X}}^\circ \cong \left(\mathcal{N}_{Z_I / D_{I''}} \setminus \bigcup_{i \in \mathcal{I}'} \mathcal{N}_{Z_I / D_{I'' \cup \{i\}}} \right) \times_{Z_I} \left(\bigtimes_{j \in I''} (\mathcal{N}_{D_j / X} |_{Z_I} \setminus Z_I) \right).$$

Analogously, in the respective cases $\mathcal{I}' \subseteq I$ and $\mathcal{I}' \not\subseteq I$ we have

$$\mathcal{N}_{\tilde{D}_I \cap E / \tilde{X}}^\circ \cong \left(\mathcal{N}_{Z_I / D_I} \setminus Z_I \right) \times_{Z_I} \left(\bigtimes_{j \in I} (\mathcal{N}_{D_j / X} |_{Z_I} \setminus Z_I) \right),$$

respectively

$$\mathcal{N}_{\tilde{D}_I \cap E / \tilde{X}}^\circ \cong \left(\mathcal{N}_{Z_I / D_I} \setminus \bigcup_{i \in \mathcal{I}' \setminus I} \mathcal{N}_{Z_I / D_{I \cup \{i\}}} \right) \times_{Z_I} \left(\bigtimes_{j \in I} (\mathcal{N}_{D_j / X} |_{Z_I} \setminus Z_I) \right).$$

As explained in Example 5.9, Zariski locally over Z_I there exist equivariant splittings of the short exact sequence

$$0 \rightarrow \mathcal{N}_{Z_I / D_{\mathcal{I}' \cup I}} \rightarrow \mathcal{N}_{Z_I / D_{I''}} \rightarrow \bigoplus_{j \in \mathcal{I}'} \mathcal{N}_{D_j / X} |_{Z_I} \rightarrow 0. \quad (5.11)$$

First we treat the case $\mathcal{I}' \subseteq I$. Zariski locally over Z_I we use splittings of (5.11) to identify

$$\mathcal{N}_{Z_I / D_I} \oplus \left(\bigoplus_{j \in \mathcal{I}'} \mathcal{N}_{D_j / X} |_{Z_I} \right)$$

with $\mathcal{N}_{Z_I / D_{I''}}$. We have also defined $Z' \subset Z_I$, closed, over which a splitting s of (5.11) exists and is unique. So

$$\begin{aligned} U := & \left(\mathcal{N}_{Z_I / D_{I''}} \setminus \left(\bigcup_{i \in \mathcal{I}'} \mathcal{N}_{Z_I / D_{I'' \cup \{i\}}} \cup s \left(\bigoplus_{i \in \mathcal{I}'} \mathcal{N}_{D_i / X} |_{Z'} \right) \right) \right) \times_{Z_I} \\ & \left(\bigtimes_{j \in I''} (\mathcal{N}_{D_j / X} |_{Z_I} \setminus Z_I) \right) \end{aligned}$$

is a well-defined open subscheme of $\mathcal{N}_{\tilde{D}_{I''} \cap E / \tilde{X}}^\circ$. To U we have, over G -invariant open subschemes of Z_I , equivariant open immersions from $\mathcal{N}_{\tilde{D}_I \cap E / \tilde{X}}^\circ$. Lemma 5.10 is applicable and yields

$$[U \hookrightarrow G]^{\text{naive}} = [\mathcal{N}_{\tilde{D}_I \cap E / \tilde{X}}^\circ \hookrightarrow G]^{\text{naive}}$$

in $\text{Burn}_n^0(G)$ (or both U and $\tilde{D}_I \cap E$ are empty, and the equality holds trivially). On the other hand, we may write

$$[\mathcal{N}_{\tilde{D}_{I''} \cap E / \tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} - [U \hookrightarrow G]^{\text{naive}}$$

as a sum over $W \subset Z_I$ of elements of $\text{Burn}_n^0(G)$, which is equal to the contribution to $[\mathcal{N}_{D_I / X}^\circ \hookrightarrow G]^{\text{naive}}$ from loci contained in $(\pi_I^\circ)^{-1}(Z_I)$. The

equality of the left- and right-hand sides of (5.10) is thus established for the summands with $\mathcal{I}' \subseteq I$.

When $\mathcal{I}' \not\subseteq I$ we take a splitting of (5.11) over invariant open $Y \subset Z_I$

$$s: \bigoplus_{j \in \mathcal{I}'} \mathcal{N}_{D_j/X}|_Y \rightarrow \mathcal{N}_{Y/D_{I''}}$$

and restrict to $\bigoplus_{j \in I'} \mathcal{N}_{D_j/X}|_Y$ to obtain an isomorphism

$$\mathcal{N}_{Y/D_I} \oplus \left(\bigoplus_{j \in I'} \mathcal{N}_{D_j/X}|_Y \right) \cong \mathcal{N}_{Y/D_{I''}}. \quad (5.12)$$

This, we claim, induces an isomorphism of $\mathcal{N}_{\tilde{D}_I \cap E/\tilde{X}}^\circ$ with $\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ$ over Y . Indeed, when $i \in I'$, the isomorphism (5.12) induces

$$\mathcal{N}_{Y/D_I} \oplus s \left(\bigoplus_{j \in I' \setminus \{i\}} \mathcal{N}_{D_j/X}|_Y \right) \cong \mathcal{N}_{Y/D_{I'' \cup \{i\}}},$$

while for $i \in \mathcal{I}' \setminus I$ we obtain

$$\mathcal{N}_{Y/D_{I \cup \{i\}}} \oplus s \left(\bigoplus_{j \in I'} \mathcal{N}_{D_j/X}|_Y \right) \cong \mathcal{N}_{Y/D_{I'' \cup \{i\}}}$$

from (5.12). For the latter, it is crucial that the splitting used to produce (5.12) is the restriction of a splitting s of (5.11). Lemma 5.10 is applicable (where the open immersions that exist locally are isomorphisms), so that

$$[\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} - [\mathcal{N}_{\tilde{D}_I \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} = 0$$

in $\text{Burn}_n^0(G)$, as desired. \square

Remark 5.11. It is instructive to examine the proof of Proposition 5.8 when Z is equal to the intersection of some of the divisors D_i . Then the left-hand term in (5.11) is always zero. The equalities

$$\begin{aligned} [\mathcal{N}_{D_I/X}^\circ \hookrightarrow G]^{\text{naive}} &= [\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} & (\mathcal{I}' \subseteq I) & \quad \text{and} \\ [\mathcal{N}_{\tilde{D}_I \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} &= [\mathcal{N}_{\tilde{D}_{I''} \cap E/\tilde{X}}^\circ \hookrightarrow G]^{\text{naive}} & (\mathcal{I}' \not\subseteq I) \end{aligned}$$

can be traced back to canonical, globally defined G -equivariant isomorphisms of the respective pairs of punctured normal bundles. A general result, from which these isomorphisms can (indirectly) be obtained is stated next.

Proposition 5.12. *Let W be a smooth quasiprojective variety with a generically free G -action, $D = D_1 \cup \dots \cup D_\ell$ a simple normal crossing divisor where each D_i is G -invariant, and a_1, \dots, a_ℓ positive integers. Let $U := W \setminus D$ and*

$$\mathcal{K} := i_* \mathcal{O}_U,$$

where $i: U \rightarrow W$ denotes the inclusion. and define the \mathcal{O}_W -subalgebra

$$\mathcal{A} := \{f \mid a_1 \text{ord}_1(f) + \cdots + a_\ell \text{ord}_\ell(f) \geq 0\}$$

of \mathcal{K} , where $\text{ord}_i(f)$ denotes the order of vanishing of $f \in \mathcal{K}_z$ along D_i and the condition is imposed at all $z \in Z := D_1 \cap \cdots \cap D_\ell$. Then $V := \text{Spec}(\mathcal{A})$ is smooth with smooth divisor $E \subset V$ defined by the sheaf of ideals

$$\mathcal{I} := \{f \mid a_1 \text{ord}_1(f) + \cdots + a_\ell \text{ord}_\ell(f) > 0\},$$

and we have a canonical G -equivariant isomorphism

$$\mathcal{N}_{E/V}^\circ \cong \mathcal{N}_{Z/W}^\circ.$$

Proposition 5.13. *Let U and U' be smooth quasiprojective varieties with generically free G -actions, of the form $U = X \setminus D$ and $U' = X' \setminus D'$ for simple normal crossing divisors*

$$D = \bigcup_{i \in \mathcal{I}} D_i, \quad \mathcal{I} := \{1, \dots, \ell\},$$

and

$$D' = \bigcup_{i \in \mathcal{I}'} D'_i, \quad \mathcal{I}' := \{1, \dots, \ell'\},$$

where X and X' have compatible G -actions, satisfy Assumption 2, and each D_i and each D'_i is G -invariant. If U and U' are equivariantly birationally equivalent, then

$$[U \curvearrowleft G] = [U' \curvearrowleft G]$$

in $\text{Burn}_n(G)$.

Proof. As in the proof of Proposition 5.8 it suffices to treat the case that X' is obtained from X by equivariant smooth blow-up, where the center of blow-up Z has normal crossing with the divisors D_i , and the components of Z form a single G -orbit. The case that Z is disjoint from U has already been treated, so we may suppose that Z meets D_I transversely, for every $I \subseteq \mathcal{I}$. Then we have $\ell' = \ell$, and for every $I \subseteq \mathcal{I}$,

$$\mathcal{N}_{D'_I/X'}^\circ \cong \mathcal{N}_{D_I/X}|_{D'_I}.$$

By Lemma 5.3,

$$[\mathcal{N}_{D_I/X} \curvearrowleft G]^{\text{naive}} = [\mathcal{N}_{D'_I/X'} \curvearrowleft G]^{\text{naive}}$$

in $\text{Burn}_n(G)$. By combining this with the equality from Theorem 5.1 we get the desired result. \square

Using Proposition 3.3 we are able to extend the definition of the class of a smooth projective G -variety in the equivariant Burnside group, so that it is not necessary to suppose that Assumption 2 is satisfied. We can extend, as well, the definition of the class of a smooth quasiprojective G -variety to eliminate the requirement of Assumption 2.

Definition 5.14. Let X be a smooth projective variety over k with a generically free G -action. The associated class in $\text{Burn}_n(G)$ is obtained by taking X' to be a smooth projective variety with G -action and equivariant birational morphism to X , such that X' satisfies Assumption 2, and setting

$$[X \hookrightarrow G] := [X' \hookrightarrow G] \in \text{Burn}_n(G).$$

Let U be a smooth quasiprojective variety over k with a generically free G -action. The associated class is obtained by taking X with G -action to be a smooth projective compactification of U with $U = X \setminus D$, such that $D = D_1 \cup \dots \cup D_\ell$ is a simple normal crossing divisor on X , where each D_i is G -invariant, and X' to be a smooth projective variety with G -action and equivariant birational morphism to X , such that X' satisfies Assumption 2 and has a simple normal crossing divisor $D' = D'_1 \cup \dots \cup D'_{\ell'}$, where each D'_i is G -invariant, and D' is the support of the pre-image of D in X' ; then

$$[U \hookrightarrow G] := [U' \hookrightarrow G] \in \text{Burn}_n(G),$$

where U' denotes the complement of D' in X' .

Theorem 5.15. *The classes in Definition 5.14 are well-defined and give rise to equivariant birational invariants of smooth projective, respectively quasiprojective varieties over k with a generically free G -action.*

Proof. For projective varieties this is clear by Proposition 3.3 and Theorem 5.1. For quasiprojective varieties we combine equivariant embedded resolution of singularities (for the existence of D as claimed) with Remark 3.4 (for X' and D') and Proposition 5.13. \square

Remark 5.16. It is easy to see that the classes $[U \hookrightarrow G]$ generate $\text{Burn}_n(G)$.

6. SPECIALIZATION

Let \mathfrak{o} be a complete DVR with residue field k . The goal of this section is to produce a specialization map

$$\text{Burn}_{n,K}(G) \rightarrow \text{Burn}_{n,k}(G),$$

where K denotes the field of fractions of \mathfrak{o} . The non-equivariant case was treated in [17]. A given smooth projective variety over K is extended to a regular model \mathcal{X} over \mathfrak{o} , where the special fiber is a strict normal crossing

divisor. Then the components of the special fiber, along with their intersections, are used to define the specialization of $[X]$. In one variant, the multiplicities of the components play no role. Another variant involves the multiplicities and is sensitive to the choice of a uniformizer of \mathfrak{o} .

We fix a choice of uniformizer $\pi \in \mathfrak{o}$. Then, from [17, §5]:

Definition 6.1. The *Burnside volume*

$$\rho_\pi: \text{Burn}_{n,K} \rightarrow \text{Burn}_{n,k}$$

is defined, for a smooth projective variety X of dimension n over K and regular model \mathcal{X} over \mathfrak{o} whose special fiber is a simple normal crossing divisor $D_1 \cup \dots \cup D_\ell$, with each D_i irreducible, by

$$\rho_\pi([X]) := \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|-1} [k(\omega_I^{-1}(1))], \quad \mathcal{I} := \{1, \dots, \ell\}.$$

Here,

$$\omega_I: \mathcal{N}_{D_I/\mathcal{X}}^\circ \rightarrow \mathbb{G}_m$$

denotes the morphism obtained from the trivialization of

$$\bigotimes_{i \in I} \mathcal{N}_{D_i/\mathcal{X}}^{\otimes d_i}|_{D_I}$$

determined by π , where d_i denotes the multiplicity of D_i in the special fiber of \mathcal{X} .

Example 6.2. Let E be an elliptic curve over $K := k((t))$ with full 2-torsion defined over K and minimal model over $k[[t]]$ of Kodaira type I_0^* . The special fiber consists of four rational curves of multiplicity 1 and a fifth rational curve of multiplicity 2. For $I = \{5\}$, $\omega_I^{-1}(1)$ is a degree 2 cover of $\mathbb{P}^1 \setminus \{4 \text{ points}\}$, while the contributions from all other I cancel. The outcome: $\rho_t([E])$ is the class of an elliptic curve over k , which changes by quadratic twist by $\alpha \in k^\times$ when we replace t by αt .

Remark 6.3. As may be deduced, e.g., from [21, Prop. 2.3.2], $\omega_I^{-1}(1)$ may as well be obtained from the normalization of

$$\text{Spec}(\mathfrak{o}[\pi^{1/d_I}]) \times_{\text{Spec}(\mathfrak{o})} \mathcal{X},$$

where d_I denotes the gcd of the integers d_i , $i \in I$, as a $\mathbb{G}_m^{|I|-1}$ -torsor over the pre-image of D_I° . In combination with Proposition 5.12, this quickly leads to the following observations:

- If m is a positive integer, then the DVR $\mathfrak{o}' := \mathfrak{o}[\pi^{1/m}]$ with field of fractions $K' := K(\pi^{1/m})$ gives rise to a commutative diagram

$$\begin{array}{ccc} \text{Burn}_{n,K} & \xrightarrow{\rho_\pi} & \text{Burn}_{n,k} \\ \downarrow & & \parallel \\ \text{Burn}_{n,K'} & \xrightarrow{\rho_{\pi^{1/m}}} & \text{Burn}_{n,k} \end{array}$$

- To obtain $\rho_\pi([X])$ we may take m such that a *semistable* model \mathcal{X} over \mathfrak{o}' exists and apply the specialization map

$$\text{Burn}_{n,K'} \xrightarrow{\rho} \text{Burn}_{n,k}$$

of [17, §3], where the uniformizing element does not play a role. (Such m exists by [15, §IV.3].)

We turn now to the equivariant case.

Definition 6.4. We define the *equivariant Burnside volume*

$$\rho_\pi^G: \text{Burn}_{n,K}(G) \rightarrow \text{Burn}_{n,k}(G)$$

by, for $H \in \{H_1, \dots, H_r\}$, smooth projective Y over K of dimension $d \leq n$ with generically free action of $N_G(H)/H$, and β a sequence of $n-d$ elements of $A := H^\vee$ which generates A , sending

$$(H, N_G(H)/H \curvearrowright K(Y), \beta)$$

to

$$\sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|-1} (H, N_G(H)/H \curvearrowright k(\omega_I^{-1}(1)), \beta), \quad \mathcal{I} := \{1, \dots, \ell\}.$$

Here we take \mathcal{Y} to be a regular model over \mathfrak{o} , with compatible $N_G(H)/H$ -action and special fiber a simple normal crossing divisor $D_1 \cup \dots \cup D_\ell$, where each D_i is G -invariant, and let

$$\omega_I: \mathcal{N}_{D_I/\mathcal{Y}}^\circ \rightarrow \mathbb{G}_m$$

denote the morphism obtained from the trivialization of

$$\bigotimes_{i \in I} \mathcal{N}_{D_i/\mathcal{Y}}^{\otimes d_i}|_{D_I},$$

determined by π , where the expression of the special fiber as $D_1 \cup \dots \cup D_\ell$ is taken, so that the components of each D_i have a common multiplicity d_i .

The verification that Definition 6.4 yields a well-defined homomorphism is straightforward. Indeed we recognize that the equivariant Burnside group splits as a direct sum according to the triviality or nontriviality

of the representation component of triples. Now we view Y as *quasiprojective* over $\text{Spec}(\mathfrak{o})$ and apply Definition 5.14 to $Y \hookrightarrow N_G(H)/H$, with compactification \mathcal{Y} . In the definition we disregard Assumption 2 and disregard all contributions from subvarieties with nontrivial generic stabilizer. We also disregard the first term from (5.2). Disregarding all contributions from subvarieties with nontrivial generic stabilizer reduces Lemma 5.7 to a triviality. The essential content is contained in Proposition 5.8, specifically (5.10). Since we are disregarding all contributions from subvarieties with nontrivial generic stabilizer, the left-hand side trivially vanishes. The (local) identifications of direct sums of normal bundles that we obtain from equivariant splittings of short exact sequences of vector bundles respect the invertible function on punctured normal bundles determined by π . So, the cancellations remain valid when the naive classes are replaced by the classes of the fibers over 1. The direct summand of $\text{Burn}_{d,k}(N_G(H)/H)$, determined by the triviality of the first component of the triples, maps to $\text{Burn}_{n,k}(G)$ by replacing the first, respectively third component in each triple by H , respectively β .

Remark 6.5. The class of an irreducible variety with G -action can specialize to an orbit of varieties. For instance, a non-hyperelliptic curve of genus 3 with unramified degree 2 cover may be presented by the defining equation

$$Q_1 Q_3 = Q_2^2,$$

in \mathbb{P}^2 , respectively

$$Q_1 = r^2, \quad Q_2 = rs, \quad Q_3 = s^2,$$

in \mathbb{P}^4 , where $Q_1, Q_2, Q_3 \in k[u, v, w]$ are homogeneous of degree 2, with $G := \mathbb{Z}/2\mathbb{Z}$ acting by $(u : v : w : r : s) \mapsto (u : v : w : -r : -s)$ (see [8]). Now we define X over $k((t))$ by replacing Q_1 , respectively Q_3 , by

$$\tilde{Q}_1 := u^2 + tQ_1, \quad \text{respectively} \quad \tilde{Q}_3 := v^2 + tQ_3.$$

For general Q_1, Q_2, Q_3 , the same equations define a regular model \mathcal{X} where the special fiber consists of two pairs of conics exchanged by G , and the equivariant Burnside volume of $[X \hookrightarrow G] = (\text{triv}, G \curvearrowright k(X), \text{triv})$ is a nonzero multiple of $(\text{triv}, G \curvearrowright k(z) \times k(z), \text{triv})$.

We formulate a version of Assumption 2 for an action of G on a regular model \mathcal{X} of a projective variety X over K . As before, we require all stabilizers of the action to be abelian. For each $H \in \{H_1, \dots, H_r\}$ and $N_G(H)$ -orbit Y of components of \mathcal{X}^H where the stabilizer at the generic points of the components is equal to H , we consider two cases:

- For Y contained in the special fiber we require the composite

$$\text{Pic}^G(\mathcal{X}) \rightarrow H^1(N_G(H), k(Y)^\times) \rightarrow H^1(H, k(Y)^\times)^{N_G(H)/H} \rightarrow H^\vee$$

- to be surjective.
- Otherwise, we require the composite

$$\mathrm{Pic}^G(\mathcal{X}) \rightarrow H^1(N_G(H), K(Y)^\times) \rightarrow H^1(H, K(Y)^\times)^{N_G(H)/H} \rightarrow H^\vee$$

to be surjective.

Theorem 6.6. *Let X be a smooth projective variety over K with a generically free G -action and regular model \mathcal{X} , projective over \mathfrak{o} , to which the G -action extends, whose special fiber is a simple normal crossing divisor $D_1 \cup \dots \cup D_\ell$, where each D_i is G -invariant and components of each D_i have a common multiplicity d_i . We suppose that the G -action on \mathcal{X} satisfies the above variant of Assumption 2. Let*

$$\omega_I: \mathcal{N}_{D_I/\mathcal{X}}^\circ \rightarrow \mathbb{G}_m$$

denote the morphism obtained from the trivialization of

$$\bigotimes_{i \in I} \mathcal{N}_{D_i/\mathcal{X}}|_{D_i},$$

determined by π . Then

$$\begin{aligned} \rho_\pi^G([X \curvearrowleft G]) = & \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|-1} \sum_{i=1}^r \\ & \sum_{\substack{V \subset \mathcal{N}_{D_I/\mathcal{X}}^\circ \text{ with} \\ \text{generic stabilizer } H_i}} (H_i, N_G(H_i)/H_i \curvearrowleft k(V \cap \omega_I^{-1}(1)), \beta_V(\mathcal{N}_{D_I/\mathcal{X}}^\circ)). \end{aligned}$$

Proof. This follows directly from the definition. \square

Remark 6.7. The G -equivariant analogues of the statements from Remark 6.3 are valid.

Corollary 6.8. *Let X and X' be smooth projective varieties over K with generically free G -actions, admitting regular models \mathcal{X} , respectively \mathcal{X}' , smooth and projective over \mathfrak{o} , to which the G -action extends. If X and X' are G -equivariantly birational over K , then the special fibers of \mathcal{X} and \mathcal{X}' are G -equivariantly birational over k .*

The following generalizes the notion of B -rational singularities of [17].

Definition 6.9. Let X_0 be a singular projective variety over k with a generically free G -action. We say that X_0 , respectively a pair (\mathcal{X}, X_0) has *BG-rational singularities* if for every projective model \mathcal{X} over \mathfrak{o} , respectively a given projective model \mathcal{X} , with G -action, smooth generic fiber X , and special fiber G -equivariantly isomorphic to X_0 , we have

$$\rho_\pi^G([X \curvearrowleft G]) = [X_0 \curvearrowleft G].$$

Example 6.10. If the only singularity of X_0 is an orbit of isolated ordinary double point singularities on which G acts simply transitively, then X_0 is BG -rational. Indeed, any projective model \mathcal{X} is resolved by a sequence of blow-ups of orbits of points, and it is straightforward to verify using Theorem 6.6 that $[X \curvearrowright G]$ specializes to $[X_0 \curvearrowright G]$.

7. EQUIVARIANT BURNSIDE GROUPS AND ORBIFOLDS

In this section, we define a natural homomorphism from the equivariant Burnside group to the Burnside group of orbifolds:

$$\kappa^G : \text{Burn}_n(G) \rightarrow \overline{\text{Burn}}_n,$$

a group we introduced in [18], as a quotient, by explicit relations, of the \mathbb{Z} -module with generators

$$([X], [\beta]),$$

where

- $[X] \in \text{Burn}_d$, $d \leq n$, and
- $[\beta] \in \overline{\mathcal{B}}_{n-d}$, a certain invariant of representations of finite abelian groups [18, Definition 3.1].

Then the class in $\overline{\text{Burn}}_n$ of an n -dimensional orbifold \mathcal{X} is defined as follows (see [18, Section 4] for terminology and precise definitions):

- After divisorialification we may assume that \mathcal{X} is *divisorial* (with respect to some finite collection of line bundles);
- Express

$$\mathcal{X} = \coprod_H \mathcal{X}_H,$$

where \mathcal{X}_H are strata characterized by the isomorphism type of their geometric stabilizer group H (an abelian group);

- Put

$$[\mathcal{X}] := \sum_H ([X_H], [N_{X,H}]) \in \overline{\text{Burn}}_n, \quad (7.1)$$

where X_H is the coarse moduli space of \mathcal{X}_H , and the representation $[N_{X,H}]$ is extracted from the normal bundle $N_{X,H} := \mathcal{N}_{\mathcal{X}_H/\mathcal{X}}$.

This class is a well-defined invariant of the birational type of \mathcal{X} .

Example 7.1. Let an abelian group H act trivially on a smooth projective d -dimensional variety Y and diagonally by characters a_1, \dots, a_{n-d} generating $A := H^\vee$ on \mathbb{A}^{n-d} . Then

$$[Y \times \mathbb{A}^{n-d}/H] = \sum_{I \subseteq \{1, \dots, n-d\}} (-1)^{|I|} ([Y], (\bar{a}_1, \dots, \bar{a}_{n-d})), \quad (7.2)$$

where \bar{a}_i denotes the character of $H_I := \bigcap_{i \in I} \ker(a_i)$ given by the class of a_i in $A_I := A/\langle a_i \rangle_{i \in I}$. We obtain (7.2) from (7.1) by repeatedly applying modified scissors relations [18, §2] to subvarieties defined by the vanishing of a coordinate of \mathbb{A}^{n-d} .

Remark 7.2. Suppose that $a_i = 0$ for some i in Example 7.1. Then $[Y \times \mathbb{A}^{n-d}/H]$ is a product with \mathbb{A}^1 . Hence (7.2) vanishes by the triviality in Burn_n of any product with \mathbb{A}^1 (cf. the proof of [18, Prop. 2.2]). Another way to obtain the vanishing is to notice that in the sum (7.2) the terms indexed by I and $I \cup \{i\}$ cancel, for $i \notin I$.

In Example 7.1 the orbifold in question is the total space of a direct sum of line bundles over $Y \times BH$. More generally, we may consider a smooth projective Deligne-Mumford stack \mathcal{X} with line bundles L_1, \dots, L_r , such that the total space of $L_1 \oplus \dots \oplus L_r$ is an orbifold. It will be convenient to require this orbifold to be divisorial. We recall a well-known fact [25, Prop. 2.1]: *Every smooth separated finite-type Deligne-Mumford stack over a field is a gerbe over an orbifold.* A smooth projective (respectively, quasiprojective) Deligne-Mumford stack \mathcal{X} is a gerbe, then, over a projective (respectively, quasiprojective) orbifold $\overline{\mathcal{X}}$.

Lemma 7.3. *Let \mathcal{X} be a smooth separated irreducible Deligne-Mumford stack of finite type over k , gerbe over the orbifold $\overline{\mathcal{X}}$, with finite abelian group H as the isomorphism type of the geometric generic point of \mathcal{X} , and let L_1, \dots, L_r be line bundles on \mathcal{X} . The following are equivalent.*

- (i) *The orbifold $\overline{\mathcal{X}}$ is divisorial, and the representation of H at the geometric generic point of \mathcal{X} determined by $L_1 \oplus \dots \oplus L_r$ is faithful.*
- (ii) *The total space of $L_1 \oplus \dots \oplus L_r$ is an orbifold and is divisorial.*
- (iii) *The total space of $L_1 \oplus \dots \oplus L_r$ is an orbifold and is divisorial with respect to the pullbacks of L_1, \dots, L_r and some finite collection of line bundles on $\overline{\mathcal{X}}$.*

Proof. The kernel of the representation in (i) is the geometric generic stabilizer of the total space of $L_1 \oplus \dots \oplus L_r$. This lets us rephrase (ii) and (iii), replacing the requirement for the total space of $L_1 \oplus \dots \oplus L_r$ to be an orbifold by the faithful representation requirement from (i). For an orbifold to be divisorial with respect to a finite collection of line bundles, we recall, means that the corresponding morphism to a product of copies of $B\mathbb{G}_m$ is representable; we allow ourselves to apply this terminology to arbitrary Deligne-Mumford stacks. This way, we can formulate variants of (ii) and (iii), let us say (ii') and (iii'), in which instead of requiring $L_1 \oplus \dots \oplus L_r$ to be divisorial we require \mathcal{X} to be divisorial. Since \mathcal{X} sits as the zero-section in $L_1 \oplus \dots \oplus L_r$, we have (ii) \Rightarrow (ii') and (iii) \Rightarrow (iii'). As

well, $L_1 \oplus \cdots \oplus L_r \rightarrow \mathcal{X}$ is representable, so (ii') \Rightarrow (ii) and (iii') \Rightarrow (iii). Trivially, (iii) \Rightarrow (ii).

We establish (i) \Rightarrow (iii') by the observation that the stabilizer at a geometric point of \mathcal{X} contains H , such that the quotient by H is the stabilizer at the corresponding geometric point of $\overline{\mathcal{X}}$. We are done, then, if we can establish (ii') \Rightarrow (i). A line bundle on \mathcal{X} comes from $\overline{\mathcal{X}}$ if and only if induces the trivial representation of H , according to the criterion of [3, Thm. 10.3] (in a form adapted to a relative notion of coarse moduli space, cf. [24, §2.3]). Since we have a faithful representation as in (i), we get a surjective homomorphism of character groups $\mathbb{Z}^r \rightarrow H^\vee$. So any line bundle on \mathcal{X} will, after adjustment by a suitable tensor combination of L_1, \dots, L_r , descend to $\overline{\mathcal{X}}$. Carrying this out for a collection of line bundles, with respect to which \mathcal{X} is divisorial, we get vector bundles, with respect to which $\overline{\mathcal{X}}$ is divisorial \square

Given $\mathcal{X}, L_1, \dots, L_r$ satisfying the equivalent conditions of Lemma 7.3, with \mathcal{X} quasiprojective of dimension $n - r$, we define, following the notation of [18, Thm. 4.1],

$$[\mathcal{X}, (L_1, \dots, L_r)] := \sum_P ([X_P], [N_{X,P} \oplus L_1 \oplus \cdots \oplus L_r]) \in \overline{\text{Burn}}_n,$$

sum over geometric stabilizer groups P of \mathcal{X} .

Lemma 7.4. *Let $\mathcal{X}, L_1, \dots, L_r$ satisfy the equivalent conditions of Lemma 7.3, with \mathcal{X} quasiprojective of dimension $n - r$, and let $D = D_1 \cup \cdots \cup D_\ell$ be a simple normal crossing divisor on \mathcal{X} . Then, with $\mathcal{U} := \mathcal{X} \setminus D$ and $\mathcal{I} := \{1, \dots, \ell\}$ we have*

$$\begin{aligned} [\mathcal{U}, (L_1|_{\mathcal{U}}, \dots, L_r|_{\mathcal{U}})] &= [\mathcal{X}, (L_1, \dots, L_r)] \\ &+ \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [D_I, (L_1|_{D_I}, \dots, L_r|_{D_I}, \dots, \mathcal{N}_{D_i/\mathcal{X}}|_{D_I}, \dots)] \end{aligned} \quad (7.3)$$

in $\overline{\text{Burn}}_n$, where the last term includes the line bundles $\mathcal{N}_{D_i/\mathcal{X}}|_{D_I}$ for all $i \in I$.

Proof. We first remark that (7.3), when \mathcal{X} is a projective variety, is essentially the formula in Burn_n for the class of the complement of a strict normal crossing divisor [18, (1.1)]. It is an easy exercise to verify that the formula is valid when \mathcal{X} is a quasiprojective variety.

To treat the case of a Deligne-Mumford stack, we consider a geometric stabilizer group P of \mathcal{X} . It is a straightforward fact that \mathcal{X}_P has normal crossing with D_1, \dots, D_ℓ . We obtain the lemma from an equality for each P . When (a connected component of) \mathcal{X}_P is contained in some D_i , there is no contribution to the left-hand side, while the right-hand side,

viewed as a single sum over all $I \subseteq \mathcal{I}$, has pairs of terms that cancel, indexed by I and $I \cup \{i\}$ for $i \notin I$. Otherwise, each L_i is generically trivial on D_I , for all I , and the equality may be rewritten as

$$([U_P], [N_{U,P} \oplus L_1 \oplus \cdots \oplus L_r]) = ([X_P], [N_{X,P} \oplus L_1 \oplus \cdots \oplus L_r]) \\ + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|}([(D_I)_P \times \mathbb{P}^{|I|}], [N_{D_I,P} \oplus L_1 \oplus \cdots \oplus L_r]), \quad (7.4)$$

where we write $(D_I)_P$ for the coarse moduli space of $D_I \cap \mathcal{X}_P$ and neglect to include in the notation that the line bundles are restricted to U , respectively, to D_I . The same representation is extracted from the vector bundles in all of the terms in (7.4). So, the equality follows from the case of a quasiprojective variety mentioned above. \square

Lemma 7.5. *Let \mathcal{X} be an n -dimensional quasiprojective orbifold over k that is divisorial with respect to some collection of line bundles, and let $D = D_1 \cup \cdots \cup D_\ell$ be a simple normal crossing divisor with complement \mathcal{U} . Then, with $\mathcal{I} := \{1, \dots, \ell\}$, the coarse moduli spaces X of \mathcal{X} , etc., satisfy*

$$[U] = [X] + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [D_I \times \mathbb{P}^{|I|}]$$

in Burn_n .

Proof. The equality holds if and only if it holds after blow-up of any smooth substack of \mathcal{X} that has normal crossings with D_1, \dots, D_ℓ (where the exceptional divisor gets added as $D_{\ell+1}$ in case of center of blow-up contained in D), cf. the proof of [17, Thm. 4]. We conclude by applying the destackification algorithm [4], which leads to a situation where X is smooth with simple normal crossing divisor determined by the D_i . Then the desired formula is [18, (1.1)] which, as mentioned in the proof of Lemma 7.4, is also valid in the setting of quasiprojective X . \square

Definition 7.6. Let K_0 be a finitely generated field over k and K/K_0 a Galois algebra for the group $N_G(H)/H$. Put

$$\kappa^G: (H, N_G(H)/H \curvearrowright K, \beta) \mapsto [Y \times \mathbb{A}^{n-d}/H],$$

where H acts trivially on Y , any smooth projective variety with function field K_0 , and via the representation β on \mathbb{A}^{n-d} .

Proposition 7.7. *We have a well-defined group homomorphism*

$$\kappa^G: \text{Burn}_n(G) \rightarrow \overline{\text{Burn}}_n.$$

Proof. Changing the triple by a conjugation relation does not change the isomorphism type of the quotient stack $[Y \times \mathbb{A}^{n-d}/H]$, hence these

relations are respected by the map in Definition 7.6. To show that the map respects **(B1)**, i.e.,

$$[Y \times \mathbb{A}^{n-d}/H] = 0 \in \overline{\text{Burn}}_n,$$

when H acts on \mathbb{A}^{n-d} with weights (a_1, \dots, a_{n-d}) , and $a_1 + a_2 = 0$, we let $\mathcal{Y} := [Y \times \mathbb{A}^{n-d-2}/H]$ with action by weights a_3, \dots, a_{n-d} , and L the line bundle given by the weight a_1 . Using Example 7.1, we verify

$$[Y \times \mathbb{A}^{n-d}/H] = [\mathcal{Y}, (L, L^\vee)] - [Y \times \mathbb{P}^2 \times \mathbb{A}^{n-d-2}/\ker(a_1)] \in \overline{\text{Burn}}_n.$$

The right-hand side vanishes, as we see by applying the final relation of [18, Defn. 3.1] with $a_2 = 0$ and $j = 2$.

Let us rewrite **(B2)** as

$$(H, N_G(H)/H \curvearrowright K, \beta) - \Theta_2 = \Theta_1.$$

To see that the map respects **(B2)**, we first verify that the image of $-\Theta_2$ in $\overline{\text{Burn}}_n$ is

$$[Y \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-d-1}/H], \quad (7.5)$$

where the action of H is by $a_1 - a_2$ on $\mathbb{A}^1 \setminus \{0\}$ and by a_2, \dots, a_{n-d} on \mathbb{A}^{n-d-1} . There are two cases:

- If $a_i \in \langle a_1 - a_2 \rangle$ for some i ; then after replacing the corresponding coordinate of \mathbb{A}^{n-d-1} by its product with a suitable power of the coordinate of $\mathbb{A}^1 \setminus \{0\}$, we have a trivial H -action on this coordinate. As in Remark 7.2, (7.5) is 0, while $\Theta_2 = 0$ by definition.
- Otherwise, Θ_2 is the class of the triple given in Definition 4.2, and we have a formula for the image in $\overline{\text{Burn}}_n$ of $-\Theta_2$ by Example 7.1. Reasoning as in Example 7.1, we obtain a formula for (7.5). Comparing, we find that the two expressions are equal.

Thus it suffices to show that

$$[Y \times \mathbb{A}^{n-d}/H] + [Y \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-d-1}/H] = [Y \times \mathbb{A}^{n-d}/H] + [Y \times \mathbb{A}^{n-d}/H]$$

holds in $\overline{\text{Burn}}_n$, where the first term on the left corresponds to the given triple, and the terms on the right correspond to the triples in Θ_1 . Equivalently, by [18, Thm. 4.1],

$$\begin{aligned} [Y \times (B\ell_0 \mathbb{A}^2) \times \mathbb{A}^{n-d-2}/H] + [Y \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-d-1}/H] \\ = [Y \times \mathbb{A}^{n-d}/H] + [Y \times \mathbb{A}^{n-d}/H]. \end{aligned}$$

The stack $[Y \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-d-1}/H]$ as well as each of the stacks $[Y \times \mathbb{A}^{n-d}/H]$ on the right-hand side may be viewed as complements of smooth divisors in $[Y \times (B\ell_0 \mathbb{A}^2) \times \mathbb{A}^{n-d-2}/H]$. We get the desired equality by Lemma 7.4. \square

In order to show that κ^G maps the class of a variety with group action to the class of the associated quotient stack (Proposition 7.9), we need the following computation, where we employ the notation

$$[X \hookrightarrow G, (L_1, \dots, L_r)]^{\text{naive}} := [L_1 \oplus \dots \oplus L_r \hookrightarrow G]^{\text{naive}} \in \overline{\text{Burn}}_n(G)$$

for a smooth quasiprojective variety X with G -action and G -linearized line bundles L_1, \dots, L_r , with X of dimension $n-r$, such that the G -action on the total space of $L_1 \oplus \dots \oplus L_r$ is generically free.

Lemma 7.8. *Let X be a smooth quasiprojective variety with G -action, and let L_1, \dots, L_r be G -linearized line bundles on X , such that the G -action on $L_1 \oplus \dots \oplus L_r$ is generically free. We suppose that G acts transitively on the set of components of X , the dimension of X is $n-r$, and for some component of X , every point has stabilizer $H \subseteq G$. Then $\kappa^G([X \hookrightarrow G, (L_1, \dots, L_r)]^{\text{naive}})$ is the class in $\overline{\text{Burn}}_n$ of the field $k(X)^G$ and the representation of H determined by $L_1 \oplus \dots \oplus L_r$.*

Proof. Let $A := H^\vee$, with $a_1, \dots, a_r \in A$ determined by L_1, \dots, L_r . With $\mathcal{I} := \{1, \dots, r\}$ and, for $I \subseteq \mathcal{I}$,

$$H_I := \bigcap_{i \in I} \ker(a_i) \quad \text{and} \quad A_I := A/\langle a_i \rangle_{i \in I},$$

we have an expression of $[X \hookrightarrow G, (L_1, \dots, L_r)]^{\text{naive}}$ as a sum of triples over $I \subseteq \mathcal{I}$, such that for all $j \in \mathcal{I} \setminus I$ we have $a_j \notin \langle a_i \rangle_{i \in I}$. The triple for a given I consists of the group H_I , a Galois algebra over $k(X)^G(t_1, \dots, t_{|I|})$, and the classes of a_j in A_I , for $j \in \mathcal{I} \setminus I$. By Example 7.1, upon application of κ^G such a triple gives rise to a sum indexed by subsets of \mathcal{I} , disjoint from I :

$$\kappa^G([X \hookrightarrow G, (L_1, \dots, L_r)]^{\text{naive}}) = \sum'_{I \subseteq \mathcal{I}} \sum_{\substack{J \subseteq \mathcal{I} \\ I \cap J \neq \emptyset}} (-1)^{|J|} ([k(X)^G], (\bar{a}_1, \dots, \bar{a}_r)), \quad (7.6)$$

where \sum' denotes the sum over I satisfying $a_j \notin \langle a_i \rangle_{i \in I}$ for all $j \in \mathcal{I} \setminus I$, and \bar{a}_i denotes the class of a_i in $A_{I \cup J}$. By Remark 7.2, if we consider some I such that $a_j \in \langle a_i \rangle_{i \in I}$ for some $j \in \mathcal{I} \setminus I$ and evaluate the inner sum of the right-hand side of (7.6) for this I , we get 0. So

$$\kappa^G([X \hookrightarrow G, (L_1, \dots, L_r)]^{\text{naive}}) = \sum_{\substack{I, J \subseteq \mathcal{I} \\ I \cap J \neq \emptyset}} (-1)^{|J|} ([k(X)^G], (\bar{a}_1, \dots, \bar{a}_r)).$$

Due to the sign, only the term with $I = J = \emptyset$ survives. \square

Proposition 7.9. *For a quasiprojective variety X of dimension n with generically free G -action and stack quotient $\mathcal{X} := [X/G]$ we have*

$$\kappa^G([X \curvearrowleft G]) = [\mathcal{X}]$$

in $\overline{\text{Burn}}_n$.

Proof. Definition 5.4 and Lemma 7.4 allow us to reduce the proposition to the claim, that for a smooth projective variety X with G -action and G -linearized line bundles L_1, \dots, L_r , such that the G -action on $L_1 \oplus \dots \oplus L_r$ is generically free and satisfies Assumption 2, with X of dimension $n-r$, we have

$$\kappa^G([X \curvearrowleft G, (L_1, \dots, L_r)]^{\text{naive}}) = [\mathcal{X}, (L_1, \dots, L_r)], \quad (7.7)$$

where the line bundles on \mathcal{X} are those determined by the G -linearized line bundles L_1, \dots, L_r .

We prove (7.7) by induction on $\dim(X)$. The case $\dim(X) = 0$ follows from Lemma 7.8. So we assume $\dim(X) > 0$.

The class in $\text{Burn}_n(G)$ on the left-hand side of (7.7) is a birational invariant, by Lemma 5.3. If we blow up X along a smooth G -invariant subvariety, then \mathcal{X} gets blown up along the corresponding smooth substack. We verify that the right-hand side of (7.7) remains unchanged, by observing that the equality [18, (4.1)] is valid when $L_1 \oplus \dots \oplus L_r$ is inserted on both sides. So, by divisorialification as in Proposition 3.3, we may reduce further to the case that X possesses a simple normal crossing divisor $D = D_1 \cup \dots \cup D_\ell$, where each D_i is G -invariant, such that $U := X \setminus D$ has constant stabilizers, as in Lemma 7.8.

With a minor adaptation to the proof of Lemma 5.7, taking W in $L_1|_{D_M} \oplus \dots \oplus L_r|_{D_M}$ instead of in D_M , we obtain the identity

$$\begin{aligned} [X \curvearrowleft G, (L_1, \dots, L_r)]^{\text{naive}} &= [U \curvearrowleft G, (L_1|_U, \dots, L_r|_U)]^{\text{naive}} \\ &+ \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}_{D_I/X}^\circ \curvearrowleft G, (L_1|_{D_I}, \dots, L_r|_{D_I})]^{\text{naive}} \\ &- \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [D_I \curvearrowleft G, (L_1|_{D_I}, \dots, L_r|_{D_I}, \dots, \mathcal{N}_{D_i/X}|_{D_I}, \dots)]^{\text{naive}}, \end{aligned}$$

where the last term includes the line bundles $\mathcal{N}_{D_i/X}|_{D_I}$ for all $i \in I$. After applying κ^G , Lemma 7.8 is applicable to the first two terms on the right, then Lemma 7.5 lets us identify their contribution as the class in Burn_{n-r} of the quotient variety of U by G , paired with the class in $\overline{\mathcal{B}}_r$ of the representation of the generic stabilizer determined by L_1, \dots, L_r . The induction hypothesis is applicable to the final term on the right, and we conclude by Lemma 7.4. \square

8. COMPARISONS

In this section, G is a finite *abelian* group. We will compare birational invariants for actions of such G introduced in [16] with the invariant in Definition 4.4, taking values in the equivariant Burnside group

$$\text{Burn}_n(G)$$

from Definition 4.2.

Let $A := G^\vee$ be the character group of G . We recall from [16]:

$$\mathcal{B}_n(G)$$

is the \mathbb{Z} -module generated by equivalence classes of faithful n -dimensional linear representations of G over k , i.e., by symbols

$$\beta := [a_1, \dots, a_n], \quad a_i \in A, \quad (8.1)$$

consisting of n -tuples of characters of G , up to order, generating A ; these are subject to relations, for all $2 \leq j \leq n$:

$$\begin{aligned} & [a_1, \dots, a_n] \\ &= \sum_{\substack{1 \leq i \leq j \\ a_i \neq a_{i'} \forall i' < i}} [a_1 - a_i, \dots, a_i, \dots, a_j - a_i, a_{j+1}, \dots, a_n]. \end{aligned} \quad (8.2)$$

We first explain how to obtain $\mathcal{B}_n(G)$ as a quotient of $\text{Burn}_n(G)$. There is a quotient group $\text{Burn}_n^G(G)$ of $\text{Burn}_n(G)$, consisting of triples with first entry G :

$$(G, \text{triv} \curvearrowright K, \beta).$$

The homomorphism

$$\text{Burn}_n(G) \rightarrow \text{Burn}_n^G(G)$$

to the quotient group annihilates all triples whose first entry is a proper subgroup of G . The next result reveals a further quotient group, isomorphic to $\mathcal{B}_n(G)$.

Proposition 8.1. *The map sending the class of a triple*

$$(G, \text{triv} \curvearrowright K, \beta) \in \text{Burn}_n^G(G),$$

with $\beta = (a_1, \dots, a_{n-d})$, and d the transcendence degree of K over k , to

$$[k' : k][a_1, \dots, a_{n-d}, 0, \dots, 0] \in \mathcal{B}_n(G),$$

where k' denotes the algebraic closure of k in K , extends to a surjective homomorphism

$$\text{Burn}_n^G(G) \rightarrow \mathcal{B}_n(G).$$

For $n \geq 2$, the group $\mathcal{B}_n(G)$ may be presented as a \mathbb{Z} -module by the generators (8.1) and just the relations (8.2) with $j = 2$.

Proof. The quantity $[k' : k]$ is unchanged when K is replaced by $K(t)$. So, to verify that the map is a homomorphism, it suffices to verify relations analogous to **(B1)** and **(B2)** in $\mathcal{B}_n(G)$. For **(B1)**, we apply (8.2) with $j = 2$ and $a_2 = 0$. The general $j = 2$ case of (8.2) takes care of **(B2)**. It is clear that the homomorphism is surjective. For the final claim, we just have to check that in the abelian group defined by generators (8.1) and relations (8.2) with $j = 2$, the relations (8.2) with $j > 2$ follow as a consequence. If any of a_1, \dots, a_j is 0, then the relation is a consequence of the relation analogous to **(B1)**, which we have already treated. Otherwise, the relation is analogous to Proposition 4.7 (ii) and thus follows from relations analogous to **(B1)** and **(B2)**. \square

A refined version of $\mathcal{B}_n(G)$ keeps information about the stable birational type of K . This refined group

$$\mathcal{B}_n(G, k)$$

is defined by associating to a function field K of transcendence degree d over k :

- The isomorphism type of $K(t_1, \dots, t_{n-1-d})$, encoding the stable birational type of a model of K , when the formation of products with \mathbb{P}^1 is restricted by the requirement of dimension $\leq n-1$.
- An element of $\mathcal{B}_{m+1}(G)$, where m is the largest integer such that $K(t_1, \dots, t_{n-1-d})$ is isomorphic over k to $L(u_1, \dots, u_m)$ for some field L , finitely generated over k .

We refer the reader to [16] for a more detailed description and the definition of the class $\beta_k(X) \in \mathcal{B}_n(G, k)$ of a smooth projective variety X with faithful G -action.

Proposition 8.2. *The surjective homomorphism from Proposition 8.1 factors through $\mathcal{B}_n(G, k)$, and the homomorphism*

$$\text{Burn}_n^G(G) \rightarrow \mathcal{B}_n(G, k) \tag{8.3}$$

is surjective. If X is a smooth projective variety with a generically free action of G , then the image of $[X \hookrightarrow G]$ in $\text{Burn}_n^G(G)$ maps to $\beta_k(X)$ under the map (8.3).

Proof. The proof is analogous to that of Proposition 8.1. \square

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