WITT RINGS, PFISTER FORMS, AND EQUIVARIANT BIRATIONAL GEOMETRY

BRENDAN HASSETT AND YURI TSCHINKEL

Abstract. We study equivariant birational geometry of quadrics with actions of finite groups and develop analogs of Witt groups and Pfister theory in the equivariant context.

1. Introduction

The birational geometry of smooth quadric hypersurfaces over a non-closed field is a beautiful and intricate subject. Generic splitting fields, essential dimension, motivic cohomology, and the Milnor conjectures interact to form a deep and varied theory. We refer the reader to [Tot09] for an overview of birational questions, [DR15, KM03] for important insights coming from versality constructions, and [EKM08, Kah08] for a comprehensive overview. At the same time, there are intriguing results for quadrics in specific dimensions, see, e.g., [Hof95, Kar04].

Our focus here is on quadrics with generically free actions of finite groups G and their equivariant birational geometry. Quadrics over fields and over classifying spaces BG have many common features, especially, where cohomological invariants are concerned. However, birational questions over BG are not as well-understood, with most results focusing on examples of small dimensions, see [HT25], [CTZ25, Section 4].

To move beyond low dimensions, we aim to translate ideas and techniques from the theory of quadratic forms (of arbitrary dimension!) over fields to the equivariant context. Here we consider Witt rings and Pfister forms. One complication is the distinction between stable and unstable notions; for us, stabilization is taking the product with a linear representation of G. At an elementary level, rational varieties over infinite fields have many rational points, which we can exploit for geometric constructions. Witt's theory of isotropic subspaces is the archetypal example: Once we have an isotropic subspace, we can produce many more of the same dimension via explicit operations. In the equivariant context, it is necessary to consider *all* such constructions simultaneously; even linear actions of non-abelian groups on projective space may lack fixed points. However, this presents no difficulty when the parameter space itself is stably linearizable.

Section 2 establishes a bridge between equivariant stable birational geometry and birational geometry over non-closed fields. We review quadrics over fields in Section 3 and wonderful compactifications in Section 4; this machinery sheds new light on multiplicative quadratic forms. Equivariant formulations are developed in Section 5. Section 6 explores birational implications of invariant isotropic subspaces; stable birational properties are revealed on restriction to 2-Sylow subgroups. Several notions of Witt rings are presented in Section 7. Stably Pfister forms are treated in Section 8; their unstable analogues are explored in Section 9.

Acknowledgments: The first author was partially supported by the Simons Foundation Award 546235 and the US National Science Foundation Grant 1929284. The second author was partially supported by NSF grant 2301983. We are grateful to Candace Bethea for conversations that informed this project, and to Barry Mazur for suggestions on a draft of this paper. We used MAGMA [BCP97] and Gemini for computations informing our results, and ChatGPT and Claude to proofread the text.

2. Equivariant birational geometry

We recall main terms and constructions in equivariant birational geometry; for a more detailed discussion, see, e.g., [HT22, Sec. 2].

Let k be a field of characteristic zero and X a smooth projective variety over k with a regular action by a finite group G. The action is *G-equivariantly stably linearizable* (or *stably rational*) if there exist G-representations V and W and a G-equivariant birational map

$$X \times V \overset{\sim}{\dashrightarrow} W.$$

When the action on X is generically free, the No-Name Lemma implies that this is equivalent to the existence of an equivariant birational map

$$X \times \mathbb{A}^d \overset{\sim}{\dashrightarrow} W,$$

where the action on $\mathbb{A}^d$ is trivial. It is also equivalent to the existence of a G-equivariant vector bundle $E \to X$ and an equivariant

$$E \overset{\sim}{\dashrightarrow} W.$$

Two smooth projective varieties X_1 and X_2 with G-actions are *G-equivariantly stably birational* if there exist representations W_1 and W_2

and a G-equivariant birational

$$X_1 \times W_1 \overset{\sim}{\dashrightarrow} X_2 \times W_2.$$

This is an equivalence relation. Again, if the actions on X_1 and X_2 are generically free, this follows from the existence of a G-equivariant birational

$$E_1 \overset{\sim}{\dashrightarrow} E_2,$$

where E_1 and E_2 are G-equivariant vector bundles over X_1 and X_2, respectively.

The work of Duncan and Reichstein [DR15] provides links between equivariant birational geometry and geometry over non-closed fields. In particular, we can test equivariant stable birationality using twisted forms over function fields. We record the following strengthening of [DR15, Th. 1.1]:

Proposition 2.1. *Fix a linear algebraic group G and geometrically irreducible varieties X_1 and X_2, with regular G-actions. Then the following are equivalent:*

1. *X_1 and X_2 are G-equivariantly stably birational;*
2. *for every G-torsor T defined over any extension K/k, the twists*
$$^{T}X_1, {}^{T}X_2$$
are stably birational over K;
3. *there exists a linear representation V of G for which G acts freely on a dense open subset $V_0 \subset V$, with field of invariants $F = k(V)^G$, such that the twists*
$$^{V_0}X_1, {}^{V_0}X_2$$
are stably birational over F.

Similar reasoning may be found in [KT26, §2].

Proof. The implication (2) $\Rightarrow$ (3) is immediate, as V_0 is a G-torsor over V_0/G. Below we use S to denote the corresponding torsor over F.

We prove (1) $\Rightarrow$ (2). Since X_1 and X_2 are equivariantly stably birational, there exist G-representations W_1 and W_2 and an equivariant birational

$$\phi : X_1 \times W_1 \overset{\sim}{\dashrightarrow} X_2 \times W_2.$$

Twisting via T, we obtain

$$^{T}\phi : {}^{T}X_1 \times_K {}^{T}W_1 \overset{\sim}{\dashrightarrow} {}^{T}X_2 \times_K {}^{T}W_2.$$

By Hilbert's Theorem 90, the twisted representations are K-rational; it follows that ${}^{T}X_1$ and ${}^{T}X_2$ are stably birational over K.

We turn to (3) $\Rightarrow$ (1). We have birational maps

$$ {}^{V_0}X_1 \times \mathbb{A}^e_F \overset{\sim}{\dashrightarrow} {}^{V_0}X_2 \times \mathbb{A}^d_F. $$

We also have

$$ {}^{V_0}X_1 \times \mathbb{A}^e_F \simeq {}^{V_0}(X_1 \times \mathbb{A}^e_F) $$

with trivial action on the last $\mathbb{A}^e_F$.

We recall [DR15, Lem. 5.1]: Suppose that X has a G-action. A birational map

$$ Y_F \overset{\sim}{\dashrightarrow} {}^{V_0}X $$

yields a G-equivariant birational

$$ V \times Y \overset{\sim}{\dashrightarrow} V \times X. $$

Applying this, we get G-equivariant birational

$$ V \times \mathbb{A}^e_k \times X_1 \overset{\sim}{\dashrightarrow} V \times \mathbb{A}^d_k \times X_2 $$

with trivial actions on the affine spaces. It follows that X_1 and X_2 are G-equivariantly stably birational. $\square$

3. Quadrics over fields

Assume that the base field k has characteristic not equal to two.

Background on quadratic and bilinear forms. Let V be a finite-dimensional vector space over k. A symmetric bilinear form

$$ b : V \times V \to k $$

is *non-degenerate* if the associated linear map

$$ \begin{array}{rcl} s : V & \to & V^\vee \\ v & \mapsto & b(v, -) \end{array} $$

is an isomorphism; its associated quadratic form is

$$ q(v) := b(v, v). $$

Its *rank* is the dimension of V. Given symmetric bilinear forms $b_i : V_i \times V_i \to k$ we may define their direct sum

$$ \begin{gathered} b_1 \oplus b_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \to k, \\ (b_1 \oplus b_2)(v_1 + v_2, v_1' + v_2') = b_1(v_1, v_1') + b_2(v_2, v_2') \end{gathered} $$

and tensor product

$$ \begin{gathered} b_1 \otimes b_2 : (V_1 \otimes V_2) \times (V_1 \otimes V_2) \to k, \\ (b_1 \otimes b_2)(v_1 \otimes v_2, v_1' \otimes v_2') = b_1(v_1, v_1') b_2(v_2, v_2'). \end{gathered} $$

A linear map $g : V \to V$ is a *similitude* with respect to b if there is a scalar $\lambda(g)$ with

$$b(g(v), g(v')) = \lambda(g)b(v, v'), \quad \text{for all } v, v' \in V.$$

These form a group $\mathrm{GO}(V, b)$ with a natural character

$$\lambda : \mathrm{GO}(V, b) \to \mathbb{G}_m;$$

its kernel is denoted $\mathrm{O}(V, b)$ and elements of determinant one is written $\mathrm{SO}(V, b)$. There is analogous notation for quadratic forms q. We may interpret $q \in \mathrm{Sym}^2(V^\vee) \otimes L$, where L is the one-dimensional representation associated with λ. In the non-degenerate case, we have [KMRT98, §12.A]

$$\det(g)^2 = \lambda(g)^{\dim(V)}.$$

Discriminant. Assuming further that $\dim(V) = 2m$, we obtain a character

$$\mathrm{disc} : \mathrm{GO}(V, b) \to \{\pm 1\}, \quad \mathrm{disc}(g) = \det(g)\lambda(g)^{-\dim(V)/2}.$$

This coincides with the determinant on restriction to $\mathrm{O}(V, b)$.

Given a non-degenerate quadratic form (V, q), an *isotropic subspace* $W \subset V$ is one where $q|W = 0$; we have $\ell := \dim(W) \leq \dim(V)/2$. The *isotropic Grassmannian*

$$\mathrm{OGr}(\ell, q)$$

is the variety of all ℓ-dimensional isotropic subspaces. This scheme is non-empty for $\ell \leq \dim(V)/2$. These come with a $\mathrm{GO}(V, q)$ action.

Clifford algebras and spin: Even-dimensional case. We recall basic facts from [EKM08, §85]: Let (V, q) be non-degenerate with $\dim(V) = 2m$ and $\mathrm{OGr}(m, q)$ its maximal isotropic Grassmannian. Geometrically, it has two connected components $\mathrm{OGr}(m, q)_\pm$, each smooth of dimension $(m-1)m/2$, permuted according to discriminant character. Over the discriminant extension, each connected component of $\mathrm{OGr}(m, q)$ is a point for $m = 1$, a conic when $m = 2$, and a Brauer-Severi threefold for $m = 3$.

By [FH91, pp. 387-390], over an algebraically closed field, we have embeddings

$$\mathrm{OGr}(m, q)_\pm \hookrightarrow \mathbb{P}(S_\pm),$$

where the $S_\pm$ are the *half-spin representations*, which are irreducible 2^{m-1}-dimensional representations of the spin group

$$1 \to \mu_2 \to \mathrm{Spin}(q) \to \mathrm{SO}(V, q) \to 1. \tag{3.1}$$

The even Clifford algebra decomposes [FH91, p. 305]

$$C_0(q) \simeq \mathrm{End}(S_+) \times \mathrm{End}(S_-).$$

Over a nonclosed field, we have:

Proposition 3.1. [EKM08, Rem. 13.9] *Let (V,q) be a non-degenerate quadratic form over a field k, with $\dim(V) = 2m$ and trivial discriminant. Then*

$$C_0(q) \simeq C^+(q) \times C^-(q)$$

and the Brauer-Severi varieties associated with $\mathbb{P}(S_\pm)$ have the same class in $\mathrm{Br}(k)$, realized via Clifford invariant $c(q) \in \mathrm{H}^2(k,\mu_2)$ arising from the connecting homomorphisms of (3.1).

We elaborate on Clifford structures and the actions of similitudes [KMRT98, §13.A]. Recall that L is the representation of $\mathrm{GO}(V,q)$ associated with λ. Given $q \in \mathrm{Sym}^2(V^\vee) \otimes L$ we may construct the *even Clifford algebra*

$$C_0(q) = \left(\oplus_{i=0}^m (V^{\otimes 2i} \otimes L^{-i})\right)/I, \quad 2m = \dim(V),$$

where the ideal I comes from the multiplication induced by

$$v^2\ell \mapsto \ell(q(v)), \quad v \in V, \ell \in L^\vee,$$

i.e.,

$$I = \left\langle v^2 \otimes \ell - \ell(q(v^2))\right\rangle.$$

Thus we have an isomorphism of $\mathrm{GO}(V,q)$-representations

$$C_0(q) = \oplus_{i=0}^m (\wedge^{2i} V \otimes L^{-i})$$

as well as an action of $\mathrm{GO}(V,q)$. When $\dim(V) = 2m$, its center is a quadratic algebra containing 1, determining the discriminant invariant [EKM08, Prop. 11.6]. .

Clifford algebras and spin: Odd-dimensional case. We now consider (V,q) with $\dim(V) = 2m+1$. Over an algebraically closed field, $C_0(q) \simeq \mathrm{End}(S)$ where S is the irreducible spin representation of the associated spin group [FH91, Prop. 20.20]. We have an embedding [FH91, p. 390]

$$\mathrm{OGr}(m,q) \hookrightarrow \mathbb{P}(S)$$

such that the square of the ample line bundle equals the polarization from the Plücker embedding.

Over a general field, $C_0(q)$ is a central simple algebra [EKM08, Prop. 11.6(2)], also called the Clifford invariant of (V,q), and $\mathrm{OGr}(m,q)$ embeds into the corresponding Brauer-Severi variety of dimension $2^m - 1$. When $m = 2$, the embedding is an isomorphism and $\mathrm{OGr}(2,q)$ is a Brauer-Severi threefold.

Splitting varieties of linear subspaces.

Proposition 3.2. *Let (V, q) be a non-degenerate quadratic form over a field k.*

- *If* $\dim(V) = 2m - 1$ *and* $\mathrm{OGr}(m-1, q)$ *has a k-rational point then it is rational.*
- *If* $\dim(V) = 2m$ *and* $\mathrm{OGr}(m, q)$ *has a k-rational point then the irreducible component of* $\mathrm{OGr}(m, q)$ *containing that point is rational over k.*

Proof. We establish the first statement: Let $L \subset V$ denote an isotropic subspace of dimension $m-1$ over k. Linear algebra gives another isotropic subspace $M \subset V$ of dimension $m-1$ such that the induced pairing between L and M is non-degenerate i.e., $M \simeq L^\vee$. Write

$$L^\perp \cap M^\perp = \mathrm{span}(w)$$

for some non-zero $w \in V$. Consider elements

$$A \in \mathrm{Hom}(L, M \oplus kw)$$

whose projections to $\mathrm{Hom}(L, M) \simeq \mathrm{Hom}(L, L^\vee)$ are anti-symmetric under duality. This anti-symmetry is equivalent to the graph

$$\Gamma_A \subset L \oplus M \oplus kw = V$$

being isotropic for q. The generic element of $\mathrm{OGr}(m-1, q)$ arises from such a graph, giving a birational

$$\mathbb{A}^{\binom{m-1}{2}+m-1} \overset{\sim}{\dashrightarrow} \mathrm{Gr}(m-1, q),$$

and the desired rationality.

Now for the second assertion: Let $L \in \mathrm{OGr}(m, q)$ be the rational point; label the irreducible components so that $L \in \mathrm{OGr}(m, q)_+$. Using linear algebra, we find an isotropic subspace $M \subset V$ of dimension m such that the induced pairing between L and M is non-degenerate, i.e. $M \simeq L^\vee$. Note that $M \in \mathrm{OGr}(m, q)_+$ if m is even and $M \in \mathrm{OGr}(m, q)_-$ when m is odd. Consider elements

$$A \in \mathrm{Hom}(L, M) \simeq \mathrm{Hom}(L, L^\vee)$$

that are anti-symmetric under duality, which is equivalent to the graph

$$\Gamma_A \subset L \oplus M = V$$

being isotropic for q. Thus $\mathrm{OGr}(m, q)_+$ is birational to an affine space of dimension $m(m-1)/2$. ☐

For odd m, the same argument with L and M interchanged implies that $\mathrm{OGr}(m, q)_-$ is also rational.

Proposition 3.3. *Let (V, q) be a non-degenerate quadratic form over a field k. Suppose $\dim(V) = 2m$ and $\mathrm{OGr}(m, q)_\pm$ are both defined over k, i.e., the discriminant of q is trivial. Then $\mathrm{OGr}(m, q)_+$ and $\mathrm{OGr}(m, q)_-$ are isomorphic to each other.*

In particular, if one component has a rational point then the the other does as well.

Proof. Diagonalize

$$q = \sum_{i=1}^{2m} a_i x_i^2, \quad 0 \neq a_i \in k$$

and consider the reflection

$$(x_1, x_2, \ldots, x_{2m}) \mapsto (-x_1, x_2, \ldots, x_{2m}).$$

This has discriminant -1 and induces the desired isomorphism. □

Remark 3.4. We show that the components are stably birational without choosing a reflection: Consider all hyperplanes $W \subset V$. Working over the dual space, we obtain a birational

$$\mathrm{OGr}(m, q)_\pm \times V^\vee \overset{\sim}{\dashrightarrow} \mathrm{OGr}(m-1, q|W) \times V^\vee.$$

Here we regard $q|W$ as a quadratic form over the function field of $V^\vee$. We conclude that the $\mathrm{OGr}(m, q)_\pm$ are mutually stably birational.

Here is an argument valid for odd m: $\mathrm{OGr}(m, q)_+ \times \mathrm{OGr}(m, q)_-$ is birational to $\mathrm{OGr}(m, q)_+ \times \mathbb{A}^{m(m-1)/2}$ and $\mathbb{A}^{m(m-1)/2} \times \mathrm{OGr}(m, q)_-$. Thus the components are mutually stably birational.

Proposition 3.5. *Let (V, q) be a non-degenerate quadratic form over a field k and ℓ a positive integer such that $2\ell < \dim(V)$. If $\mathrm{OGr}(\ell, q)$ has a k-rational point then it is rational over k.*

Proof. Let $X = \{q = 0\} \subset \mathbb{P}(V)$ and $P = \mathbb{P}(L) \subset X$ be the subspace associated with the k-rational point $[L] \in \mathrm{OGr}(\ell, q)$. Projection from P gives a dominant rational map

$$\pi : \mathrm{OGr}(\ell, q) \dashrightarrow \mathrm{Gr}(\ell, V/L).$$

Given $L' \subset V/L$ of dimension ℓ, the induced extension

$$0 \to L \to E \to L' \to 0$$

has dimension 2ℓ. The restriction $Q = q|E$ is a quadratic form of rank 2ℓ, with tautological maximal isotropic subspace L. The generic fiber of π is isomorphic to $\mathrm{OGr}(\ell, Q)_\circ$ over the function field $k(\mathrm{Gr}(\ell, V/L))$. Here $\circ$ denotes one component of the orthogonal Grassmannian – the one parametrizing subspaces intersecting L in a subspace of even dimension. This contains L if and only if ℓ is even.

The second part of Proposition 3.2 implies that $\mathrm{OGr}(\ell, q)$ is rational over the function field of $\mathrm{Gr}(\ell, V/L)$, hence rational over k. $\square$

Subspaces of dimension ℓ contains subspaces of smaller dimensions, thus we have:

Corollary 3.6. *Retain the notation of Proposition 3.5. If* $\mathrm{OGr}(\ell, q)$ *has a* k*-rational point then* $\mathrm{OGr}(\ell', q)$ *is rational, or a disjoint union of two rational components, for each* $\ell' \leq \ell$.

Witt rings over fields. Consider non-degenerate quadratic forms (V, q) over k, under the operations of direct sum and tensor product. Recall that a quadratic form is *hyperbolic* if it may be expressed as a direct sum of two-dimensional vector spaces with hyperbolic form xy. Recall fundamental results of Witt: Every non-degenerate form admits an expression

$$(V, q) = (V_{an}, q_{an}) \oplus_{\perp} H,$$

where the first summand is *anisotropic* – has no non-trivial isotropic subspace – and the second summand is hyperbolic. Moreover, the anisotropic summand is unique up to isomorphism; indeed, if r is the maximal dimension of an isotropic subspace of q then H is a direct sum of r hyperbolic planes. The results above (like Corollary 3.6) imply that *all* such decompositions are parametrized by a rational variety or the disjoint union of two rational varieties.

For any non-degenerate quadratic form (V, q), the direct sum

$$(V, q) \oplus_{\perp} (V, -q)$$

is hyperbolic, with the diagonal subspace isotropic. The tensor product of any form with a hyperbolic form is hyperbolic.

Definition 3.7. [EKM08, §2] The *Witt ring* $\mathrm{Witt}(k)$ is the collection of isomorphism classes of non-degenerate quadratic forms under direct sum and tensor product, modulo the ideal of hyperbolic forms.

Each non-zero element of $\mathrm{Witt}(k)$ is represented by an anisotropic quadratic form.

The *augmentation ideal* $I \subset \mathrm{Witt}(k)$ consists of all the quadratic forms of even dimension. The associated graded ring

$$\mathrm{Witt}(k)/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

encodes many key invariants: The discriminant of an even-rank form is captured by I/I^2 and the Clifford invariant by I^2/I^3.

4. Compactifications

Let (V, q) be a non-degenerate quadratic form over k with $\dim(V) = 2m > 0$. The projectivization of $\mathrm{SO}(V, q) \subset \mathrm{End}(V)$ is the *projective orthogonal group*

$$\mathrm{PSO}(V, q) \subset \mathbb{P}(\mathrm{End}(V)).$$

It is the adjoint form of the special orthogonal group. We interpret the similitude character

$$\lambda \in \Gamma(\mathcal{O}_{\mathrm{PSO}(V,q)}(2)),$$

as it is a homogeneous quadratic form in the matrix entries.

Geometric normal forms. We sketch the geometry of closure of the orthogonal group, assuming k is algebraically closed. More formal descriptions will follow after we introduce the wonderful compactification.

Since q is hyperbolic, we may choose coordinates

$$x_{1'}, \ldots, x_{m'}, x_{1''}, \ldots, x_{m]]}$$

such that

$$q := x_{1'}x_{1''} + \cdots + x_{m'}x_{m''}.$$

The maximal torus of $\mathbb{G}_m \times \mathrm{SO}(V, q)$ acts by

$$(x_{1'}, \ldots, x_{m'}, x_{1''}, \ldots, x_{m''}) \mapsto (t_1x_{1'}, \ldots, t_mx_{m'}, \lambda t_1^{-1}x_{1''}, \ldots, \lambda t_m^{-1}x_{m''}).$$

There is a natural homomorphism

$$\mathbb{G}_m \times \mathrm{SO}(V, q) \to \mathrm{GO}(V, q).$$

The Weyl group $\mathrm{W}(\mathsf{D}_m)$ acts on the maximal torus of $\mathrm{SO}(V, q)$ via signed permutations, exchanging the indices $\{1, \ldots, m\}$ and inverting various t_i.

The eigenvectors of a generic semisimple element of $\mathrm{SO}(V, q)$ determines a collection of 2^{m-1} decompositions into complementary isotropic subspaces

$$V = W' \oplus W'',$$
$$W' = \{x_{1''} = \cdots = x_{m''} = 0\}, \quad W'' = \{x_{1'} = \cdots = x_{m'} = 0\}.$$

Their orbits under $\mathrm{W}(\mathsf{D}_m)$ depends on the parity of m. For odd m, W' and W'' come from distinct varieties $\mathrm{OGr}(m, q)_+$ and $\mathrm{OGr}(m, q)_-$, thus each decomposition admits an action by the discriminant. The group $C_2^{m-1} \subset \mathrm{W}(\mathsf{D}_m)$ acts simply transitively on the collection. For even m, W' and W'' come from the same component of the variety of maximal isotropic subspaces, and C_2^{m-1} has two orbits.

We consider non-zero limiting elements in the closure

$$\mathrm{PSO}(V, q) \subset \overline{\mathrm{PSO}}(V, q) \subset \mathbb{P}(\mathrm{End}(V));$$

these may be understood as where $\lambda = 0$. Such elements are conjugate to

$$(x_{1'}, \dots, x_{m'}, x_{1''}, \dots, x_{m''}) \mapsto (t_1 x_{1'}, \dots, t_j x_{j'}, 0, \dots, 0),$$

for $j = 1, \dots, m$. In other words, up to the action of the Weyl group and exchanging $x_{i'}$ and $x_{i''}$, these are where

$$\lambda = t_{j+1} = \cdots = t_m = 0$$

and

$$x_{(j+1)'} = \cdots = x_{m'} = x_{1''} = \cdots = x_{m''} = 0.$$

These have rank $\leq m$ and we describe those of rank m:

Proposition 4.1. *The variety of rank-m elements*

$$\{A \in \overline{\mathrm{PSO}}(V, q) : \mathrm{rank}(A) = m\}$$

has two irreducible components, of dimension

$$2m^2 - m - 1 = \dim(\mathrm{PSO}(V, q)) - 1,$$

described as follows: Generic elements A correspond to decompositions into isotropic subspaces

$$V = W' \oplus W''$$

invariant under A, where $A|W' = 0$ and $A|W''$ is invertible (resp. $A|W'' = 0$ and $A|W'$ is invertible).

If m is odd then W' and W'' come from distinct components $\mathrm{OGr}(m, q)_{\pm}$; when m is even they come from same component of the maximal isotropic Grassmannian. The resulting divisors are PGL_m-bundles over

$$\mathrm{OGr}(m, q)_+ \times \mathrm{OGr}(m, q)_-,$$

when m is odd, and over

$$\mathrm{OGr}(m, q)_+ \times \mathrm{OGr}(m, q)_+ \text{ and } \mathrm{OGr}(m, q)_- \times \mathrm{OGr}(m, q)_-$$

when m is even.

We close with two remarks about the strata of rank $< m$:

- they have dimension strictly smaller than the divisorial strata of Proposition 4.1;
- for matrices A of rank $< m$ the subspaces W' and W'' are not uniquely determined by the matrix.

Wonderful constructions. We consider compactifications of $\mathrm{PSO}(V,q)$, bi-equivariant under the action:

$$\begin{aligned}\mathrm{PSO}(V,q)^2 \times \mathrm{PSO}(V,q) &\to \mathrm{PSO}(V,q)\\ ((g_1,g_2),g) &\mapsto g_1 g g_2^{-1}.\end{aligned}$$

These were pioneered by De Concini-Procesi [DCP83] and subsequently extended to general fields [Str87, DCS99]. Details on their Picard groups, from the perspective of spherical varieties, may be found in [Bri89] and [Bri07].

Consider the simply-connected cover of $\mathrm{PSO}(V,q)$ over an algebraic closure and fix positive and simple roots. Let $\boldsymbol{\lambda}$ denote a dominant weight with associated representation $V_{\boldsymbol{\lambda}}$ and endomorphisms $\mathrm{End}(V_{\boldsymbol{\lambda}})$. Let

$$X_{\boldsymbol{\lambda}}(V,q) \subset \mathbb{P}(\mathrm{End}(V_{\boldsymbol{\lambda}}))$$

denote the closure of the identity under the $\mathrm{PSO}(V,q) \times \mathrm{PSO}(V,q)$ action on

$$\mathrm{End}(V_{\boldsymbol{\lambda}}) = V_{\boldsymbol{\lambda}} \otimes V_{\boldsymbol{\lambda}}^{\vee}.$$

When $\boldsymbol{\lambda}$ is regular – a positive sum of the fundamental weights, in the interior of the Weyl chamber – we obtain the *wonderful compactification* [DCP83, 3.4]

$$\overline{X}(V,q) := X_{\boldsymbol{\lambda}}(V,q) \supset \mathrm{PSO}(V,q).$$

For weights $\boldsymbol{\lambda}'$ along the boundary of the Weyl chamber, we obtain bi-equivariant morphisms

$$X_{\boldsymbol{\lambda}}(V,q) \to X_{\boldsymbol{\lambda}'}(V,q).$$

For $V_{\boldsymbol{\lambda}'} = V$, the $2m$-dimension standard representation, the distinguished orbit in $\mathbb{P}(\mathrm{End}(V))$ is the projective similitude group. This yields an explicit iterated blowup of its closure

$$\beta : \overline{X}(V,q) \xrightarrow{\sim} X_{\boldsymbol{\lambda}'}(V,q) = \overline{\mathrm{PSO}}(V,q) \subset \mathbb{P}(\mathrm{End}(V)).$$

We record standard facts about $\overline{X}(V,q)$ over algebraically closed fields [DCP83, 3.1]:

- $\overline{X}(V,q)$ is smooth and projective;
- the boundary $\overline{X}(V,q) \setminus \mathrm{PSO}(V,q)$ is a normal crossings divisor with irreducible components D_i indexed by the simple roots and strata/orbits indexed by subsets of those roots;
- the minimal stratum $Y \simeq (\mathrm{PSO}(V,q)/B)^2$, the product of complete isotropic flag varieties.

Furthermore, analyzing the restriction homomorphism

$$\mathrm{Pic}(\overline{X}(V,q)) \to \mathrm{Pic}(\mathrm{PSO}(V,q)/B)^2,$$

we have [DCP83, §8] and [Bri07, 2.1.3]:

- $\mathrm{Pic}(\overline{X}(V,q))$ is isomorphic to the character group Λ of the simply-connected cover of $\mathrm{PSO}(V,q)$, and is realized as the graph of an explicit isomorphism

$$\mathrm{Pic}(\mathrm{PSO}(V,q)/B) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{PSO}(V,q)/B).$$

- The Picard group is freely generated by the "color" divisors $C_1,\ldots,C_m$, closures of codimension-one components of

$$\mathrm{PSO}(V,q) \setminus (B^- \times_T B),$$

where B^- is the "opposite" Borel with $B^- \cap B = T$ the maximal torus. The color divisors are indexed by fundamental weights $\omega_1,\ldots,\omega_m$ of the simply-connected cover of $\mathrm{PSO}(V,q)$ [Bri07, 2.2.4].
- Nef divisors are non-negative linear combinations of $C_1,\ldots,C_m$ [Bri89, 2.6].
- The cone (but not necessarily the monoid) of effective divisors is generated by $D_1,\ldots,D_m$, realized as simple roots [Bri07, 2.3.5].

Given a weight $\boldsymbol{\mu} \in \Lambda$, let $L_{\boldsymbol{\mu}}$ be the associated line bundle on $\overline{X}(V,q)$. Brion [Bri07, 3.2.4] computes the total coordinate ring

$$R(\overline{X}(V,q)) = \oplus_{\boldsymbol{\mu}\in\Lambda}\Gamma(\overline{X}(V,q), L_{\boldsymbol{\mu}}).$$

For our purposes, it suffices to mention that for a dominant weight $\boldsymbol{\lambda}$, there is distinguished summand

$$\mathrm{End}(V_{\boldsymbol{\lambda}}) \subset \Gamma(\overline{X}(V,q), L_{\boldsymbol{\lambda}})$$

associated with the morphism

$$\overline{X}(V,q) \to X_{\boldsymbol{\lambda}}(V,q) \subset \mathbb{P}(\mathrm{End}(V_{\boldsymbol{\lambda}})).$$

In particular, we can compute the image of β^* on the Picard group.

We explore the geometry of the boundary divisors under β. The simple roots $\alpha_1,\ldots,\alpha_{m-1},\alpha_m$ may be ordered so that the first $m-2$ are the strand of the Dynkin diagram and α_{m-1},α_m are the two nodes at the end, exchanged by the involution of the diagram. (This assignment is ambiguous for $m=4$ due to triality.)

Proposition 4.2. *The divisors $D_1,\ldots,D_{m-2}$ are exceptional for β. The divisors D_{m-1} and D_m correspond to projections onto pairs of maximal isotropic subspaces*

$$\mathrm{OGr}(m,q)_+ \times \mathrm{OGr}(m,q)_- \textit{ (odd } m) \quad \textit{or} \quad \mathrm{OGr}(m,q)_\pm^2 \textit{ (even } m)$$

composed with isomorphisms of one of those subspaces. The pull back of the similitude divisor $\{\lambda = 0\}$ *under* β *has class*

$$2(D_1 + \cdots + D_{m-2}) + D_{m-1} + D_m,$$

so the exceptional divisors all appear with multiplicity two.

Proof. The description of D_{m-1} and D_m are given in Proposition 4.1. The remaining strata $D_1, \dots, D_{m-2}$ correspond to lower-rank matrices.

In light of the descriptions of effective and nef divisors above, the last assertion is a computation with root systems. We use the notation of [FH91, 19.2] with fundamental weights

$$\omega_i = \begin{cases} L_1 + \cdots + L_i & i \leq m-2 \\ (L_1 + \cdots + L_{m-1} - L_m)/2 & i = m-1 \\ (L_1 + \cdots + L_{m-1} + L_m)/2 & i = m \end{cases}$$

and simple roots

$$\alpha_i = L_i - L_{i+1},\ i = 1, \dots, m-1, \quad \alpha_m = L_{m-1} + L_m.$$

These correspond to the divisors C_i and D_i respectively. Thus we find

$$\begin{aligned} \beta^*\{\lambda = 0\} = 2L_1 = 2(L_1 - L_2) + \cdots + 2(L_{m-2} - L_{m-1}) \\ +(L_{m-1} - L_m) + (L_{m-1} + L_m) \\ = 2D_1 + \cdots + 2D_{m-2} + D_{m-1} + D_m. \end{aligned}$$

□

Remark 4.3 (Extension to orthogonal groups). Recall that the groups $\mathrm{O}(V, q)$, $\mathrm{GO}(V, q)$, and the projectivization $\mathrm{PGO}(V, q)$ have two connected components, distinguished by the discriminant character disc. Concretely, when q is hyperbolic we can express

$$\mathrm{O}(V, q) = \mathrm{SO}(V, q) \rtimes \langle \iota \rangle$$

where ι is an involution exchanging $\{x_{i+}, x_{i-}\}$ for some index i. The construction of $X_{\boldsymbol{\lambda}}(V, q)$ yields compactifications of $\mathrm{PGO}(V, q)$ with two components isomorphic to $X_{\boldsymbol{\lambda}}(V, q)$. Thus $\mathrm{PGO}(V, q)$ admits a wonderful compactification consisting of two copies of $\overline{X}(V, q)$. These are all equivariant for the $\mathrm{PGO}(V, q) \times \mathrm{PGO}(V, q)$ action

$$((g_1, g_2), g) \mapsto g_1 g g_2^{-1}.$$

5. Equivariant quadratic forms

From now on, k is algebraically closed of characteristic zero unless specified otherwise.

Let G be a finite group and consider a projective quadric hypersurface $X \subset \mathbb{P}^n$ admitting a regular action of G. Using the extension

$$1 \to \mathbb{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$$

we extract the following data:

- an extension

$$1 \to \mathbb{G}_m \to \widehat{G} \to G \to 1; \tag{5.1}$$

- a linear representation of dimension $n+1$

$$\rho : \widehat{G} \to \mathrm{GL}(V)$$

  inducing the G-action on $\mathbb{P}^n = \mathbb{P}(V)$;
- a G-quadratic form, i.e., a G-invariant element $q \in \mathrm{Sym}^2(V^\vee) \otimes L$, where L is a one-dimensional representation with associated character $\lambda : \widehat{G} \to \mathbb{G}_m$, such that

$$X = \{q = 0\}, \quad q(\rho(g) \cdot v) = \lambda(g) q(v).$$

The obstruction to the extension (5.1) splitting is a class

$$\alpha \in \mathrm{H}^2(G, \mathbb{G}_m)$$

known as the *Amitsur invariant.* If $\alpha \neq 0$ then $\mathbb{P}^n$ and its G-subvarieties are not stably linearizable; see [BP13, §2], [BCDP23, App.A], [HT22, §2], and [HT23, §3.5] for more context. From now on, we impose the vanishing of this invariant:

Definition 5.1. A *G-quadric* is a quadric hypersurface $X \subset \mathbb{P}(V)$, where $\rho : G \to \mathrm{GL}(V)$ is a linear representation and X is invariant under the action of G.

The G-quadratic form q is equivalent to a G-linear isomorphism

$$s : V \longrightarrow V^\vee \otimes L,$$

symmetric in that it is unchanged on applying $\mathrm{Hom}(-, L)$, via the functional relation

$$q(v) = s(v) \cdot v.$$

Thus these correspond to G-representations in the group of similitudes. Write $\lambda : G \to \mathbb{G}_m$ for the character associated with L.

Example 5.2. Let $G = C_n = \langle\gamma\rangle$ be a cyclic group of order n, and $V = \langle v_1, v_2\rangle = \{xv_1 + yv_2\}$ a representation

$$\gamma \cdot x = \zeta x, \quad \gamma \cdot y = \zeta^a y, \quad \zeta = \exp(2\pi i/n),$$

with quadratic form $q = xy$ with $\lambda(\gamma) = \zeta^{a+1}$.

Assumption 5.3. *We will usually assume that X is* smooth*, or equivalently, $s : V \xrightarrow{\sim} V^\vee \otimes L$ or q is* non-degenerate*.*

Remark 5.4. For any character $\chi : G \to \mathbb{G}_m$, the representations ρ and $\rho \otimes \chi$ yield the same projective representation $G \to \mathrm{PGL}(V)$. Once the G-action on $X \subset \mathbb{P}^n = \mathbb{P}(V)$ is specified, ρ is uniquely determined up to a character.

We define basic invariants of smooth G-quadrics $X \subset \mathbb{P}(V)$. The *rank* of X is $\dim(V)$, which has the same parity as $\dim(X)$.

Similitude class. Consider $V \otimes N$, where N is a one-dimensional representation with character ν. The resulting $(V \otimes N, q)$ with

$$q \in \mathrm{Sym}^2(V^\vee) \otimes L = \mathrm{Sym}^2((V \otimes N)^\vee) \otimes (N^2 \otimes L),$$

has character $\nu^2\lambda$.

The *similitude class* of $X = \{q = 0\}, q \in \mathrm{Sym}^2(V^\vee) \otimes L$ is the element

$$\begin{aligned}[][\lambda] \in \mathrm{cok}&\left(\mathrm{Hom}(G, \mathbb{G}_m) \xrightarrow{\cdot 2} \mathrm{Hom}(G, \mathbb{G}_m)\right) \\ &= \ker\left(\mathrm{H}^2(G, \mu_2) \longrightarrow \mathrm{H}^2(G, \mathbb{G}_m)\right),\end{aligned}$$

where λ is the character associated with L.

The class vanishes if and only if there exists a representation $\rho : G \to \mathrm{GL}(V)$ and an embedding $X \hookrightarrow \mathbb{P}(V)$ such that X is defined by a G-invariant quadratic form $q \in \mathrm{Sym}^2(V^\vee)$.

Discriminant. Write $X = \{q = 0\} \subset \mathbb{P}(V)$, for $q \in \mathrm{Sym}^2(V^\vee) \otimes L$. Taking determinants gives an isomorphism

$$\det(V)^2 \simeq L^{\dim(V)}.$$

When $\mathrm{rank}(X)$ is even, we obtain a character

$$\left(\det(V) \otimes L^{-\dim(V)/2}\right) : G \to \mu_2$$

and an invariant

$$\mathrm{disc}(X) := [\det(V) \otimes L^{-\dim(V)/2}] \in \mathrm{Hom}(G, \mu_2),$$

the *G-discriminant* of X. The G-discriminant is independent of the choice of V and L, i.e., unchanged under

$$V \mapsto W \otimes N, \quad L \mapsto M \otimes N^2.$$

Remark 5.5. When $\dim(V) = 2m$ and q is non-degenerate then the discriminant encodes the action of G on $\mathrm{OGr}(m, q)_\pm$ cf. [EKM08, §85].

Clifford invariant. Let $X = \{q = 0\} \subset \mathbb{P}(V)$ be a smooth G-quadric with $\dim(V) = 2m$ and trivial discriminant. The embeddings

$$\mathrm{OGr}(m, q)_\pm \hookrightarrow \mathbb{P}(S_\pm)$$

induce projective representations of G. Let $\alpha_\pm \in \mathrm{H}^2(G, \mathbb{G}_m)[2]$ denote their Amitsur invariants, the obstructions to lifting $S_\pm$ to linear representation of G, the *G-Clifford invariants*.

Remark 5.6. Proposition 3.1 shows that the classes $\alpha_\pm$ are related to the classical Clifford invariant of [EKM08, §14], defined as the Brauer class of $C^+(q) \simeq \mathrm{End}(S_+) \otimes \mathrm{End}(S_-)$ (see [EKM08, 13.9]).

Remark 5.7. Parimala and Srinivas [PS92] define cycle classes for Brauer algebras with involution that yield the Clifford invariant as a special case. There is even a refinement with values in $\mathrm{H}^2(G, \mu_2)$. Such refinements do not exist over $I_2(X)$ – quadratic forms of even rank and trivial discriminant – for a general base X.

Let $X \subset \mathbb{P}(V)$ be a smooth G-quadric with $\dim(V) = 2m + 1$.

Example 5.8. We give an example of a quadric threefold that is linearizable but has non-trivial Clifford invariant. Consider the dihedral group

$$\left\langle \sigma, \tau : \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^3 \right\rangle$$

with representation V

$$\sigma \mapsto \begin{pmatrix} i & 0 \\ 0 & i^3 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives a projective representation of

$$C_2 \times C_2 = \left\langle \sigma, \tau : \sigma^2 = \tau^2 = 1 \right\rangle.$$

Let C be the corresponding conic, realized in $\mathbb{P}(\mathrm{Sym}^2(V))$, with a non-trivial Amitsur invariant. Consider $\mathbb{P}(\mathrm{Sym}^2(V) \oplus \mathbf{1})$ which is birational to a quadric threefold X blown up at a fixed point p

$$\mathrm{Bl}_C(\mathbb{P}(\mathrm{Sym}^2(V) \oplus \mathbf{1})) \simeq \mathrm{Bl}_p(X).$$

We blow down the proper transform of $\mathbb{P}(\mathrm{Sym}^2(V))$ to get p. Thus $X = \{q = 0\}$ for some $C_2 \times C_2$-invariant quadratic form.

The variety $\mathrm{OGr}(2, q)$ is a Brauer-Severi threefold, birational to a $\mathbb{P}^2$-bundle over C. Thus Clifford invariants can be non-trivial for quadrics that are linearizable.

6. Invariant isotropic subspaces

Definitions and basic properties. Let $X = \{q = 0\} \subset \mathbb{P}(V)$ be a G-quadric, where V is a linear representation of G. Write L for the one-dimensional representation associated to q and

$$s : V \xrightarrow{\sim} V^\vee \otimes L$$

the associated symmetric isomorphism of G-representations. Let $W \subset V$ be a G-stable isotropic subspace.

Example 6.1. The singular locus of G-invariant quadric hypersurface $X \subset \mathbb{P}(V)$ is the projectivization of an isotropic subspace.

Definition 6.2. A G-quadric $X = \{q = 0\} \subset \mathbb{P}(V)$ is *anisotropic* if there exists no sub-representation $0 \neq W \subset V$ such that $q|W = 0$.

We consider examples for regular representations:

Example 6.3. Suppose that G is an abelian 2-group that is not 2-elementary, i.e., $2G \neq 0$. Then $V := k[G]$ is isotropic.

We may as well assume that $G = \langle g \rangle$ is cyclic of order $2^n \geq 4$ with quadratic form

$$q = x_0^2 + x_1 x_{2^n-1} + \cdots + x_{2^{n-1}-1} x_{2^{n-1}+1} + x_{2^{n-1}}^2,$$

where the x_i are characters of G. The subspace

$$W = \{x_0 = x_1 = \ldots = x_{2^{n-1}} = 0\}$$

is isotropic.

Proposition 6.4. *Retain the notation introduced above and assume that $W \subset V$ is irreducible and isotropic. Then there exists an isotropic irreducible $W' \subset V$ such that the induced pairing on $W \times W'$ is non-degenerate. Moreover $W' \simeq W^\vee \otimes L$.*

Our proof does not require that k be algebraically closed.

Proof. Consider the quotient $V^\vee \twoheadrightarrow W^\vee$ and choose a G-equivariant splitting, yielding a copy of $W^\vee \otimes L \subset V^\vee \otimes L$. Consider $W'' := s^{-1}(W^\vee \otimes L)$ and the induced L-valued bilinear form on $W \times W'' \subset V$. We already know this is isotropic on the first factor and induces the natural L-valued bilinear pairing on $W \times W''$. It remains to compute this in W''; we are done if it is isotropic.

Suppose W'' is *not* isotropic. Then, by Schur's Lemma, we would have a symmetric isomorphism

$$W'' \xrightarrow{\sim} (W'')^\vee \otimes L,$$

or equivalently, a symmetric

$$t : W^\vee \otimes L \xrightarrow{\sim} W.$$

Schur's Lemma guarantees this is the unique non-trivial homomorphism between these representations, up to scalars. Thus we have

$$W \oplus W'' \simeq W \oplus W.$$

with quadratic form

$$q(w_1, w_2) = c_1[t^{-1}(w_1) \cdot w_2 + t^{-1}(w_2) \cdot w_1] + c_2[t^{-1}(w_2) \cdot w_2],\ w_1, w_2 \in W,$$

for some nonzero constants c_1, c_2. Applying row-and-column operations, we obtain

$$W' \simeq W \hookrightarrow W \oplus W''$$

that is isotropic and pairs non-degenerately with the first factor. □

Definition 6.5. Fix a character $\lambda : G \to \mathbb{G}_m$ with associated representation L. Given a G-representation W, the *hyperbolic lattice*

$$H_W(\lambda) = W \oplus (W^\vee \otimes L),$$

with bilinear form

$$b(w_1, f_1; w_2, f_2) = \frac{1}{2}(f_1(w_2) + f_2(w_1)).$$

This is non-degenerate and the summands W and $W^\vee \otimes L$ are isotropic subspaces.

Note that any hyperbolic lattice is a direct sum of hyperbolic lattices arising from irreducible representations.

Classification results. Here we extend the analysis of [HT25, §3]. The base field k is algebraically closed.

Proposition 6.6. *Consider a smooth G-quadric*

$$X = \{q = 0\} \subset \mathbb{P}(V)$$

with associated symmetric $s : V \xrightarrow{\sim} V^\vee \otimes L$ with character $\lambda : G \to \mathbb{G}_m$. The irreducible representations $W \subset V$ are of three types:

1. *W has a symmetric homomorphism $W \xrightarrow{\sim} W^\vee \otimes L$.*
2. *W has a skew-symmetric $W \xrightarrow{\sim} W^\vee \otimes L$.*
3. *W admits no such isomorphism.*

The isotypic component $W^m \subset V$, where m is the multiplicity of W in V, in each case takes the form:

1. *$q|W^m$ is non-degenerate and isomorphic to $H_W(\lambda)^n$ if $m = 2n$ and $H_W(\lambda)^n \oplus W$ if $m = 2n + 1$;*
2. *$m = 2n$ and $q|W^{2n} \simeq H_W(\lambda)^n$;*

(3) *$W^\vee \otimes L$ also appears in V with multiplicity m and*
$$q|(W^m \oplus (W^\vee \otimes L)^m) \simeq H_W(\lambda)^m.$$
Moreover, these decompositions are unique up to automorphisms of q.

Remark 6.7. This should be understood as an equivariant analog of the Witt decomposition theorems. While the hyperbolic summands are not canonical they are determined up to automorphisms. The trichotomy of irreducbile orthogonal representations goes back to Frobenius and Schur [Isa06, p. 58].

Proof. The trichotomy – that the three cases are disjoint – follows from Schur's Lemma. The proof of Proposition 6.4 guarantees the existence of an irreducible summand pairing non-degenerately with W. In the first case, this is isomorphic to W; however, observe that $W^{\oplus 2}$ always has an isotropic copy of W. Any sum
$$(W, r) \oplus (W, cr), \quad c \in k, r = q|W,$$
admits an isotropic subspace, i.e., the image of $w \mapsto (\sqrt{-c} \cdot w, w)$. For the second case, W is necessarily isotropic and the $q|W^m$ is an $m \times m$ skew-symmetric matrix (a_{ij}), where the a_{ij} are the skew pairings $\wedge^2 W \to L$. For the parity of m, observe that skew-symmetric matrices always have even rank. The reduction theory of symplectic forms implies that the matrix is equivalent to
$$\begin{pmatrix} 0 & Ia \\ -Ia & 0 \end{pmatrix}$$
where the entries are an $n \times n$ identity matrix I times a fixed $a : \wedge^2 W \xrightarrow{\sim} L$. This is symmetric once we take into account that the isomorphisms are all skew! In the third case, W is also isotropic and the claim follows directly from Proposition 6.4. □

We obtain the following consequences:

Corollary 6.8. *Let $X = \{q = 0\} \subset \mathbb{P}(V)$ be a smooth G-quadric. We may express*
$$V = V_a \oplus_\perp V_h$$
where $q|V_a$ is anisotropic and $q|V_h$ is hyperbolic. Write
$$X_a := \{q|V_a = 0\} = X \cap \mathbb{P}(V_a)$$
for the anisotropic section of X, which is determined up to isomorphism.

Our next result extends [HT25, Prop. 3.7]:

Corollary 6.9. *Fix a character $\lambda : G \to \mathbb{G}_m$ with associated representation L. Each anisotropic $X = \{q = 0\} \subset \mathbb{P}(V)$ with character λ expressed uniquely as a sum*

$$(V, q) \simeq \oplus_{i=1}^{n} (W_i, r_i)$$

where the W_i are distinct irreducible representations of G and r_i encodes a symmetric $W_i \xrightarrow{\sim} W_i^\vee \otimes L$.

Example 6.10 (Regular representations). Given a finite group G, let $V = k[G]$ be the regular representations with coordinates

$$\{x_g, g \in G\}, \quad x_g(g') = \delta_{g,g'}.$$

Consider the G-invariant non-degenerate quadratic form

$$q = \sum_{g \in G} x_g^2.$$

Its discriminant is the sign character for the permutation representation of G on itself.

Now let W be an irreducible representation of G and consider the isotypic component

$$W^{\dim(W)} \subset V.$$

We apply Proposition 6.6 and its corollaries: The anisotropic part $V_a \subset V$ is the sum of irreducible self-dual representations of odd dimension, each with multiplicity one. When G is a 2-group – so all the irreducible representations satisfy $\dim(W) = 2^n$ – V_a is the direct sum of the characters $G \to \mu_2$.

Birational implications. We recall birational properties of quadrics with invariant isotropic subspaces, following [HT25, §3] and especially Theorem 3.5.

Remark 6.11. Suppose that $X \subset \mathbb{P}(V)$ is a smooth G-quadric of positive dimension.

- If X has a fixed point then it is linearizable;
- If X has an invariant isotropic subspace then the G-action on X is stably linearizable.

More precisely, suppose an invariant isotropic subspace W yields

$$V = H_W(\lambda) \oplus H_W(\lambda)^\perp = W \oplus (W^\vee \otimes L) \oplus H_W(\lambda)^\perp$$

via Proposition 6.4 and Definition 6.5. If the G-action on

$$\mathbb{P}(W^\vee \oplus H_W(\lambda)^\perp)$$

is generically free then X is linearizable.

Splitting and varieties of linear subspaces. Let V be a representation of a finite group G, L a character of G, and $X = \{q = 0\} \subset \mathbb{P}(V)$ a smooth G-quadric arising from a symmetric $s : V \xrightarrow{\sim} V^\vee \otimes L$.

Let $\mathrm{OGr}(\ell, q)$ denote the variety of isotropic subspaces of dimension ℓ for q, with the induced G-action. Remark 3.4 yields:

Proposition 6.12. *Suppose that* $\dim(V) = 2m$ *with trivial discriminant, so that* $\mathrm{OGr}(m, q)_\pm$ *both have* G*-actions. Then they are equivariantly stably birational.*

This is weaker than Proposition 3.3 but all we can expect in the equivariant context:

Example 6.13. Let $G = C_5$ act on $\mathbb{P}^1_{u,v}$ and $\mathbb{P}^1_{x,y}$

$$[u, v] \mapsto [\zeta u, \zeta^4 v]$$
$$[x, y] \mapsto [\zeta^2 x, \zeta^3 y]$$

where ζ is a primitive fifth root of unity. These are not isomorphic as G-varieties. Consider the product

$$X = \mathbb{P}^1_{u,v} \times \mathbb{P}^1_{x,y} \subset \mathbb{P}(V)$$

where V is the tensor product of the two-dimensional representations above, with weights $\{\zeta^3, \zeta^4, \zeta, \zeta^2\}$. The variety of lines

$$\mathrm{OGr}(2, q) = \mathbb{P}^1_{u,v} \sqcup \mathbb{P}^1_{x,y}$$

hence the $\mathrm{OGr}(2, q)_\pm$ are not isomorphic C_5-varieties. They are stably birational by the No Name Lemma.

Our next result generalizes Corollary 3.6:

Proposition 6.14. *Let* $X = \{q = 0\} \subset \mathbb{P}(V)$ *be a* G*-quadric and assume* G *acts generically freely on* $\mathrm{OGr}(\ell, q)$*. For each* $\ell' < \ell$*,* $\mathrm{OGr}(\ell', q) \times \mathrm{OGr}(\ell, q)$ *is stably linearizable over* $\mathrm{OGr}(\ell, q)$

Our reasoning is similar to [JS25, Sec. 2.2].

Proof. Let $\mathcal{S} \to \mathrm{OGr}(\ell, q)$ denote the universal isotropic sub-bundle and consider ℓ'-dimensional subspaces of the quotient

$$\gamma : \mathrm{Gr}(\ell', V \otimes \mathcal{O}_{\mathrm{OGr}(\ell,q)}/\mathcal{S}) \to \mathrm{OGr}(\ell, q),$$

with universal bundle $\mathcal{L}'$. We obtain an associated rank-$(\ell' + \ell)$ extension

$$0 \to \gamma^*\mathcal{S} \to \mathcal{E} \to \mathcal{L}' \to 0$$

contained in the trivial bundle with fiber V. Let $Q = q|E$ a quadratic form degenerate along $\mathcal{K} := \gamma^*\mathcal{S} \cap (\mathcal{L}')^\perp$, a rank-$(\ell - \ell')$ subbundle. Thus we have a non-degenerate quadratic form Q' on $\mathcal{E}/\mathcal{K}$ – of rank

$2\ell'$ – with a tautological maximal isotropic subspace $\gamma^*\mathcal{S}/\mathcal{K}$. Thus the space of all such maximal isotropic subspaces is stably linearizable over $\mathrm{Gr}(\ell', V \otimes \mathcal{O}/\mathcal{S})$ by Proposition 3.2 (see also the proof of Proposition 3.5). Fixing an ℓ'-dimensional isotropic subspace over Q', the ℓ'-dimensional isotropic subspaces of Q mapping onto that subspace are parametrized by $\mathrm{Gr}(\ell', \ell)$.

Now $\mathrm{Gr}(\ell', q) \times \mathrm{Gr}(\ell, q)$ can be described – equivariant birationally over $\mathrm{Gr}(\ell, q)$ – as a tower of bundles

$$\begin{tikzcd} \mathrm{OGr}(\ell', q) \times \mathrm{OGr}(\ell, q) \arrow[rr, dashed, "\sim"] \arrow[dr] & & \mathcal{A} \arrow[dl] \\ & \mathrm{OGr}(\ell, q) & \end{tikzcd}$$

where $\mathcal{A} \to \mathrm{OGr}(\ell, q)$ factors as follows: a $\mathrm{Gr}(\ell', \ell)$ bundle, over a split $\mathrm{OGr}(\ell', 2\ell')$ bundle, over a $\mathrm{Gr}(\ell', \dim(V) - \ell)$ bundle. It follows that $\mathrm{OGr}(\ell', q) \times \mathrm{OGr}(\ell, q)$ is stably linearizable over $\mathrm{OGr}(\ell, q)$. □

Remark 6.15. Here is an alternative proof: After stabilization, an ℓ-dimensional subspace guarantees an ℓ'-dimensional subspace for $\ell' < \ell$. Over fields, rationality of $\mathrm{OGr}(\ell', q)$ follows from Proposition 3.5. The equivariant statement follows then from Proposition 2.1.

The same reasoning yields the following equivariant version of Proposition 3.5:

Corollary 6.16. *Suppose that $X = \{q = 0\} \subset \mathbb{P}(V)$ is a G-quadric and* $\mathrm{OGr}(\ell, q)$ *admits a G-fixed point or even an equivariant morphism from a stably linearizable G-variety. Then* $\mathrm{OGr}(\ell', q)$ *is stably linearizable for each $\ell' \leq \ell$.*

Relations between quadrics. We note one last classical fact:

Proposition 6.17. *Let X and Y be smooth G-quadrics of positive dimensions d and e. Assume there exists a G-equivariant rational map $X \dashrightarrow Y$ and the action on X is generically free. Then there exists a G-equivariant birational map*

$$Y \times X \overset{\sim}{\dashrightarrow} \mathbb{P}^e \times X.$$

Suppose the actions on X and Y are both generically free and we have equivariant rational maps in both directions

$$X \dashrightarrow Y, \quad Y \dashrightarrow X. \tag{6.1}$$

Then we obtain a G-equivariant stable birational equivalence

$$Y \times \mathbb{P}^d \overset{\sim}{\dashrightarrow} \mathbb{P}^e \times X.$$

This conclusion holds also under a weakening of (6.1), i.e., that there are rational maps

$$X \times \mathbb{P}^n \dashrightarrow Y, \quad Y \times \mathbb{P}^m \dashrightarrow X.$$

Proof. The rational map from X to Y means that projection

$$Y \times X \to X$$

has a rational section. Now a quadric bundle with a section is birational over X to a projective bundle on X. This bundle is birational to the trivial bundle $\mathbb{P}^e \times X$ by the No-Name Lemma, as the G-action on X is generically free. Applying this reasoning twice gives the stable birational equivalence. Repeating this argument using the rational maps from the products gives

$$Y \times \mathbb{P}^d \times \mathbb{P}^m \times \mathbb{P}^n \overset{\sim}{\dashrightarrow} X \times \mathbb{P}^e \times \mathbb{P}^m \times \mathbb{P}^n$$

and our final assertion. □

Proposition 6.18. *Let $X \subset \mathbb{P}(V)$ be a smooth G-quadric and $Y \subset X$ is a G-invariant smooth linear section of codimension c. Assume that*

- *the G-action on Y is generically free;*
- *for some $\ell > c$, there is an equivariant rational map $X \dashrightarrow \mathrm{OGr}(\ell, q)$, or even an equivariant $X \times \mathbb{P}^n \dashrightarrow \mathrm{OGr}(\ell, q)$ for some n.*

Then X and Y are G-equivariantly stably birational.

Proof. As the action is generically free on Y, the same is true for X. We follow the argument for Proposition 6.17; clearly, $X \times Y$ is stably birational to Y. A dimension count shows that projective subspaces parametrized by $\mathrm{OGr}(\ell, q)$ intersect Y non-trivially, in isotropic subspaces. Remark 6.11 gives that $X \times Y$ is equivariantly stably birational to X as well. Thus X and Y are stably birational to each other. □

Remark 6.19. Proposition 6.18 only applies when $2\dim(Y) \geq \dim(X)$. Otherwise, we cannot find the required isotropic subspaces in X!

Reductions to 2-Sylow subgroups. Results of Duncan and Reichstein [DR15, Th. 10.2] (see also [HT25, Sec. 5.3]) imply that a G-quadric X is stably linearizable if and only if it is stably linearizable for any 2-Sylow subgroup $G_2 \subset G$. Similar results apply for stable birational equivalence:

Proposition 6.20. *Let X and Y be smooth quadrics with generically free actions of a finite group G. These are stably birational over G iff they are stably birational over a 2-Sylow subgroup.*

In analyzing equivariant stable birational equivalence of quadrics, we need only consider actions by 2-groups.

Proof. The forward implication is immediate; we focus on the reverse direction.

Consider the stable birational equivalence over G_2

$$X \times V \overset{\sim}{\dashrightarrow} Y \times W.$$

This yields G_2-equivariant rational maps

$$X \dashrightarrow Y, \quad Y \dashrightarrow X.$$

The projections

$$X \times Y \to X, \quad X \times Y \to Y$$

therefore have G_2-equivariant rational sections Σ_X and Σ_Y. Let $G\cdot\Sigma_X$ and $G\cdot\Sigma_Y$ denote the closures of the G-orbits of these rational sections. These have odd degree dividing $[G : G_2]$ over X and Y respectively.

By Proposition 2.1, we are reduced to proving the following statement: Given an extension K/k and a G-torsor T over K, the twists ${}^TX$ and ${}^TY$ are stably birational over K.

Consider ${}^T(X \times Y)$, which is isomorphic to ${}^TX \times {}^TY$, by [DR15, Cor. 3.4(a)]. We also have ${}^T(G\cdot\Sigma_X)$ and ${}^T(G\cdot\Sigma_Y)$, which are still odd-degree multisections over ${}^TX$ and ${}^TY$ respectively. Springer's Theorem implies that the projections of ${}^T(X \times Y)$ to ${}^TX$ and ${}^TY$ have sections defined over K. Thus ${}^T(X \times Y)$ is stably birational to both ${}^TX$ and ${}^TY$ over K, and ${}^TX$ and ${}^TY$ are stably birational to each other (see the proof of Prop. 6.17). □

7. Equivariant Witt rings

Fix a character $\lambda : G \to \mathbb{G}_m$. Consider pairs (V, q), where V is a G-representation and $q \in \mathrm{Sym}^2(V^\vee) \otimes L$ is a G-quadratic form with character $\lambda_q = \lambda$. A *homomorphism* of such pairs (V_1, q_1) and (V_2, q_2) is a homomorphism of representations $\phi : V_1 \to V_2$ such that $q_2 \circ \phi = q_1$. The *sum* of (V_1, q_1) and (V_2, q_2) with $\lambda_{q_1} = \lambda_{q_2} = \lambda$ is defined

$$(V_1, q_1) \oplus (V_2, q_2) = (V_1 \oplus V_2, q_1 \oplus q_2);$$

we have $\lambda_{q_1 \oplus q_2} = \lambda$.

Definition 7.1. Let G be a finite group and $\lambda : G \to \mathbb{G}_m$ a character. The *Witt group* $\mathrm{Witt}(G, \lambda)$ is the semigroup of non-degenerate (V, q) with $\lambda_q = \lambda$ with sum operation as defined above, modulo the hyperbolic forms. By convention, $\mathrm{Witt}(G, \lambda) = 0$ if there are no such pairs.

Corollaries 6.8 and 6.9 guarantee the group structure is well-defined. Concretely, it is the $\mathbb{F}_2$ vector space generated by isomorphism classes of irreducible representations W with a symmetric

$$W \xrightarrow{\sim} W^\vee \otimes L$$

where L is the one-dimensional representation associated with λ.

Suppose that (V, q) and (W, r) are quadratic forms with associated symmetric homomorphisms

$$s : V \longrightarrow V^\vee \otimes L, \quad t : W \longrightarrow W^\vee \otimes M.$$

Tensoring gives a symmetric

$$s \otimes t : V \otimes W \longrightarrow (V \otimes W)^\vee \otimes (L \otimes M),$$

which is an isomorphism if s and t are isomorphisms. We write

$$(V, q) \otimes (W, r) = (V \otimes W, q \otimes r)$$

where $q \otimes r$ is the quadratic form associated with $s \otimes t$, with values in $L \otimes M$. Tensoring with a hyperbolic form yields a hyperbolic form, thus tensor product yields a well-defined

$$\mathrm{Witt}(G, \lambda) \times \mathrm{Witt}(G, \mu) \longrightarrow \mathrm{Witt}(G, \lambda\mu),$$

where $\lambda, \mu : G \to \mathbb{G}_m$ are the characters associated with L and M. Let $\mathbf{1} \in \mathrm{Witt}(G, 1)$ denote the non-degenerate quadratic form on the trivial irreducible representation, which serves as the unity.

Definition 7.2. The *Witt ring* is the $\mathrm{Hom}(G, \mathbb{G}_m)$-graded algebra

$$\mathrm{Witt}(G) = \oplus_\lambda \mathrm{Witt}(G, \lambda) \tag{7.1}$$

We are primarily interested in the projective geometry of quadrics. Consider a character

$$\nu : G \to \mathbb{G}_m$$

with associated one-dimensional representation N. Given a G-equivariant (V, q) with character λ, tensoring gives $(V \otimes N, q)$ with character $\lambda\nu^2$. This is compatible with direct sums and takes hyperbolic forms to hyperbolic forms. Let $\mathrm{PWitt}(G)$ denote the quotient of $\mathrm{Witt}(G)$ by this action of $\mathrm{Hom}(G, \mathbb{G}_m)$. Note that

$$\begin{aligned} &\mathrm{PWitt}(G) = \oplus_{[\lambda]} \mathrm{PWitt}(G, [\lambda]) \\ &[\lambda] \in \mathrm{cok}\left(\mathrm{Hom}(G, \mathbb{G}_m) \xrightarrow{\cdot 2} \mathrm{Hom}(G, \mathbb{G}_m)\right) \end{aligned}$$

i.e., with terms indexed by similitude class. Each summand is the quotient of $\mathrm{Witt}(G, \lambda)$ under the action of $\mathrm{Hom}(G, \mu_2)$, interpreted one-dimensional representations with the canonical invariant quadratic form.

Remark 7.3. The *character grading* (7.1) is distinct from filtrations associated with the *augmentation ideal* of virtual quadratic forms of rank zero.

Cyclic groups. Let $G = \langle g \rangle$ be a cyclic group of order 2^n with $n \geq 1$; fix a primitive 2^n-th root of unity ζ. For $i \in \mathbb{Z}/2^n\mathbb{Z}$, let L_i denote the one-dimensional representation with coordinate x_i, where g acts via ζ^i.

The quadric in the regular representation

$$X = \{x_0^2 + x_1 x_{2^n-1} + \cdots + x_{2^{n-1}}^2 = 0\} \subset \mathbb{P}(\oplus_{i=0}^{2^n-1} L_i)$$

is linearizable due to the existence of fixed points, e.g., $[0,1,0,\ldots,0]$. The discriminant is non-trivial so X has no G-invariant maximal isotropic subspace. It does have isotropic subspaces of dimension $2^{n-1}-1$, e.g.,

$$x_0 = x_1 = \cdots = x_{2^{n-1}} = 0.$$

Thus all G-invariant linear sections $Y \subset X$ with $\dim(Y) \geq 2^{n-1}$ are equivariantly stably birational to X, by Proposition 6.18, hence stably linearizable.

Invariant quadratic forms cover the degree-zero piece of the grading (7.1). The covariant quadratic forms with

$$\lambda : G \to \mathbb{G}_m, \quad \lambda(g) = \zeta,$$

may be written

$$\sum_{i=0}^{2n-1-1} b_i x_i x_{1-i},$$

all of which are hyperbolic.

The Witt ring $\mathrm{Witt}(G)$ only has terms of degree zero. Setting

$$\mathbf{1} = (L_0, x_0^2), \quad w = (L_{2^{n-1}}, x_{2^{n-1}}^2)$$

$$\mathrm{Witt}(G) = \mathbb{F}_2[w]/\left\langle w^2 - \mathbf{1} \right\rangle.$$

Note that the Witt ring does not coincide with the $\mathbb{F}_2$-cohomology of G. Recall that [CTVEZ03, Prop. 4.5.1]:

$$\mathrm{H}^*(G, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2[z], \deg(z) = 1 & \text{if } |G| = 2 \\ \mathbb{F}_2[\eta, z]/\left\langle \eta^2 \right\rangle, \deg(\eta) = 1, \deg(z) = 2 & \text{if } |G| = 2^n > 2. \end{cases}$$

2-elementary groups. Let $G = \langle g_1, \ldots, g_n \rangle$ be the group C_2^n. Write $L_{i_1\ldots i_n}$, $i_j = 0,1$, for the one-dimensional representation with coordinate $x_{i_1\ldots i_n}$, such that g_j acts via $(-1)^{i_j}$. Invariant quadratic forms take the form

$$\sum_{I=(i_1\ldots i_n)} a_I x_I^2,$$

where the sum is over a subset of characters. These are anisotropic provided the form is non-degenerate, i.e., the coefficients a_I are all non-zero – see Section 8 for discussion. Fixing a non-trivial

$$\lambda \in \mathrm{Hom}(G, \mu_2) \simeq \mathrm{Hom}(G, \mathbb{G}_m),$$

covariant forms of degree λ may be written

$$\sum_I b_I x_\lambda x_{\lambda - I}.$$

These forms contain no squares, are hyperbolic, and admit fixed points, e.g., where all but one of the variables vanish. Setting

$$w_1 = (L_{10\ldots0}, x^2_{10\ldots0}),\ w_2 = (L_{010\ldots0}, x^2_{010\ldots0}),\ \ldots,\ w_n = (L_{0\ldots01}, x^2_{0\ldots01}),$$

we find

$$\mathrm{Witt}(G) = \mathbb{F}_2[w_1, \ldots, w_n] / \left\langle w_1^2 - \mathbf{1}, \ldots, w_n^2 - \mathbf{1} \right\rangle.$$

The augmentation ideal is

$$\langle w_1 - \mathbf{1}, \ldots, w_n - \mathbf{1} \rangle.$$

Again, the Witt ring is not isomorphic to

$$\mathrm{H}^*(G, \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, \ldots, x_n].$$

The ring $\mathrm{Witt}(G)$ is an Artinian local $\mathbb{F}_2$-algebra; it is a complete intersection and thus Gorenstein, with socle generated by

$$\prod_{j=1}^n (w_j - \mathbf{1}).$$

This element arises from the quadratic form on the regular representation of G

$$q = \sum_{I=(i_1,\ldots,i_n)} (-1)^{|I|} x_I^2, \quad |I| = i_1 + \cdots + i_n.$$

Stable Witt rings. From the stable birational perspective, demanding a G-invariant isotropic subspace is overly restrictive. Suppose that $X = \{q = 0\}$ is a smooth irreducible quadric hypersurface with a generically-free action of G. Recall [DR15, §10] that the following conditions are equivalent (see also Proposition 2.1):

- X is stably linearizable;
- $\mathrm{OGr}(\ell, q)$ has a stably linearizable component for some $\ell > 0$;
- for each G-torsor T over a field K, the twist ${}^T X$ has K-rational points and is K-rational.

To form the *stable* Witt ring we should dispose of equivariant quadratic forms satisfying any of these conditions.

Definition 7.4. The *stable Witt ring*

$$\mathrm{SWitt}(G) = \oplus_\lambda \mathrm{SWitt}(G,\lambda)$$

is the quotient of $\mathrm{Witt}(G)$ obtained by setting the following quadratic forms equal to zero: Non-degenerate (V,q), with $\dim(V) = 2m$ and trivial discriminant, such that $\mathrm{OGr}(m,q)_\pm$ is stably linearizable.

Proposition 7.5. *The stable Witt ring is well-defined. Each element of* $\mathrm{SWitt}_\lambda(G)$ *may be represented by a covariant anisotropic quadratic form over the function field of a representation of* G*.*

Proof. The two components of the maximal isotropic Grassmannian are equivariantly stably birational by Proposition 6.12, so the choice of component does not affect the definition.

The existence of G-invariant maximal isotropic subspaces implies stable linearizability of the components of maximal isotropic subspaces by Corollary 6.16. Thus elements trivial in $\mathrm{Witt}(G)$ are evidently trivial in $\mathrm{SWitt}(G)$.

We check that the stipulated elements form an ideal. Suppose that (V,q) and (W,r) represent elements of $\mathrm{Witt}(G,\lambda)$, of dimensions $2m$ and $2n$, such that $\mathrm{OGr}(m,q)_+$ and $\mathrm{OGr}(n,r)_+$ are stably linearizable. Taking the direct sum $(V\oplus W, q\oplus r)$, we get a G-equivariant

$$\mathrm{OGr}(m,q)\times \mathrm{OGr}(n,r) \to \mathrm{OGr}(m+n, q\oplus r)$$

hence the components of the target are also stably linearizable by Corollary 6.16. Now assume that (V',q') represents an arbitrary element of $\mathrm{Witt}(G,\mu)$ of dimension n'. We have an equivariant morphism

$$\mathrm{OGr}(m,q) \to \mathrm{OGr}(mn', q\otimes q')$$

taking the isotropic subspace Λ to $\Lambda\otimes V'$, which is isotropic by the dimension of $q\otimes q'$. Again, Corollary 6.16 gives the desired conclusion.

For the last statement, take (V,q) and choose ℓ maximal such that $\mathrm{OGr}(\ell,q)$ admits an equivariant rational map from a linear representation of G. The equivariant version of Witt's Lemma (Proposition 6.4), applied over that representation, gives the desired quadratic form: the orthogonal complement of the ℓ-dimensional isotropic subspace over a stably linearizable base. Proposition 6.14 guarantees this is anisotropic. □

Example 7.6. The simplest situation where these notions differ is for groups G with representations

$$\rho : G \to \mathrm{GL}_2$$

injecting to PGL_2 but lacking fixed points. The symmetric group $G = \mathfrak{S}_3$ is an example. Let $\mathbf{1}$ denote the trivial representation, L

the sign representation, and V_2 the two-dimensional irreducible representation; all have non-degenerate $\mathfrak{S}_3$-invariant quadratic forms. Fix corresponding generators

$$\mathbf{1}, w, v \in \mathrm{Witt}(\mathfrak{S}_3)$$

so that

$$\mathrm{Witt}(\mathfrak{S}_3) = \mathbb{F}_2[w,v] / \left\langle w^2 - \mathbf{1}, wv - v, v^2 - v - w - \mathbf{1} \right\rangle .$$

Consider the non-degenerate quadratic form

$$(V, q), \quad V = V_2 \otimes V_2 \simeq V_2 \oplus \mathbf{1} \oplus L,$$

where q is the tensor square of the standard invariant form on V_2. This is non-trivial in $\mathrm{Witt}(\mathfrak{S}_3)$: The locus

$$X = \{q = 0\} \simeq \mathbb{P}(V_2) \times \mathbb{P}(V_2)$$

which lacks invariant isotropic subspaces. On the other hand, $\mathrm{OGr}(2, q)_{\pm} \simeq \mathbb{P}(V_2)$ which is linearizable; thus (V, q) is trivial in $\mathrm{SWitt}(\mathfrak{S}_3)$.

Another example in this vein may be found after Remark 9.5.

Notes on the literature. Since Milnor's work [Mil70], cohomological invariants have shaped our conceptual understanding of quadratic form over fields. Invariants of G-equivariant quadratic forms on *real* representations, taking values in $\mathrm{H}^*(G, \mathbb{F}_2)$, have been studied as well. We refer the reader to [GKT89] for axiomatic approaches to Stiefel-Whitney classes and Kahn [Kah83b, Kah83a, Kah91] for computations for regular representations of 2-groups and discussion of filtrations in terms of these classes. Another approach to equivariant Stiefel-Whitney classes, via intersection theory and the Grothendieck-Riemann-Roch, may be found in [Koz84, Koz89]. Guillot [Gui10] has computed invariants in numerous examples, using the comprehensive tables of group cohomology with $\mathbb{F}_2$ coefficients in [CTVEZ03]. Totaro [Tot14] considers cycle-class maps from the perspective of equivariant Chow groups.

Based on our computations above, there is not a direct dictionary between Witt rings and group cohomology, as one might expect from the theory over fields.

8. Equivariant stably Pfister forms

Twisting construction. Let (V, q) be a non-degenerate G-quadratic form with $\dim(V) = n$; write $X = \{q = 0\} \subset \mathbb{P}(V)$ for the associated projective quadric hypersurface and $\mathrm{Cone}(X) \subset V$ for the cone over X. Choose an irreducible G-variety U such that G acts freely on an open dense $U_0 \subset U$ with field of invariants $F = k(U)^G$. We have a G-invariant hypersurface

$$\mathrm{Cone}(X) \times U_0 \subset V \times U_0.$$

Taking quotients yields an inclusion of twists

$$^{U_0}(\mathrm{Cone}(X) \times U_0) := G\backslash(\mathrm{Cone}(X) \times U_0) \subset G\backslash(V \times U_0) =: {}^{U_0}V.$$

Restricting to the generic point,

$$V' := ({}^{U_0}V)|_F \simeq \mathbb{A}^n_F,$$

by Hilbert Theorem 90 (or the No-Name Lemma), and

$$Z := {}^{U_0}(\mathrm{Cone}(X) \times U_0)|_F$$

is a non-degenerate affine quadric hypersurface over F. Write $Z = \{q' = 0\}$, where q' is a non-degenerate quadratic form over F, well-defined up to a scalar. When q is G invariant, we may take $q' = q$, re-interpreted as a quadratic form on $\mathbb{A}^n_F$.

In summary, twisting takes a G-quadratic form to a quadratic form over a G-invariant field F, determined up to scalar; there is a canonical choice provided the quadratic form is invariant.

Compatibility with purely transcendental extensions. From now on, U is a linear representation of G acting freely on a nonempty $U_0 \subset U$. Recall that *general Pfister forms* are non-zero multiples of Pfister forms [EKM08, §6A].

Proposition 8.1. *Let (V, q) be a non-degenerate G-quadratic form. Suppose U_1 and U_2 are linear representations of G that are generically free, with function fields of invariants F_1 and F_2. Write (V_1', q_1') and (V_2', q_2') for the corresponding non-degenerate quadratic forms over F_1 and F_2.*

- *(V_1', q_1') has an isotropic r-dimensional linear subspace over F_1 iff (V_2', q_2') has an isotropic r-dimensional linear subspace over F_2.*
- *(V_1', q_1') is general Pfister over F_1 iff (V_2', q_2') is general Pfister over F_2.*

Our strategy is similar to the approach in [KT26, §2].

Proof. The representation $U_1 \oplus U_2$ is also generically free; choose open $U_{12;0} \subset U_{12}$ mapping dominantly to $U_{1;0} \subset U_1$ and $U_{2;0} \subset U_2$. The group G acts freely on all these open subsets. By the No-Name Lemma,

$$G\backslash U_{1;0} \longleftarrow G\backslash U_{12;0} \longrightarrow G\backslash U_{2;0}$$

are birational to affine-space bundles. Thus the invariant field F_{12} is purely transcendental over F_1 and F_2.

The first assertion follows because the existence of isotropic subspaces does not change under purely transcendental field extension [EKM08, 1.21,7.15].

For the second assertion, we use the results of [EKM08, §23] especially Cor. 23.4: General Pfister forms fall into two disjoint classes:

- hyperbolic forms in 2^s variables;
- anisotropic forms in $2m > 0$ variables admitting a rational map
$$X = \{q = 0\} \dashrightarrow \mathrm{OGr}(m, q)$$
from the projective quadric to the variety of maximal isotropic subspaces.

In the former case, the first assertion gives our result. We therefore focus on the anisotropic case. Proposition 3.2 implies that the following conditions are equivalent, for $m > 1$ over a field k:

- the existence of a rational map
$$X \dashrightarrow \mathrm{OGr}(m, q);$$
- the existence of a rational point of $\mathrm{OGr}(m, q)$ over $k(X)$;
- X is stably birational to $\mathrm{OGr}(m, q)_\pm$.

When $m = 1$, $X = \mathrm{OGr}(1, q)$ and these conditions are automatic. Note that the choice of $\mathrm{OGr}(m, q)_\pm$ is immaterial by Proposition 3.3.

Since passing to a purely transcendental extension does not change the dimensions of isotropic subspaces, all three conditions are stable under such extensions. □

Stably Pfister equivariant quadratic forms.

Definition 8.2. The G-quadratic form (V, q) is *stably Pfister* if (V', q') is a general Pfister form over $F = k(U)^G$ for some generically-free linear representation U of G.

Abusing terminology, we say that $X = \{q = 0\} \subset \mathbb{P}(V)$ is stably Pfister as well. Proposition 8.1 means this is independent of the choice of representation U.

Recall from Proposition 2.1 – see [DR15, §10] – that X is stably linearizable if and only if every twist admits a rational point. Thus the following conditions are equivalent for $m > 1$:

(1) X and $\mathrm{OGr}(m, q)_\pm$ are G-equivariantly stably birational;
(2) for every G-torsor T defined over any extension K/k, the twists
$${}^T X, \quad {}^T\mathrm{OGr}(m, q)_\pm$$

are stably birational over K;

(3) there exists a linear representation U of G for which G acts freely on a dense open subset $U_0 \subset U$, with field of invariants $F = k(U)^G$, such that the twists

$$^{U_0}X, \quad {}^{U_0}\mathrm{OGr}(m,q)_\pm$$

are stably birational over F.

One motivation is to emulate the theory of *Pfister neighbors*: Given an anisotropic Pfister form q in 2^n variables over k and a subform Q in at least $2^{n-1}+1$ variables, then the quadric hypersurfaces

$$\{Q=0\} =: Y \subset X := \{q=0\}$$

are stably birational over k [EKM08, 23.10]. We obtain many stable birational equivalences of quadrics with non-trivial birational geometry. Proposition 6.18 is the corresponding equivariant statement. Our notion of "stably Pfister" gives stable birational equivalence among neighbors, provided the actions are generically free.

We record some basic properties:

Proposition 8.3. *Let (V,q) be a non-degenerate G-quadratic form. If (V,q) is stably Pfister for G then it is stably Pfister for subgroups of G.*

Suppose that (V,q) and (V',q') are non-degenerate quadratic forms for groups G and G'. Assume that

$$X = \{q=0\} \subset \mathbb{P}(V), \quad X' = \{q'=0\} \subset \mathbb{P}(V')$$

are stably Pfister for G and G', respectively. Then

$$X \otimes X' = \{q \otimes q' = 0\} \subset \mathbb{P}(V \otimes V')$$

is stably Pfister for $G \times G'$.

Recall the Pfister condition is compatible with field extensions and tensor products. Indeed, this follows from equivalences supporting Definition 8.2: An equivariant form is stably Pfister iff all its twists are Pfister. However, the definition of Pfister forms over fields [EKM08, §4.B] makes clear that tensor products of Pfister forms are also Pfister.

Examples.

2-elementary groups. Let $G = C_2^n$ and consider the natural diagonal form on the regular representation

$$q = \sum_{I=(i_1 \ldots i_n)} (-1)^{|I|} x_I^2.$$

For example, when $n=3$ we have

$$q = x_{000}^2 - x_{100}^2 - x_{010}^2 - x_{001}^2 + x_{110}^2 + x_{101}^2 + x_{011}^2 - x_{111}^2.$$

Let G act on $\mathbb{A}^n_{t_1,\dots,t_n}$, where the ith factor of C_2^n acts via ± 1 on t_i and trivially on the other coordinates; write

$$T := \{t_1 \cdots t_n \neq 0\} \subset \mathbb{A}^n$$

for the locus where G acts freely. Set $a_i = t_i^2$ so we may write

$$^T q = \sum_I a^I x_I^2,$$

e.g., for $n = 3$

$$x_{000}^2 - a_1 x_{100}^2 - a_2 x_{010}^2 - a_3 x_{001}^2 + a_1 a_2 x_{110}^2 + a_1 a_3 x_{101}^2 + a_2 a_3 x_{011}^2 - a_1 a_2 a_3 x_{111}^2.$$

This is a generic Pfister form over the field $F = k(a_1, \dots, a_n)$. Thus $X = \{q = 0\}$ is stably Pfister as a G-equivariant quadric hypersurface.

These forms are not stably linearizable – anisotropic over the function field: Equivariantly, this follows because $X = \{q = 0\}$ has no fixed points under G, see, e.g., [RY00, §5 and Appendix] relating fixed points and stable linearizability for abelian groups. Over function fields this is classical; one reference is the theory of normic forms [Lan52, p. 377].

Proposition 6.18 immediately gives

Proposition 8.4. *Consider a quadric*

$$Y_S = \{\sum_{I \in S} (-1)^{|I|} x_I^2 = 0\} \subset \mathbb{P}^{|S|-1}$$

where $S \subset \{0,1\}^n$ satisfies $|S| > 2^{n-1}$. Then Y_S is G-stably birational to X.

Again the G-actions on X and Y_S lack fixed points so they are not stably linearizable.

Pfister surfaces. Let (V, q) be a G-invariant quadratic form of dimension $\dim(V) = 4$; assume that the discriminant vanishes and the resulting action on $X = \{q = 0\}$ is generically free.

Then $X \simeq R_+ \times R_-$ where R_+ and R_- are conics, associated with projective representations

$$\rho_\pm : G \to \mathrm{PGL}_2$$

whose Amitsur invariants α_+ and α_- sum to zero. Since these are two-torsion, they are equal; the resulting class $\alpha \in \mathrm{H}^2(G, \mathbb{G}_m)[2]$ is the *Clifford invariant* of X under G. The projections

$$\pi_\pm : X \to R_\pm$$

guarantee that X is stably Pfister – provided that the surface is not stably linearizable.

Proposition 8.5. *The G-surface X is stably linearizable iff $\alpha = 0$.*

Proof. If $\alpha = 0$ then X is a product of linearizable $\mathbb{P}^1$'s, hence is stably linearizable. When $\alpha \neq 0$ then X cannot be stably linearizable; it dominates a variety with non-trivial Amitsur invariant. (See [BCDP23, Prop. A7] and [HT23, §7] for context and detail.) □

This yields examples of G-surfaces that are stably Pfister where G is non-abelian; however, they are tightly tied to 2-elementary examples:

Example 8.6. The representation of the dihedral group of order eight

$$(s,t) \mapsto (t,s), \quad (s,t) \mapsto (is, i^3 t)$$

gives a projective representation

$$\rho : C_2 \times C_2 \to \mathrm{PGL}_2.$$

Its tensor square $\rho^{\otimes 2}$:

$$(s_+s_-, s_+t_-, t_+s_-, t_+t_-) \mapsto (t_+t_-, t_+s_-, s_+t_-, s_+s_-)$$
$$(s_+s_-, s_+t_-, t_+s_-, t_+t_-) \mapsto (-s_+s_-, s_+t_-, t_+s_-, -t_+t_-)$$

is a linear representation of $C_2 \times C_2$. Setting

$$x_{00} = s_+t_- + t_+s_-, \quad x_{10} = s_+t_- - t_+s_-,$$
$$x_{01} = s_+s_- + t_+t_-, \quad x_{11} = s_+s_- - t_+t_-,$$

we see this is the standard regular representation of $C_2 \times C_2$.

9. Analysis of unstable Pfister forms

Can we make sense of equivariant Pfister forms without stabilizing? We propose a notion (Definition 9.1) with some reasonable properties. However, formulating the notion of "anisotropic" is subtle; see Remark 9.5.

Throughout, we take G be a finite group, (V, q) a non-degenerate G-quadratic form, and $X = \{q = 0\} \subset \mathbb{P}(V)$ the associated smooth quadric.

Multiplicative forms. We propose a definition inspired by Pfister's original formulation [Pfi95, ch. 2]: Let (V, q) be a non-degenerate quadratic form with similitude group $\mathrm{GO}(V, q)$ and character

$$\lambda : \mathrm{GO}(V, q) \to \mathbb{G}_m.$$

It is strictly multiplicative if there exists a rational map

$$\tau : V \dashrightarrow \mathrm{GO}(V, q)$$

such that

$$q(\tau(x) \cdot y) = \lambda(\tau(x)) q(y) = q(x) q(y), \quad x, y \in V.$$

We refer the reader to [EKM08, §23] for how various definitions of Pfister forms are related.

For invariant forms, there is a natural equivariant analog:

Definition 9.1. Let G be a finite group and (V, q) a non-degenerate G-invariant quadratic form. It is *equivariantly multiplicative* if there exists a G-equivariant rational map

$$\tau : V \dashrightarrow \mathrm{GO}(V, q)$$

such that

$$q(\tau(x) \cdot y) = q(x)q(y), \quad x, y \in V, \tag{9.1}$$

i.e. $\lambda(\tau(x)) = q(x)$.

The natural action on similitudes

$$(g, \tau) \mapsto g\tau g^{-1}$$

is compatible with this formula:

$$q(\tau(g\cdot x)\cdot y) = q((g\tau(x)g^{-1})\cdot y) = q(\tau(x)\cdot g^{-1}y) = q(x)q(g^{-1}y) = q(x)q(y).$$

The "Twisting Construction" of Section 8 gives the following: Suppose that (V, q) is G-invariant and equivariantly multiplicative. Let U_0 be an affine variety with free G-action and invariant field $F = k(U_0)^G$. The resulting quadratic form over $(V', q' = q)$ over F is strictly multiplicative because τ descends to F. Thus we obtain:

Proposition 9.2. *Equivariantly multiplicative forms are stably Pfister. Thus the underlying vector space V satisfies $\dim(V) = 2^e$ for some integer e and the discriminant is trivial when $e > 1$.*

Question 9.3. To what extent are stably Pfister forms equivariantly multiplicative?

Proposition 9.4. *Suppose (V, q) is equivariantly multiplicative of rank $2m$ and the G-action on X is generically free. Then there is an equivariant rational map $X \dashrightarrow \mathrm{OGr}(m, q)$.*

We refer the reader to Remark 9.5 for discussion of the converse.

Proof. This is tautologically true for $m = 1$. When $m > 1$, X is irreducible and any map factors through $\mathrm{OGr}(m, q)_\pm$. We have already discussed the $m = 2$ case: X is a quadric surface and $\mathrm{OGr}(2, q)_\pm$ parametrize its rulings. Thus we focus on $m > 2$.

We use the notations of Section 4. The similitude form λ yields an effective divisor

$$\Lambda = \{\lambda = 0\} \subset \mathbb{P}(\mathrm{End}(V))$$

with proper transform $\Lambda' \subset \overline{X}(V, q)$. Proposition 4.1 implies that Λ' has two irreducible components D_{m-1} and D_m, each of multiplicity one. Proposition 4.2 gives that the difference

$$\Lambda - \Lambda' = 2E,$$

where E is the reduced exceptional divisor of

$$\overline{X}(V, q) \xrightarrow{\beta} \overline{\mathrm{PSO}}(V, q) \subset \mathbb{P}(\mathrm{End}(V)).$$

The rational map τ induces

$$\tau' : V \dashrightarrow \overline{X}(V, q),$$

which is defined at the generic point of the hypersurface $\{q = 0\}$ due to the properness of the target. We are abusing notation, using $\overline{X}(V, q)$ for the component of the wonderful compactification of $\mathrm{PGO}(V, q)$ containing the image of τ; see Remark 4.3 for further context. The hypersurface $\mathrm{Cone}(X) = \{q = 0\}$ is mapped into Λ' by the functional relation (9.1), which guarantees that λ has multiplicity one along our hypersurface. The No-Name Lemma gives a rational section to the cone

$$\mathrm{Cone}(X) \setminus 0 \to X$$

and an induced

$$\tau'' : X \dashrightarrow \overline{X}(V, q).$$

We claim that X is mapped to one of the two irreducible components of Λ' dominating divisors in the closure of $\mathrm{PSO}(V, q) \subset \mathbb{P}(\mathrm{End}(V))$. Proposition 4.1 says that these are birational to projective bundles over

$$\mathrm{OGr}(m, q)_+ \times \mathrm{OGr}(m, q)_-, \text{ for } m \text{ odd},$$

or

$$\mathrm{OGr}(m, q)_\pm \times \mathrm{OGr}(m, q)_\pm, \text{ for } m \text{ even}.$$

Composing τ'' with the projection to one of the factors yields

$$X \dashrightarrow \mathrm{OGr}(m, q)_+ \text{ or } \mathrm{OGr}(m, q)_-,$$

as desired. To see this, note that all the other irreducible components of Λ' have *even* multiplicity. Thus if τ' took X to these components – missing the two distinguished components – then the pull-back of λ to V would necessarily vanish to order ≥ 2 along X. This would contradict the fact that λ pulls back to q, the defining equation of X. $\square$

Competing notions of isotropy. Let (V, q) be a non-degenerate quadratic form in $2m$-variables over a field k, with associated hypersurface $X \subset \mathbb{P}(V)$. Assuming it is anisotropic, it is proportional to a Pfister form if and only if there is a rational map

$$X = \{q = 0\} \dashrightarrow \mathrm{OGr}(m, q). \tag{9.2}$$

Does this make sense equivariantly? We interpret "anisotropic" as lacking a G-invariant isotropic linear subspace. We want that the resulting class has properties similar to Pfister quadrics over fields.

Remark 9.5. There exists a non-degenerate G-quadratic form (V, q) of dimension $2m$, anisotropic in the sense above, that admits an equivariant rational map (9.2) but is not stably Pfister.

Consider the Frobenius group

$$G = C_5 \rtimes C_4, \quad C_5 = \langle \sigma \rangle, C_4 = \langle \tau \rangle,$$

with the cyclic group C_4 acting via automorphisms on C_5

$$\sigma^5 = \tau^4 = 1, \quad \tau\sigma\tau^{-1} = \sigma^2.$$

Let $W = \langle w_1, w_2, w_3, w_4 \rangle$ denote the irreducible four-dimensional representation

$$\begin{aligned} \sigma \cdot w_i &= \zeta^{2^{i-1}} w_i, \quad \zeta = e^{2\pi i/5} \\ \tau \cdot w_1 &= w_4, \quad \tau \cdot w_i = w_{i-1}, \ i = 2, 3, 4. \end{aligned}$$

The projective space $\mathbb{P}(W)$ has no fixed points under G.

Consider the G-quadratic form (V, q):

$$V = \bigwedge^2 W, \quad q : \bigwedge^2 W \to \bigwedge^4 W$$

i.e., squaring of elements. The form q is σ-invariant and transforms by the sign character on $\langle \tau \rangle$. Write

$$X := \mathrm{Gr}(2, W) = \{q = 0\} \subset \mathbb{P}(\bigwedge^2 W),$$

a quadric fourfold with generically-free G-action. We have a decomposition of G-representations

$$\bigwedge^2 W \simeq W \oplus \langle w_1 \wedge w_3 + i w_2 \wedge w_4 \rangle \oplus \langle w_1 \wedge w_3 - i w_2 \wedge w_4 \rangle,$$

where the final two summands are non-isomorphic and not isotropic for q. Projection from this two-dimensional representation gives a rational map

$$\psi : X \dashrightarrow \mathbb{P}(W),$$

a conic fibration over the generic point of $\mathbb{P}(W)$. Note that X is anisotropic in that it has no G-invariant isotropic subspaces.

In this situation

$$\mathrm{OGr}(3,q) = \mathbb{P}(W) \sqcup \mathbb{P}(W^\vee)$$

with isotropic spaces given by lines through $w \neq 0 \in W$ and lines contained in $w^\vee = 0$. Thus ψ induces

$$X \dashrightarrow \mathrm{OGr}(3,q)$$

i.e., X is hyperbolic over its function field. However, since q has six variables we cannot reasonably interpret it to be equivariantly Pfister.

We close with two remarks:

- $X = \{q = 0\} = \mathrm{Gr}(2,W)$ is stably linearizable; its universal rank-two bundle is a $\mathbb{P}^2$-bundle over $W \setminus 0$.
- The components of $\mathrm{OGr}(3,q)$ are linearizable.

Thus q is "stably" isotropic and hyperbolic.

Examples of equivariantly multiplicative actions.

2-elementary groups.

Example 9.6. Let $G = C_2$ with regular representation W, where the coordinate x_0 is invariant and x_1 anti-invariant. Then $q = x_0^2 - x_1^2$ is strictly multiplicative

$$(x_0^2 - x_1^2)(y_0^2 - y_1^2) = (x_0y_0 + x_1y_1)^2 - (x_1y_0 + x_0y_1)^2.$$

The mapping

$$\tau(x_0,x_1)\cdot(y_0,y_1) = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

is C_2-equvariant.

Example 9.7. Consider the $G = C_2^2$-equivariant form

$$q = x_{00}^2 - x_{10}^2 - x_{01}^2 + x_{11}^2,$$

where the subscripts are characters $G \to \mu_2$. Setting

$$\begin{aligned} z_{00} &= +x_{00}y_{00} - x_{10}y_{10} + x_{01}y_{01} + x_{11}y_{11} \\ z_{10} &= -x_{10}y_{00} + x_{00}y_{10} + x_{11}y_{01} + x_{01}y_{11} \\ z_{01} &= +x_{01}y_{00} - x_{11}y_{10} + x_{00}y_{01} + x_{10}y_{11} \\ z_{11} &= -x_{11}y_{00} + x_{01}y_{10} + x_{10}y_{01} + x_{00}y_{11} \end{aligned}$$

we get the desired multiplicative relation

$$z_{00}^2 - z_{10}^2 - z_{01}^2 + z_{11}^2 \mapsto (x_{00}^2 - x_{10}^2 - x_{01}^2 + x_{11}^2)(y_{00}^2 - y_{10}^2 - y_{01}^2 + y_{11}^2).$$

The matrix expressing the z-variables in terms of the y-variables has determinant q^2.

Remark 9.8. The choice of signs for the Pfister form is not natural: The action of $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ on C_2^2 alters the form e.g. to

$$x_{00}^2 - x_{10}^2 + x_{01}^2 - x_{11}^2 \text{ or } x_{00}^2 + x_{10}^2 - x_{01}^2 - x_{11}^2.$$

The form

$$\hat{q} = x_{00}^2 + x_{10}^2 + x_{01}^2 + x_{11}^2$$

is invariant. Writing

$$\begin{aligned}
z_{00} &= +x_{00}y_{00} + x_{10}y_{10} - x_{01}y_{01} - x_{11}y_{11} \\
z_{10} &= -x_{10}y_{00} + x_{00}y_{10} - x_{11}y_{01} + x_{01}y_{11} \\
z_{01} &= +x_{01}y_{00} + x_{11}y_{10} + x_{00}y_{01} + x_{10}y_{11} \\
z_{11} &= +x_{11}y_{00} - x_{01}y_{10} - x_{10}y_{01} + x_{00}y_{11}
\end{aligned}$$

induces

$$(z_{00}^2 + z_{10}^2 + z_{01}^2 + z_{11}^2) \mapsto (x_{00}^2 + x_{10}^2 + x_{01}^2 + x_{11}^2)(y_{00}^2 + y_{10}^2 + y_{01}^2 + y_{11}^2).$$

The transformation

$$\begin{pmatrix} x_{00} & x_{10} & -x_{01} & -x_{11} \\ -x_{10} & x_{00} & -x_{11} & x_{01} \\ x_{01} & x_{11} & x_{00} & x_{10} \\ x_{11} & -x_{01} & -x_{10} & x_{00} \end{pmatrix}$$

has determinant $\hat{q}^2$.

However, the construction here is also non-canonical. For example, reversing the signs of both x_{01} and y_{01} gives the matrix:

$$\begin{pmatrix} x_{00} & x_{10} & -x_{01} & -x_{11} \\ -x_{10} & x_{00} & x_{11} & -x_{01} \\ -x_{01} & x_{11} & -x_{00} & x_{10} \\ x_{11} & x_{01} & x_{10} & x_{00} \end{pmatrix} \tag{9.3}$$

and then reversing the signs of x_{10} and y_{10} gives:

$$\begin{pmatrix} x_{00} & x_{10} & -x_{01} & -x_{11} \\ x_{10} & -x_{00} & x_{11} & -x_{01} \\ -x_{01} & -x_{11} & -x_{00} & -x_{10} \\ x_{11} & -x_{01} & -x_{10} & x_{00} \end{pmatrix}.$$

We can also reverse the sign of any row of the matrix, e.g., switching the third row of (9.3) gives

$$\begin{pmatrix} x_{00} & x_{10} & -x_{01} & -x_{11} \\ -x_{10} & x_{00} & x_{11} & -x_{01} \\ x_{01} & -x_{11} & x_{00} & -x_{10} \\ x_{11} & x_{01} & x_{10} & x_{00} \end{pmatrix}.$$

Rank two forms. We consider the functional equation for rank-two quadratic forms (V, q). Here G is cyclic or dihedral

$$C_r = \langle \sigma : \sigma^r = 1 \rangle \qquad D_r = \left\langle \sigma, \varrho : \sigma^r = \varrho^2 = 1, \varrho\sigma\varrho^1 = \sigma^{-1} \right\rangle.$$

In suitable coordinates, $q = x_0 x_1$ with

$$\sigma : (x_0, x_1) \mapsto (\zeta x_0, \zeta^{-1} x_1), \quad \zeta \text{ primitive } r\text{th root of unity}$$

and

$$\varrho : (x_0, x_1) \mapsto (x_1, x_0).$$

The similitude group

$$\mathrm{GO}(V, q) \simeq \mathbb{G}_m^2 \rtimes C_2$$

where the first factor acts diagonally and the second interchanges x_0 and x_1. The diagonal sub-torus of the orthogonal group acts trivially on the identity component of $\mathrm{GO}(V, q)$. When τ maps V to the identity component, we have

$$\tau(x_0, x_1) = \begin{pmatrix} \tau_{11}(x_0, x_1) & 0 \\ 0 & \tau_{22}(x_0, x_1) \end{pmatrix}, \tag{9.4}$$

where the $\tau_{ii}(x_0, x_1)$ are invariant under $C_r = G \cap \mathbb{G}_m$.

Now for $G = C_r$ we may take

$$\tau_{11}(x_0, x_1) = x_0 x_1, \quad \tau_{22}(x_0, x_1) = 1$$

which is equivariantly multiplicative. Note, however, that when r is even then τ is an even function of (x_0, x_1), unchanged under $\pm I$. When G contains ϱ, we have $\tau_{11}(x_1, x_0) = \tau_{22}(x_0, x_1)$. In the odd case, with $r = 2m + 1$, we may take

$$\tau(x_0, x_1) = \begin{pmatrix} x_0^{m+1}/x_1^m & 0 \\ 0 & x_1^{m+1}/x_0^m \end{pmatrix}$$

which gives multiplicativity for the dihedral group D_{2m+1}. Here τ is odd as a function of (x_0, x_1).

Proposition 9.9. *Let $G = \langle \sigma, \varrho \rangle \simeq C_2 \times C_2$, with $\sigma = -I$. Then (V, q) is not equivariantly multiplicative. The same holds for all dihedral groups D_{2r} with $r > 0$.*

Proof. First, ϱ exchanges the two components of $\mathrm{GO}(V,q)$ and commutes with σ. Thus we may assume that τ takes V to the identity component as in (9.4). As above $\tau_{22}(x_0,x_1)=\tau_{11}(x_1,x_0)$, hence

$$\tau_{11}(x_0,x_1)\tau_{11}(x_1,x_0)=x_0x_1.$$

Consider the quadratic field extension

$$L:=k(x_0^2,x_0x_1)/k(x_0^2+x_1^2,x_0x_1)=:K$$

associated with invariants under C_2 and $C_2\times C_2$. We may write $L=K(\gamma)$ with $\gamma=x_0^2$ satisfying

$$\gamma^2-\gamma(x_0^2+x_1^2)+(x_0x_1)^2.$$

Writing $\tau_{11}=(u+\gamma v)/w$ with u,v,w polynomials in $x_0^2+x_1^2$ and x_0x_1, we get the homogeneous equation

$$u^2-(x_0^2+x_1^2)uv+(x_0x_1)^2v^2=x_0x_1w^2,$$

a smooth conic over K. Now K is purely transcendental with generators $A=x_0x_1$ and $B=x_0^2+x_1^2$ and our conic diagonalizes to

$$\hat{u}^2+(4B^2-A^2)\hat{v}^2=Aw^2.$$

Localizing along $\langle 2B-A\rangle$, where ϖ is the uniformizer of the resulting DVR, yields

$$\hat{u}^2+\varpi(\varpi+2A)\hat{v}^2=Aw^2.$$

The resulting residue is $\sqrt{-A}$, which is non-trivial.

This precludes equivariant multiplicativity for larger groups $G\supset\langle\sigma,\varrho\rangle$. $\square$

Inductive results. We retain the standing assumptions of Section 9. Our next result is inspired by the original argument that Pfister forms are strictly multiplicative [Pfi95, ch. 2]:

Proposition 9.10. *Assume (V,q) is non-degenerate, G-invariant, and equivariantly multiplicative with rational map*

$$\tau:V\dashrightarrow\mathrm{GO}(V,q)$$

that is odd as a function of V. Let $W=\mathrm{span}(1,\alpha)$ be the regular representation of C_2, with α a non-trivial character of C_2. Write $V\otimes W$ for the resulting representation of $G\times C_2$ and

$$\widehat{q}(x\otimes 1+x'\otimes\alpha)=q(x)-q(x')$$

for the induced quadratic form.

Then this is equivariant multiplicative for $G\times C_2$, i.e. there is a linear transformation

$$\widehat{\tau}(x\otimes 1+x'\otimes\alpha):V\otimes W\to V\otimes W,$$

with coefficients rational functions in x and x', with

$$\widehat{q}(\widehat{\tau}(x\otimes 1+x'\otimes\alpha)\cdot(y\otimes 1+y'\otimes\alpha)) = \widehat{q}(x\otimes 1+x'\otimes\alpha)\widehat{q}(y\otimes 1+y'\otimes\alpha).$$

Moreover, $\widehat{\tau}$ is odd as a function on $V\otimes W$.

Proof. Set

$$\widehat{\tau}(x\otimes 1+x'\otimes\alpha) = \begin{pmatrix} \tau(x) & -\tau(x') \\ -\tau(x') & \frac{q(x)}{q(x')}\tau(x')\tau(x)^{-1}\tau(x') \end{pmatrix} \tag{9.5}$$

a matrix expressed in terms of $\{1,\alpha\}$. We expand out

$$\widehat{q}\left(\widehat{\tau}(x\otimes 1+x'\otimes\alpha)\cdot(y\otimes 1+y'\otimes\alpha)\right),$$

using the functional relation along with the fact that q is quadratic, as follows:

$$\begin{aligned} &q(\tau(x)y-\tau(x')y') - q(-\tau(x')y+\frac{q(x)}{q(x')}\tau(x')\tau(x)^{-1}\tau(x')y') \\ &= q(\tau(x)y-\tau(x')y') - q(x')q(-y+\frac{q(x)}{q(x')}\tau(x)^{-1}\tau(x')y') \\ &= q(\tau(x)y-\tau(x')y') - \frac{q(x')}{q(x)}q(-\tau(x)y+\frac{q(x)}{q(x')}\tau(x')y') \\ &= q(\tau(x)y-\tau(x')y') - \frac{1}{q(x)q(x')}q(-q(x')\tau(x)y+q(x)\tau(x')y'). \end{aligned}$$

Writing $b(,)$ for the symmetric bilinear form associated with q, this becomes

$$\begin{aligned} &q(\tau(x)y) - 2b(\tau(x)y,\tau(x')y') + q(\tau(x')y') \\ &- \frac{q(q(x')\tau(x)y) - 2b(q(x')\tau(x)y, q(x)\tau(x')y') + q(q(x)\tau(x')y')}{q(x)q(x')}. \end{aligned}$$

The bilinear terms cancel, so clearing denominators yields the simplification

$$q(\tau(x)y)+q(\tau(x')y')-q(\tau(x)y)\frac{q(x')}{q(x)}-q(\tau(x')y')\frac{q(x)}{q(x')}.$$

Applying the functional relation again gives

$$q(x)q(y)+q(x')q(y')-q(y)q(x')-q(y')q(x) = (q(x)-q(x'))(q(y)-q(y')),$$

as desired.

Equivariance of $\widehat{\tau}$ under the action of G follows from the equivariance of τ. Equivariance under C_2 follows because τ is odd:

$$\begin{pmatrix} \tau(x) & -\tau(-x') \\ -\tau(-x') & \frac{q(x)}{q(x')}\tau(-x')\tau(x)^{-1}\tau(-x') \end{pmatrix} = \begin{pmatrix} \tau(x) & \tau(x') \\ \tau(x') & \frac{q(x)}{q(x')}\tau(x')\tau(x)^{-1}\tau(x') \end{pmatrix};$$

this is the conjugation of our original matrix by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Formula (9.5) also shows that $\widehat{\tau}$ is odd as a function of (x, x'). $\square$

Proposition 8.3 shows that stably Pfister equivariant quadratic forms behave well under tensor product.

Question 9.11. Suppose that (V, q) and (W, r) are equivariantly multiplicative for G and H respectively. Is $(V \otimes W, q \otimes r)$ equivariantly multiplicative for $G \times H$?

References

[BCDP23] Jérémy Blanc, Ivan Cheltsov, Alexander Duncan, and Yuri Prokhorov. Finite quasisimple groups acting on rationally connected threefolds. *Math. Proc. Cambridge Philos. Soc.*, 174(3):531–568, 2023.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[BP13] Fedor Bogomolov and Yuri Prokhorov. On stable conjugacy of finite subgroups of the plane Cremona group, I. *Cent. Eur. J. Math.*, 11(12):2099–2105, 2013.

[Bri89] Michel Brion. Groupe de Picard et nombres caractéristiques des variétés sphériques. *Duke Math. J.*, 58(2):397–424, 1989.

[Bri07] Michel Brion. The total coordinate ring of a wonderful variety. *J. Algebra*, 313(1):61–99, 2007.

[CTVEZ03] Jon F. Carlson, Lisa Townsley, Luis Valeri-Elizondo, and Mucheng Zhang. *Cohomology rings of finite groups*, volume 3 of *Algebra and Applications*. Kluwer Academic Publishers, Dordrecht, 2003. With an appendix: Calculations of cohomology rings of groups of order dividing 64 by Carlson, Valeri-Elizondo and Zhang.

[CTZ25] Ivan Cheltsov, Yuri Tschinkel, and Zhijia Zhang. Equivariant unirationality of Fano threefolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, to appear, 2025. `arXiv:2502.19598`.

[DCP83] C. De Concini and C. Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 1983.

[DCS99] C. De Concini and T. A. Springer. Compactification of symmetric varieties. volume 4, pages 273–300. 1999. Dedicated to the memory of Claude Chevalley.

[DR15] Alexander Duncan and Zinovy Reichstein. Versality of algebraic group actions and rational points on twisted varieties. *J. Algebraic Geom.*, 24(3):499–530, 2015. With an appendix containing a letter from J.-P. Serre.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American*

Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2008.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GKT89] J. Gunarwardena, B. Kahn, and C. Thomas. Stiefel-Whitney classes of real representations of finite groups. *J. Algebra*, 126(2):327–347, 1989.

[Gui10] Pierre Guillot. The computation of Stiefel-Whitney classes. *Ann. Inst. Fourier (Grenoble)*, 60(2):565–606, 2010.

[Hof95] Detlev W. Hoffmann. Isotropy of 5-dimensional quadratic forms over the function field of a quadric. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, volume 58 of *Proc. Sympos. Pure Math.*, pages 217–225. Amer. Math. Soc., Providence, RI, 1995.

[HT22] Brendan Hassett and Yuri Tschinkel. Equivariant geometry of odd-dimensional complete intersections of two quadrics. *Pure Appl. Math. Q.*, 18(4):1555–1597, 2022.

[HT23] Brendan Hassett and Yuri Tschinkel. Torsors and stable equivariant birational geometry. *Nagoya Math. J.*, 250:275–297, 2023.

[HT25] Brendan Hassett and Yuri Tschinkel. Equivariant Geometry of Low-dimensional Quadrics. *Taiwanese J. Math.*, 29(6):1381–1402, 2025.

[Isa06] I. Martin Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].

[JS25] Lena Ji and Fumiaki Suzuki. Arithmetic and birational properties of linear spaces on intersections of two quadrics. *J. Éc. polytech. Math.*, 12:1161–1196, 2025.

[Kah83a] Bruno Kahn. La deuxième classe de Stiefel-Whitney d'une représentation régulière. I. *C. R. Acad. Sci. Paris Sér. I Math.*, 297(6):313–316, 1983.

[Kah83b] Bruno Kahn. La deuxième classe de Stiefel-Whitney d'une représentation régulière. II. *C. R. Acad. Sci. Paris Sér. I Math.*, 297(11):573–576, 1983.

[Kah91] Bruno Kahn. The total Stiefel-Whitney class of a regular representation. *J. Algebra*, 144(1):214–247, 1991. With an appendix in French.

[Kah08] Bruno Kahn. *Formes quadratiques sur un corps*, volume 15 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2008.

[Kar04] Nikita A. Karpenko. Izhboldin's results on stably birational equivalence of quadrics. In *Geometric methods in the algebraic theory of quadratic forms*, volume 1835 of *Lecture Notes in Math.*, pages 151–183. Springer, Berlin, 2004.

[KM03] Nikita Karpenko and Alexander Merkurjev. Essential dimension of quadrics. *Invent. Math.*, 153(2):361–372, 2003.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[Koz84] Andrzej Kozlowski. The Evens-Kahn formula for the total Stiefel-Whitney class. *Proc. Amer. Math. Soc.*, 91(2):309–313, 1984.

[Koz89] Andrzej Kozlowski. Transfers in the group of multiplicative units of the classical cohomology ring and Stiefel-Whitney classes. *Publ. Res. Inst. Math. Sci.*, 25(1):59–74, 1989.

[KT26] Andrew Kresch and Yuri Tschinkel. Linearizability notions in equivariant birational geometry, 2026. `arXiv:2606.10965`.

[Lan52] Serge Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952.

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/70.

[Pfi95] Albrecht Pfister. *Quadratic forms with applications to algebraic geometry and topology*, volume 217 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[PS92] Raman Parimala and Vasudevan Srinivas. Analogues of the Brauer group for algebras with involution. *Duke Math. J.*, 66(2):207–237, 1992.

[RY00] Zinovy Reichstein and Boris Youssin. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Canad. J. Math.*, 52(5):1018–1056, 2000. With an appendix by János Kollár and Endre Szabó.

[Str87] Elisabetta Strickland. A vanishing theorem for group compactifications. *Math. Ann.*, 277(1):165–171, 1987.

[Tot09] Burt Totaro. Birational geometry of quadrics. *Bull. Soc. Math. France*, 137(2):253–276, 2009.

[Tot14] Burt Totaro. *Group cohomology and algebraic cycles*, volume 204 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.

Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912, USA
Email address: `brendan_hassett@brown.edu`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA, Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA
Email address: `tschinkel@cims.nyu.edu`