BIRATIONAL GEOMETRY OF ACTIONS ON DEL PEZZO SURFACES

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Abstract. We complete the classification of regular generically free actions of finite groups on del Pezzo surfaces, up to birational equivalence. As a byproduct, we settle several open problems in equivariant birational geometry, e.g., we classify birationally rigid actions on del Pezzo surfaces.

1. Introduction

Let k be an algebraically closed field of characteristic zero and

$$\mathrm{Cr}_2 = \mathrm{Cr}_2(k) = \langle \mathsf{PGL}_3(k), \iota \rangle,$$

the plane Cremona group; here

$$\iota : (x, y, z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$$

is the Cremona involution. The classification of *finite subgroups* of Cr_2, up to conjugation, has a long history: from the classification of involutions [BB00], cyclic groups [dF04], abelian groups [Bla06], to the fundamental paper by Dolgachev–Iskovskikh [DI09] classifying finite subgroups of Cr_2. We summarize the methods that allowed one to classify finite groups that can act on rational surfaces:

- An embedding of a finite group $G \hookrightarrow \mathrm{Cr}_2$ is given by a regular generically free action on a smooth projective rational surface S. By the G-equivariant Minimal Model Program, we can reduce to the case when S is either a del Pezzo surface, with invariant Picard rank $\mathrm{rkPic}(S)^G = 1$, or a G-equivariant conic bundle $\pi : S \to \mathbb{P}^1$ and $\mathrm{rkPic}(S)^G = 2$.
- There is an almost complete classification of finite groups acting regularly and generically freely on del Pezzo surfaces [DI09]. The classification of automorphisms of conic bundles is more involved, because there are infinitely many families.

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The related problem of classification of *actions* (raised in, e.g., [PV92]) can be approached via the following steps, see [CTZ25, Section 2]:

- **(A)** classification of $G \subseteq \mathrm{Aut}(S)$, up to conjugation in $\mathrm{Aut}(S)$,
- **(B)** classification of $G \subseteq \mathrm{Aut}(S)$, up to conjugation in Cr_2,
- **(AA)** classification of actions, up to conjugation in $\mathrm{Aut}(S)$,
- **(BA)** classification of actions, up to conjugation in Cr_2.

Assuming that both **(A)** and **(B)** are solved for G, we consider homomorphisms

$$\phi_1, \phi_2 : G \hookrightarrow \mathrm{Aut}(S),$$

with $\phi_1(G) = \phi_2(G)$, such that the *actions* given by ϕ_1 and ϕ_2 are different, i.e., not conjugated in $\mathrm{Aut}(S)$. If they are are conjugated in Cr_2, then

$$[S \circlearrowleft \phi_1(G)] = [S \circlearrowleft \phi_2(G)] \in \mathrm{Burn}_2(G), \tag{1.1}$$

where $\mathrm{Burn}_2(G)$ is the equivariant Burnside group, introduced in [KT22]. The converse does not always hold, see [TYZ24, Example 7.2], but it holds for *intransitive* actions on $S = \mathbb{P}^2$, by [CTZ25, Theorem 1.1].

Let $\mathrm{Aut}^G(S)$ and $\mathrm{Bir}^G(S)$ be the normalizers of G in the groups of biregular, respectively, birational automorphisms of S. We have natural homomorphisms

$$\mathrm{Aut}^G(S) \hookrightarrow \mathrm{Bir}^G(S) \xrightarrow{\bar{\beta}} \mathrm{Out}(G), \tag{1.2}$$

where $\mathrm{Out}(G)$ is the group of outer automorphisms of G, see [CTZ25, Section 2] for more details. If

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S))$$

then Problems **(AA)** and **(BA)** coincide and are purely group-theoretic. Therefore, we mostly focus on birational issues, when the images are different. In [CTZ25], we addressed Problem **(BA)** for *linear actions*, i.e., for subgroups of PGL_3. Here, we turn to *nonlinear* actions, more precisely, regular actions on del Pezzo surfaces that are not (projectively) linearizable [TYZ24]. We leave out nonlinear actions on conic bundles.

Let S be a del Pezzo surface of anticanonical degree

$$d(S) := (-K_X)^2 \leq 9,$$

by definition, smooth. Let $G \subseteq \mathrm{Aut}(S)$ be a finite subgroup such that $\mathrm{rkPic}(S)^G = 1$. Then

$$d(S) \in \{1, 2, 3, 4, 5, 6, 8, 9\},$$

If $d(S) = 9$, then $S = \mathbb{P}^2$, if $d(S) = 8$, then $S = \mathbb{P}^1 \times \mathbb{P}^1$. If $d(S) \neq 9$, then the G-action on S is linearizable if and only if it is projectively linearizable, i.e., there exists a G-equivariant birational map $S \dashrightarrow \mathbb{P}^2$. The linearization problem has been settled in [PSY24]. We record known results from [Isk96, DI09, PSY24, CTZ25]:

- If $d(S) = 1$, then the G-action on S is not linearizable, and
$$\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S),$$
- If $d(S) = 2$ or 3, then the G-action on S is not linearizable, and
$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$
- If $d(S) = 4$, then the G-action on S is not linearizable.
- If $d(S) = 5$, then $\mathrm{Aut}(S) \simeq \mathfrak{S}_5$, and G is one of the following subgroups:
$$\mathfrak{C}_5, \quad \mathfrak{D}_5, \quad \mathfrak{F}_5 = \mathsf{AGL}_1(\mathbb{F}_5), \quad \mathfrak{A}_5, \quad \text{or} \quad \mathfrak{S}_5.$$
The G-action on S is linearizable if and only if $G \simeq \mathfrak{C}_5$ or $\mathfrak{D}_5$. Furthermore, if $G \simeq \mathfrak{A}_5$ or $G \simeq \mathfrak{S}_5$, then $\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S)$. If $G \simeq \mathfrak{F}_5$, then $\mathrm{Out}(G)$ is trivial.
- If $d(S) = 6$, then the G-action on S is linearizable if and only if $G \simeq \mathfrak{C}_6$ or $\mathfrak{S}_3$.
- If $d(S) = 8$, then the G-action on S is linearizable if and only if $S^G \neq \varnothing$, i.e., G fixes a point in S.
- The case $d(S) = 9$ has been treated in [CTZ25].

In particular, Problem **(BA)** is already solved for $d(S) \in \{1, 2, 3, 5, 9\}$. In this paper, we solve Problem **(BA)** in the remaining cases:

- $S = \mathbb{P}^1 \times \mathbb{P}^1$,
- S is the del Pezzo surface of degree 6,
- S is a del Pezzo surface of degree 4.

In detail, in Section 2, we complete the classification of finite group actions on $S = \mathbb{P}^1 \times \mathbb{P}^1$, started in [DI09, PSY24], and use it to describe generators of $\mathrm{Bir}^G(S)$. In Section 3, we do the same for the del Pezzo surface of degree 6. We consider del Pezzo surfaces of degree 4 in Section 4.

Our analysis implies several results of independent interest. The following completes investigations that started in [Seg43, Man66, DI09, Che08, Che14, Sak19, Pin24, Yas25, PSY24]:

Theorem 1.1. *Let S be a del Pezzo surface and $G \subseteq \mathrm{Aut}(S)$ a finite subgroup such that $\mathrm{rkPic}(S)^G = 1$. Then S is G-birationally rigid if and only if one of the following holds:*

- $d(S) \leq 3$,
- $d(S) = 4$, *and* $G \not\simeq \mathfrak{C}_2^2$, $\mathfrak{D}_4$, $\mathfrak{C}_8$, $\mathfrak{C}_2 \times \mathfrak{C}_6$, $\mathfrak{C}_3 \rtimes \mathfrak{C}_4$, $\mathfrak{C}_3 \rtimes \mathfrak{D}_4$,
- $d(S) = 5$, *and* $G \simeq \mathfrak{A}_5, \mathfrak{S}_5$,
- $d(S) = 6$, *and* $G \not\simeq \mathfrak{C}_3 \times \mathfrak{S}_3$, $\mathfrak{C}_2 \times \mathfrak{S}_3$, $\mathfrak{S}_3$, $\mathfrak{C}_6$,
- $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$, *$S$ has no orbits of length 1 or 2, and* $G \not\simeq \mathfrak{D}_6, \mathfrak{F}_5$,
- $S \simeq \mathbb{P}^2$, *S has no orbits of length 1, and* $G \not\simeq \mathfrak{A}_4, \mathfrak{S}_4$.

The condition $\mathrm{rkPic}(S)^G = 1$ means that S is a G-Mori fiber space (over a point), and G-birational rigidity means that S is the unique G-Mori fiber space that is G-birational to S. In [DI09], Dolgachev and Iskovskikh listed all G that can act on a del Pezzo surface S with $\mathrm{rkPic}(S)^G = 1$. We present a refined classification in Appendix A, correcting minor inaccuracies.

Theorem 1.1 yields the main result of [Yas25]:

Corollary 1.2. *Let S be a del Pezzo surface and $H \subset G \subseteq \mathrm{Aut}(S)$ finite subgroups such that*

$$\mathrm{rkPic}(S)^H = \mathrm{rkPic}(S)^G = 1.$$

If S is H-birationally rigid, then S is G-birationally rigid.

Another two byproducts of our analysis are generalizations of the following classical theorem of Segre and Manin [Seg43, Man66].

Theorem 1.3. *Let G be a finite group acting regularly and generically freely on del Pezzo surfaces S and S', with $d(S) = d(S') \leq 3$, and*

$$\mathrm{rkPic}(S)^G = \mathrm{rkPic}(S')^G = 1.$$

Then S and S' are G-birational if and only if they are G-biregular.

In [ST25], Shramov and Trepalin proved that the Segre–Manin theorem holds for del Pezzo surfaces of degree 4, see [Ela26] for a categorical proof. In Section 4, we give an alternative proof of their result. Note that a similar result holds for the del Pezzo surface of degree 5. In Section 3, we prove:

Theorem 1.4. *Let G be a finite group acting regularly and generically freely on del Pezzo surfaces S and S', with $d(S) = d(S') = 6$, and*

$$\mathrm{rkPic}(S)^G = \mathrm{rkPic}(S')^G = 1.$$

Suppose that $G \not\simeq \mathfrak{C}_3 \rtimes \mathfrak{C}_6$. Then S and S' are G-birational if and only if they are G-biregular.

The exceptional case in Theorem 1.4 was found in [Yas25]. Note however, that the statement fails in the arithmetic setting [KY25]. Moreover, the Segre–Manin theorem fails for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. However, it follows from [Sak19, CTZ25] that the Segre–Manin theorem holds for transitive actions on $\mathbb{P}^2$. In Section 2, we generalize this as follows:

Theorem 1.5. *Let G be a finite group acting regularly and generically freely on del Pezzo surfaces S and S' such that $S \simeq S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, the group G does not fix points in S and S', and*

$$\mathrm{rkPic}(S)^G = \mathrm{rkPic}(S')^G = 1.$$

Then S and S' are G-birational if and only if they are G-biregular.

For linearizable actions on $\mathbb{P}^1 \times \mathbb{P}^1$, the assertion of Theorem 1.5 fails in general. For instance, if $G \simeq \mathfrak{D}_4$ acts on $S = \mathbb{P}^1 \times \mathbb{P}^1$ as

$$\begin{aligned} ([x_1 : x_2], [y_1 : y_2]) &\mapsto ([y_1 : y_2], [x_1 : x_2]), \\ ([x_1 : x_2], [y_1 : y_2]) &\mapsto ([-x_1 : x_2], [y_1 : y_2]), \end{aligned}$$

and G acts on $S' = \mathbb{P}^1 \times \mathbb{P}^1$ as

$$\begin{aligned} ([x_1 : x_2], [y_1 : y_2]) &\mapsto ([y_1 : y_2], [x_1 : x_2]), \\ ([x_1 : x_2], [y_1 : y_2]) &\mapsto ([ix_1 : x_2], [-iy_1 : y_2]), \end{aligned}$$

then both G-actions are linearizable, so it follows from [CTZ25] that S and S' are G-birational, but S and S' are not G-biregular, because they have different fixed loci stratification. In the arithmetic case, the situation is very similar [Tre23, JLT22].

The following result has the same flavor as Theorems 1.4 and 1.5: it establishes that, apart from the exceptions listed, birationality of actions coincides with isomorphism of actions.

Theorem 1.6. *Let S be a del Pezzo surface of degree $d(S) \leq 8$, and $G \subseteq \mathrm{Aut}(S)$ a finite subgroup such that $\mathrm{rkPic}(S)^G = 1$, and the G-action on S is not linearizable. Then*

$$\frac{|\bar{\beta}(\mathrm{Bir}^G(S))|}{|\bar{\beta}(\mathrm{Aut}^G(S))|} \leq 2.$$

Moreover, equality holds if and only if

- *$d(S) = 4$, and $G \simeq \mathfrak{C}_8$,*

- $d(S)=6$, *and* $G\simeq\mathfrak{S}_3^2$,
- $S\simeq\mathbb{P}^1\times\mathbb{P}^1$, $G\simeq\mathfrak{C}_2\times\mathfrak{D}_n$ *with* n *odd, and* G *is conjugated to the subgroup generated by*

$$(x,y)\mapsto(\zeta_n x,\zeta_n y),\quad (x,y)\mapsto(\frac{1}{x},\frac{1}{y}),\quad (x,y)\mapsto(y,x).$$

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2. Quadric

Let $S=\mathbb{P}^1\times\mathbb{P}^1$ and

$$G\subset\mathrm{Aut}(S)=(\mathsf{PGL}_2)^2\rtimes\mathfrak{C}_2$$

be a finite subgroup such that $\mathrm{rkPic}(S)^G=1$. Then G projects nontrivially to $\mathfrak{C}_2$, so that

$$G\simeq(H\times_Q H)\rtimes\mathfrak{C}_2,\quad H\subset\mathsf{PGL}_2,$$

where Q is a group and

$$H\times_Q H=\{(h,h')\in H\times H:\gamma(h)=\gamma'(h')\},$$

for surjective homomorphisms $\gamma,\gamma':H\to Q$. Linearizability of actions on quadric surfaces has been settled in [PSY24]. We have:

- H is cyclic $\Rightarrow$ the G-action is linearizable.
- H is dihedral $\Rightarrow$ the G-action is not linearizable.
- $H=\mathfrak{A}_4,\mathfrak{S}_4$ or $\mathfrak{A}_5$ $\Rightarrow$ the G-action is birationally rigid.

We focus on nonlinearizable actions, in particular, we assume that $S^G=\varnothing$ and H is not cyclic. We aim to determine the image of

$$\bar{\beta}:\mathrm{Bir}^G(S)\to\mathrm{Out}(G).$$

We summarize the main steps:

- Reduction to the case when H is dihedral.
- Extraction of rigid and solid G-actions.
- Classification of non-solid actions.

Reduction to the dihedral case. If S does not contain G-orbits of length less than 6, then the classification of G-Sarkisov links implies that S is either

- G-birationally superrigid, or
- G-birationally rigid, with $\mathrm{Bir}^G(S)$ generated by $\mathrm{Aut}^G(S)$, Geiser involutions, and Bertini involutions.

If G has a G-orbit of length 4, whose points are in general position, then $\mathrm{Bir}^G(S)$ also contains a G-birational involution as in:

Lemma 2.1 ([Yas25, Proposition 5.1])**.** *If there is a G-Sarkisov link*

$$\begin{array}{ccccc} & & \tilde{S} & & \\ & \overset{\pi}{\swarrow} & & \overset{\pi'}{\searrow} & \\ S & & & & S \end{array}$$

where π and π' are blowups of G-orbits of length 4 and $\tilde{S}$ a smooth del Pezzo surface of degree 4, then there exists a birational involution $\tau \in \mathrm{Bir}^G(\tilde{S})$ and the following G-commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \phi\ } & \tilde{S} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \overset{\tau}{\dashrightarrow} & S \end{array}$$

where ϕ is a biregular involution of $\tilde{S}$. Moreover, τ commutes with G.

Proof. We give a slightly modified proof of [Yas25, Proposition 5.1]. By [DI09, Section 6.1], there is a natural embedding $\mathrm{Aut}(\tilde{S}) \hookrightarrow W_S$, and the Weyl group $W_S \simeq \mathfrak{C}_2^4 \rtimes \mathfrak{S}_5$ acts transitively on the set of (-1)-curves in $\tilde{S}$. Note also that W_S contains a unique normal subgroup $H \simeq \mathfrak{C}_2^4$, and $H \subset \mathrm{Aut}(\tilde{S})$. Let E_1, E_2, E_3, E_4 be the π-exceptional curves, and Γ the stabilizer in W_S of the curve $E_1 + E_2 + E_3 + E_4$. Then $\Gamma \simeq \mathfrak{S}_4 \times \mathfrak{C}_2$, and $G \subset \Gamma$, by construction. Using `Magma`, one checks that H contains an involution ϕ that commutes with Γ, as claimed. $\square$

Definition 2.2. An involution $\tau \in \mathrm{Bir}^G(S)$ as in Lemma 2.1 will be called a *Yasinsky involution* of S.

Using the classification of G-Sarkisov links again, we observe: if S does not have G-orbits of length 2, 3 or 5, then either S is G-birationally superrigid, and therefore

$$\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S),$$

or S is G-birationally rigid and $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Geiser involutions, Bertini involutions, and Yasinsky involutions, which also follows from [Yas25, Corollary 5.2]. Moreover, by [CTZ25, Remark 2.3] and Lemma 2.1, all these birational involutions lie in the kernel of the homomorphism $\bar{\beta} \colon \mathrm{Bir}^G(S) \to \mathrm{Out}(G)$. This yields:

Corollary 2.3 (cf. [PSY24, Proposition 6.12]). *If $H = \mathfrak{A}_4, \mathfrak{S}_4$ or $\mathfrak{A}_5$, then S is G-birationally rigid, and*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

Geometry in the dihedral case. From now on, we assume that H is dihedral. Then each factor of $S = \mathbb{P}^1 \times \mathbb{P}^1$ has an H-orbit of length 2, which is unique if $H \not\simeq \mathfrak{C}_2^2$. Without loss of generality, we may assume that these orbits consist of the points $P_1 = [0:1]$ and $P_2 = [1:0]$. Put

$$L_{11} := \mathrm{pr}_1^*(P_1), \quad L_{12} := \mathrm{pr}_1^*(P_2), \quad L_{21} := \mathrm{pr}_2^*(P_1), \quad L_{22} := \mathrm{pr}_2^*(P_2),$$

where pr_j are projections to the factors. Then G preserves the torus

$$S^\circ := S \setminus (L_{11} \cup L_{12} \cup L_{21} \cup L_{22}) \simeq \mathbb{G}_m^2,$$

and G is contained in its normalizer in $\mathrm{Aut}(S)$; we have an exact sequence

$$1 \to G_T \to G \xrightarrow{\nu} \mathfrak{D}_4, \tag{2.1}$$

where G_T is a subgroup of the group of translations in $\mathbb{G}_m^2$ and $\mathfrak{D}_4$ acts transitively on the set of curves $L_{11}, L_{12}, L_{21}, L_{22}$.

Since we assume that $\mathrm{rkPic}(S)^G = 1$ and G does not fix points in S, the group G acts transitively on $L_{11}, L_{12}, L_{21}, L_{22}$, and $\nu(G)$ is:

$$\mathfrak{C}_2^2, \quad \mathfrak{C}_4, \quad \text{or} \quad \mathfrak{D}_4.$$

Rigid groups. Using the classification of G-Sarkisov links [Isk96] or, more explicitly, [Yas25, Corollary 5.2], we obtain the following:

Lemma 2.4. *If $\nu(G) \simeq \mathfrak{C}_4$ or $\mathfrak{D}_4$ as in* (2.1)*, and $|G_T| \notin \{1, 2, 3, 5\}$, then S is G-birationally rigid, and*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

Proof. We claim that S does not contain G-orbits of length 2, 3 or 5. Indeed, the boundary $L_{11} \cup L_{12} \cup L_{21} \cup L_{22}$ does not contain a G-orbits of length 2, 3 or 5, because $\nu(G)$ permutes $L_{11}, L_{12}, L_{21}, L_{22}$ transitively. On the other hand, G_T acts on $S^\circ \simeq \mathbb{G}_m^2$ by translations, so it does not contain G-orbits of length 2, 3 or 5, since $|G_T| \notin \{1, 2, 3, 5\}$.

Since S does not contain G-orbits of length 2, 3 or 5, as already mentioned, S is G-birationally rigid, and $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Geiser involutions, Bertini involutions, and Yasinsky involutions, so

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)),$$

by [CTZ25, Remark 2.3] and Lemma 2.1. □

Recall that G is not cyclic – otherwise it would fix a point in S. Thus, if $\nu(G) = \mathfrak{C}_4$ and $|G_T| \in \{1,2,3,5\}$, then, up to conjugation, one of the following holds:

(1) $G \simeq \mathfrak{F}_5 \simeq \mathfrak{C}_5 \rtimes \mathfrak{C}_4$ and G is generated by
$$(x,y) \mapsto (\zeta_5 x, \zeta_5^2 y) \quad \text{and} \quad (x,y) \mapsto (\frac{1}{y}, x).$$
(2) $G \simeq \mathfrak{C}_2 \times \mathfrak{C}_4$ is generated by
$$(x,y) \mapsto (-x,-y) \quad \text{and} \quad (x,y) \mapsto (\frac{1}{y}, x).$$

In the first case, S is G-solid [Wol18], and the G-action on S is unique, since $\mathrm{Out}(\mathfrak{F}_5)$ is trivial. In the second case, $\mathrm{Out}(\mathfrak{C}_2 \times \mathfrak{C}_4) \simeq \mathfrak{C}_2^2$, and
$$\mathrm{Aut}^G(S) \simeq (\mathfrak{C}_2 \times \mathfrak{C}_4) \rtimes \mathfrak{D}_4,$$
with GapID `(64,138)`, generated by
$$(x,y) \mapsto (-x,y),$$
$$(x,y) \mapsto (ix,iy),$$
$$(x,y) \mapsto (\frac{1}{x}, y),$$
$$(x,y) \mapsto (y,x),$$
which implies that the G-action on S is also unique. In this case, G is conjugated in $\mathrm{Aut}(S)$ to the subgroup G' generated by
$$(x,y) \mapsto (\frac{1}{x}, \frac{1}{y}) \text{ and } (x,y) \mapsto (y,-x).$$
Since $\nu(G') \simeq \mathfrak{C}_2^2$, S is not G-solid. In Lemma 2.15, we describe the generators of $\mathrm{Bir}^G(S)$ in a more general setting.

Keeping in mind Lemma 2.4, we need a classification for small G_T. Recall that $G_T \subset \mathbb{G}_m^2$ acts on $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ by scaling the first coordinates on each factor.

Lemma 2.5. *If $\nu(G) = \mathfrak{D}_4$ as in* (2.1)*, then*
- *either $G_T = \langle(\zeta_n,1),(1,\zeta_n)\rangle \simeq \mathfrak{C}_n^2$ for some n, or*
- *n is even and $G_T = \langle(\zeta_n,\zeta_n),(\zeta_n^2,1)\rangle \simeq \mathfrak{C}_n \times \mathfrak{C}_{n/2}$.*

Proof. Let $g \in G_T$ be an element of the largest order, say n. Then $g = (\zeta_n^a, \zeta_n^b)$, where $\gcd(a,b,n) = 1$, and $G_T \subset \langle(\zeta_n,1),(1,\zeta_n)\rangle \subset \mathbb{G}_m^2$. Using the conjugation action of $\nu(G)$ on G_T, we see that $(\zeta_n^c, \zeta_n) \in G_T$, for some c, and also that $(\zeta_n^c, \zeta_n^{-1}) \in G_T$, so that
$$(\zeta_n^c, \zeta_n)(\zeta_n^c, \zeta_n^{-1})^{-1} = (1, \zeta_n^2) \in G_T,$$

and $(\zeta_n^2, 1) \in G_T$ as well. If c is even, then $(1, \zeta_n) \in G_T$, which gives $G_T = \langle (\zeta_n, 1), (1, \zeta_n) \rangle$. If c is odd, then $(\zeta_n, \zeta_n) \in G_T$, which gives

$$\langle (\zeta_n, \zeta_n), (\zeta_n^2, 1) \rangle \subset G_T.$$

If n is odd, this gives

$$G_T = \langle (\zeta_n, 1), (1, \zeta_n) \rangle.$$

If n is even, then

$$\langle (\zeta_n, \zeta_n), (\zeta_n^2, 1) \rangle \simeq \mathfrak{C}_n \times \mathfrak{C}_{n/2}$$

is a subgroup of index 2 in $\langle (\zeta_n, 1), (1, \zeta_n) \rangle$. $\square$

Applying Lemma 2.4, we obtain:

Corollary 2.6. *If $\nu(G) = \mathfrak{D}_4$ and $|G_T| \neq 1, 2$, then S is G-birationally rigid, and $\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S))$.*

Proof. Similar to the proof of Lemma 2.4. $\square$

Assume that $\nu(G) = \mathfrak{D}_4$ and $|G_T| = 1, 2$, and $S^G = \varnothing$. Then, up to conjugation, one of the following holds:

(1) $G \simeq \mathfrak{D}_4$, generated by

$$(x, y) \mapsto (-y, -x) \quad \text{and} \quad (x, y) \mapsto (\frac{1}{y}, x).$$

(2) $G \simeq \mathfrak{C}_2 \times \mathfrak{D}_4 \simeq \mathfrak{C}_2^2 \rtimes \mathfrak{C}_2^2$, generated by

$$(x, y) \mapsto (-x, -y), \quad (x, y) \mapsto (y, x), \quad \text{and} \quad (x, y) \mapsto (\frac{1}{y}, x).$$

(3) $G \simeq \mathfrak{C}_2.\mathfrak{D}_4 \simeq \mathfrak{C}_2^2 \rtimes \mathfrak{C}_4$, with GapID `(16,3)`, generated by

$$(x, y) \mapsto (-x, -y), \quad (x, y) \mapsto (y, -x), \quad \text{and} \quad (x, y) \mapsto (\frac{1}{y}, x).$$

In Case (1), $\mathrm{Out}(\mathfrak{D}_4) \simeq \mathfrak{C}_2$, and $\mathrm{Aut}^G(S) \simeq (\mathfrak{C}_2 \times \mathfrak{C}_4) \rtimes \mathfrak{C}_2^2$, with GapID `(32,49)`, that is generated by G, and $(x, y) \mapsto (-x, y)$, and $(x, y) \mapsto (ix, iy)$, which implies that the G-action on S is unique. In this case, G is conjugated in $\mathrm{Aut}(S)$ to the subgroup G' generated by

$$(x, y) \mapsto (\frac{1}{x}, \frac{1}{y}) \text{ and } (x, y) \mapsto (iy, ix).$$

Since $\nu(G') \simeq \mathfrak{C}_2^2$, S is not G-solid (we will describe the generators of $\mathrm{Bir}^G(S)$ in Lemma 2.14).

In Case (2), $\mathrm{Out}(\mathfrak{C}_2\times\mathfrak{D}_4)\simeq\mathfrak{C}_2\times\mathfrak{D}_4$, and $\mathrm{Aut}^G(S)\simeq(\mathfrak{C}_2\times\mathfrak{C}_4)\times\mathfrak{D}_4$, with GapID `(64,138)`. In this case, G is conjugated in $\mathrm{Aut}(S)$ to the subgroup G' generated by

$$(x,y)\mapsto(\frac{1}{x},\frac{1}{y}),\quad (x,y)\mapsto(y,x)\text{ and }(x,y)\mapsto(-x,y).$$

Since $\nu(G')\simeq\mathfrak{C}_2^2$, S is not G-solid. In Lemma 2.12, we will show that $\mathrm{Bir}^G(S)=\mathrm{Aut}^G(S)$.

In Case (3), S contains a G-orbit of length 4, of points

$$(i,i),\quad (i,-i),\quad (-i,i),\quad (-i,-i),$$

and G acts on these points as the cyclic group $\mathfrak{C}_4$, so G is conjugated in $\mathrm{Aut}(S)$ to a subgroup G' with $\nu(G')\simeq\mathfrak{C}_4$. Thus S is G-birationally rigid, and $\bar\beta(\mathrm{Bir}^G(S))=\bar\beta(\mathrm{Aut}^G(S))$, by Lemma 2.4. Arguing as in the proof of Lemma 2.4, one can show that S is G-birationally superrigid.

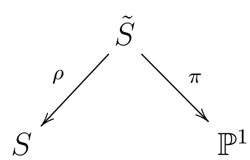
Non-solid groups. If $\nu(G)\simeq\mathfrak{C}_2^2$, then S is not G-solid, because S has at least two G-orbits of length 2. One of them is formed by points

$$L_{11}\cap L_{21}=(0,0)\quad\text{and}\quad L_{12}\cap L_{22}=(\infty,\infty),$$

and another by

$$L_{12}\cap L_{21}=(\infty,0)\quad\text{and}\quad L_{11}\cap L_{22}=(0,\infty).$$

Blowing up one of these orbits $\rho\colon\tilde S\to S$, we obtain a G-Sarkisov link:



where π is a G-conic bundle. In particular, S is not G-solid, and therefore not G-birationally rigid.

Proposition 2.7. *Suppose that $\nu(G)=\mathfrak{C}_2^2$ as in (2.1), and $S^G=\varnothing$. Then, up to conjugation in $\mathrm{Aut}(S)$, one of the following holds:*

(1) G is generated by

$$\mathrm{diag}(\zeta_n,1),\quad \mathrm{diag}(1,\zeta_n),\quad (\frac{1}{x},\frac{1}{y}),\quad (y,x),$$

for $n\geq 2$.

(2) G is generated by

$$\mathrm{diag}(\zeta_n, 1), \quad \mathrm{diag}(1, \zeta_n), \quad (\frac{1}{x}, \frac{1}{y}), \quad (\zeta_{2n}y, \zeta_{2n}x),$$

for some even n.

(3) G is generated by

$$\mathrm{diag}(\zeta_n^r, 1), \quad \mathrm{diag}(1, \zeta_n^r), \quad \mathrm{diag}(\zeta_n, \zeta_n), \quad (\frac{1}{x}, \frac{1}{y}), \quad (y, x),$$

for some $r, n \geq 2$, where $r \mid n$.

(4) and G is generated by

$$\mathrm{diag}(\zeta_n^r, 1), \quad \mathrm{diag}(1, \zeta_n^r), \quad \mathrm{diag}(\zeta_n, \zeta_n), \quad (\frac{1}{x}, \frac{1}{y}), \quad (\zeta_{2n}y, \zeta_{2n}x),$$

for some even $n \geq 2$ and $r \mid n$, $r \geq 2$.

(5) G is generated by

$$\mathrm{diag}(\zeta_n^r, 1), \quad \mathrm{diag}(1, \zeta_n^r), \quad \mathrm{diag}(\zeta_n, \zeta_n), \quad (\frac{1}{x}, \frac{1}{y}), \quad (y, \zeta_{2n}^r x),$$

for some even $r, n \geq 2$ where $r \mid n$.

(6) G is generated by

$$\mathrm{diag}(\zeta_n^r, 1), \quad \mathrm{diag}(1, \zeta_n^r), \quad \mathrm{diag}(\zeta_n, \zeta_n), \quad (\frac{1}{x}, \frac{1}{y}), \quad (\zeta_{2n}y, \zeta_{2n}^{1+r} x),$$

for some even $n, r \geq 2$, where $r \mid n$.

Proof. First observe that up, to scaling by the torus, we may assume that G is generated by G_T,

$$\sigma : (x, y) \to (\frac{1}{x}, \frac{1}{y}), \quad \text{and} \quad \tau : (x, y) \to (by, ax),$$

for some $a, b \in k^\times$. It follows that

$$\tau^2 = \mathrm{diag}(ab, ab) \in G_T, \quad \text{and} \quad (\sigma\tau)^2 = \mathrm{diag}(b/a, a/b) \in G_T. \tag{2.2}$$

When $G_T = 1$, we know that $ab = 1$ and $a/b = 1$. It follows that $(a, b) = (1, 1)$ or $(-1, -1)$. These two choices are conjugated in $\mathrm{Aut}(S)$ by $(x, y) \mapsto (-x, y)$. In this case, G fixes a point in S, contradicting our assumption.

When $G_T \neq 1$, Goursat's lemma implies the exact sequence

$$1 \to H \times_Q H \to G \to \mathfrak{C}_2 \to 1,$$

where $H \times_Q H$ acts on $\mathbb{P}^1 \times \mathbb{P}^1$ without switching the factors. Since $G_T \neq 1$, we know that H is a dihedral group $\mathfrak{D}_n$ of order $2n$, $n \geq 2$.

Note that $\mathfrak{D}_2 \simeq \mathfrak{C}_2^2$. We know that $Q \neq 1$ since otherwise $\nu(G) = \mathfrak{D}_4$. We have the following possibilities

$$Q = \mathfrak{C}_2, \quad \mathfrak{D}_r, \quad \text{where} \quad r \mid n.$$

Let $\varphi_1, \varphi_2 : \mathfrak{D}_n \to Q$ be the corresponding homomorphisms defining $H \times_Q H$, $K_1 = \ker(\varphi_1)$ and $K_2 = \ker(\varphi_2)$. Recall that $K_1 \times K_2$ is a subgroup of $H \times_Q H$. Since $\mathrm{rkPic}(S)^G = 1$, we know that $K_1 = K_2$.

When $Q = \mathfrak{C}_2$, if n is even and $K_1 = K_2 = \mathfrak{D}_{\frac{n}{2}}$, then $\bar{G} = \mathfrak{D}_4$. It follows that $K_1 = K_2 = \mathfrak{C}_n$, and $G_T = K_1 \times K_2$ is generated by

$$\mathrm{diag}(\zeta_n, 1), \quad \mathrm{diag}(1, \zeta_n).$$

It follows that $a = \zeta_{2n}^{r_1}$, $b = \zeta_{2n}^{r_2}$, for some integers r_1, r_2. Up to multiplying by an element in G_T, we may assume that $r_1, r_2 \in \{0, 1\}$. From (2.2), we see that $r_1 = r_2 = 0$ or 1. We obtain Case (2) and (3).

When $Q = \mathfrak{D}_r$, where $r \mid n$, we have $K_1 = K_2 = \mathfrak{C}_{\frac{n}{r}}$, and G_T is generated by

$$\mathrm{diag}(\zeta_n^r, 1), \quad \mathrm{diag}(1, \zeta_n^r), \quad \mathrm{diag}(\zeta_n, \zeta_n).$$

As above, we know that $a = \zeta_{2n}^{r_1}$, $b = \zeta_{2n}^{r_2}$ and $r_1 + r_2$ is even. Up to multiplying by elements in G_T, we may assume that $0 \leq r_1 \leq r_2 < r$, and $r_1 \in \{0, 1\}$. Recall that $(\sigma\tau)^2 = \mathrm{diag}(\zeta_{2n}^{r_2 - r_1}, \zeta_{2n}^{r_1 - r_2}) \in G_T$. It follows that $\zeta_n^{r_1 - r_2}$ is a power of ζ_n^r, and $r_1 - r_2 = 0 \pmod r$. Thus, we have at most four possibilities

$$(a, b) = (1, 1), \quad (\zeta_{2n}, \zeta_{2n}), \quad (1, \zeta_{2n}^r), \quad (\zeta_{2n}, \zeta_{2n}^{1+r}),$$

where the latter two are only possible when r is even. When n is odd, the first two cases give conjugated subgroups in $\mathrm{Aut}(S)$, which correspond to Case (3). When n is even, the four possible cases give non-isomorphic groups in general, which correspond to Cases (3)–(6). □

Using Proposition 2.7, we proceed to analyze different actions up to conjugation in the Cremona group. For this, we need to describe generators of $\mathrm{Bir}^G(S)$ in each case. Recall that S contains two G-orbits of length 2: $(0, 0) \cup (\infty, \infty)$ and $(0, \infty) \cup (0, \infty)$. Moreover, these are the only G-orbits of length 2 except for one case:

Lemma 2.8. *Suppose that G is the group described in Case (3) in Proposition 2.7, with $n = r = 2$. Then $G \simeq \mathfrak{C}_2^3$, $\mathrm{Out}(G) = \mathsf{GL}_3(\mathbb{F}_2)$, and*

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)) = \mathfrak{S}_4.$$

In particular, G gives rise to 7 non-birational actions on S.

Proof. We use the Burnside formalism of [KT22]. There are four involutions in G fixing a conic, with a residual $\mathfrak{C}_2^2$-action, giving rise to incompressible symbols

$$(\mathfrak{C}_2, \mathfrak{C}_2^2 \circlearrowright k(\mathbb{P}^1), (1)).$$

These four involutions sum to 0 in G. Up to $\mathsf{GL}_3(\mathbb{F}_2)$-equivalence, there are 7 choices of such 4 involutions, giving rise to 7 non-birational actions since their corresponding classes in $\mathrm{Burn}_2^{\mathrm{inc}}(G)$ are different. On the other hand, one can check that $\mathrm{Aut}^G(S)$ contains

$$(x,y) \mapsto (\frac{x-1}{x+1}, \frac{y-1}{y+1}), \quad (x,y) \mapsto (\frac{1}{x}, y), \quad (x,y) \mapsto (\zeta_4 x, \zeta_4 y).$$

Together with G, they generate a group of order 192 with GapID `(192,955)`. The image of this group under $\bar{\beta}$ in $\mathrm{Out}(G)$ is $\mathfrak{S}_4$. Recall that $|\mathsf{GL}_2(\mathbb{F}_3)| = 168$ and by the Burnside formalism the remaining 7 actions are not birational to each other. It follows that

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)) = \mathfrak{S}_4.$$

□

In fact, arguing as in the proof of Lemma 2.13 below, we can also show that $\mathrm{Aut}^G(S) = \mathrm{Bir}^G(S)$ in the case of Lemma 2.8. Similarly, S does not have G-orbits of length 3, except for one classical case [Isk03, Isk08].

Lemma 2.9. *If G is the group in Case (3) in Proposition 2.7, with $n = r = 3$, then $G \simeq \mathfrak{D}_6$, $\bar{\beta}(\mathrm{Aut}^G(S))$ is trivial, and*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \mathrm{Out}(G) \simeq \mathfrak{C}_2.$$

Proof. Recall that $G \simeq \mathfrak{D}_6 \simeq \mathfrak{C}_2 \times \mathfrak{S}_3$. Let $\{a, b, c\}$ be a set of generators of G, where a is in the center of G, b is an order 2 element in $\mathfrak{S}_3$, and c is an order 3 element in $\mathfrak{S}_3$. Let $\varphi_1 : G \to \mathrm{Aut}(S)$ be the homomorphism given by

$$(\varphi_1(a))(x,y) = (-x,-y), \quad (\varphi_1(b))(x,y) = (\frac{1}{x}, \frac{1}{y}),$$

$$(\varphi_1(c))(x,y) = (y,x),$$

and $\varphi_2 : G \to \mathrm{Aut}(S)$ the map given by

$$\varphi_2(a) = \varphi_1(a), \quad \varphi_2(b) = \varphi_1(ab), \quad \varphi_1(c) = \varphi_1(c).$$

Then $\mathrm{Out}(G)$ swaps the actions given by φ_1 and φ_2. The map (2.5) in $\mathrm{Bir}^G(S)$, with $n = 2$, also swaps φ_1 and φ_2. Thus

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \mathrm{Out}(G).$$

Using Proposition 2.7, we obtain that $\mathrm{Aut}^G(S)$ is generated by G and $\mathrm{diag}(-1,-1)$, which implies that $\bar{\beta}(\mathrm{Aut}^G(S))$ is trivial. $\square$

In the following, we assume that G is not one of the groups described in Lemmas 2.8 and 2.9. In particular, $(0,0)\cup(\infty,\infty)$ and $(0,\infty)\cup(\infty,0)$ are the only G-orbits of length 2. Fix

$$\chi \in \mathrm{Bir}^G(S).$$

If $\chi \notin \mathrm{Aut}^G(S)$, it can be decomposed into a sequence of G-Sarkisov links. On the other hand, every G-Sarkisov link that starts at S is given by blowing up $\rho\colon \tilde{S} \to S$ a G-orbit $\Sigma \subset S$ such that

$$|\Sigma| \in \{2,4,5,6,7\},$$

and $\tilde{S}$ is a del Pezzo surface.

In fact, $|\Sigma| \neq 5$. Indeed, if $|\Sigma| = 5$, it follows that $G \simeq \mathfrak{C}_2 \times \mathfrak{D}_5$ is the group described in Case (3) in Proposition 2.7, with $n = r = 5$. In this case, the only G-orbit of length 5 in S is contained in a curve of degree $(1,1)$, so blowing it up we do not obtain a del Pezzo surface.

If $|\Sigma| = 4, 6, 7$, the G-Sarkisov link results in a Yasinsky, Geiser, Bertini involution of the surface S, respectively. Composing χ with these involutions (if any), we may assume that $|\Sigma| \notin \{4,6,7\}$. Thus, $|\Sigma| = 2$, and

$$\Sigma = (0,0)\cup(\infty,\infty) \quad \text{or} \quad \Sigma = (0,\infty)\cup(\infty,0).$$

Then $\tilde{S}$ is a smooth del Pezzo surface of degree 6 with $\mathrm{Pic}(\tilde{S})^G \simeq \mathbb{Z}^2$, and the corresponding G-Sarkisov link is:

(2.3)
$$\begin{array}{ccc} & \tilde{S} & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ S & \overset{\phi}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

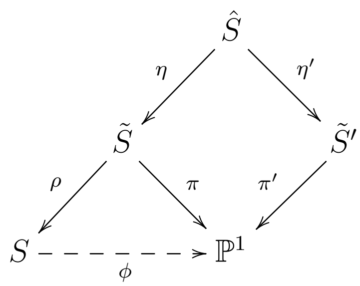
where π is a conic bundle, and ϕ the rational map given by

$$(x,y) \mapsto \begin{cases} \dfrac{x}{y} \text{ if } \Sigma = (0,0)\cup(\infty,\infty), \\ xy \text{ if } \Sigma = (0,\infty)\cup(\infty,0). \end{cases}$$

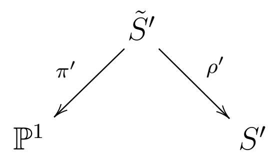
The second G-Sarkisov link used in the decomposition of χ starts at $\tilde{S}$, and is determined by a G-orbit $\tilde{\Sigma} \subset \tilde{S}$ that satisfies the following conditions:

($\diamondsuit$) each smooth fiber of π contains at most one point in $\tilde{\Sigma}$,

($\heartsuit$) no points of $\tilde{\Sigma}$ are contained in singular fibers of π, Let $\eta\colon \hat{S} \to \tilde{S}$ be the blow up of $\tilde{\Sigma}$. We expand (2.3) to the G-commutative diagram:

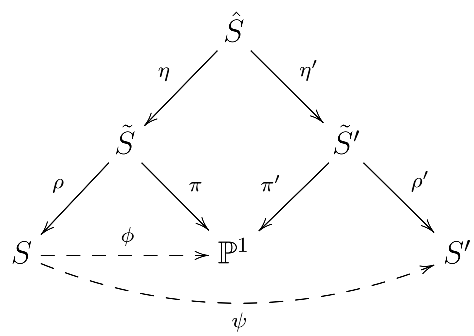


where η' is the contraction of the strict transforms of the fibers of π that contain points of $\tilde{\Sigma}$, and π' is a conic bundle. Moreover, it follows from [Isk80, Theorem 5] that $\tilde{S}'$ is a del Pezzo surface of degree 6 with $\mathrm{Pic}(\tilde{S}')^G \simeq \mathbb{Z}^2$. Thus, the third G-Sarkisov link used in the decomposition of χ is



where $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathrm{Pic}(S')^G = \mathbb{Z}$, and ρ' is a blow up of a G-orbit of length 2. Let $\psi\colon S \dashrightarrow S'$ be the constructed G-birational map. Then ψ is a composition of three G-Sarkisov links, which are combined in the following G-equivariant commutative diagram:

(2.4)



By construction,

$$\chi = \chi' \circ \psi,$$

where $\chi'\colon S' \dashrightarrow S$ is a G-birational map that can be decomposed into a fewer number of G-Sarkisov links than the original χ.

If the surfaces S and S' in (2.4) are G-biregular, we may choose

$$\psi \in \mathrm{Bir}^G(S).$$

A priori, this may not be the case, cf. [Tre23, Theorem 2.9] for the arithmetic counterpart. However, the classification in Proposition 2.7 implies that this is actually the case:

Proposition 2.10. *The surfaces S and S' in* (2.4) *are G-biregular.*

The proof of this result follows from a case by case analysis: we find explicit formulas for ψ in each case and verify that $\psi \in \mathrm{Bir}^G(S)$. This is done in Lemmas 2.12, 2.13, 2.14, 2.15, 2.16 below.

Note that the construction of ψ may fail for some groups G listed in Proposition 2.7. Namely, ψ in (2.4) is determined by the choice of

$$\Sigma = (0,0) \cup (\infty,\infty) \quad \text{or} \quad \Sigma = (0,\infty) \cup (\infty,0),$$

and the choice of a G-orbit $\tilde{\Sigma} \subset \tilde{S}$ satisfying both ($\diamondsuit$) and ($\heartsuit$). It can happen that $\tilde{S}$ simply does not contain G-orbits that satisfy these conditions.

Remark 2.11. It is easy to check whether or not $\tilde{S}$ contains a G-orbit that satisfies ($\diamondsuit$) and ($\heartsuit$). Since ϕ in (2.4) is G-equivariant, it induces an exact sequence of groups:

$$1 \to G_\phi \to G \to G_{\mathbb{P}^1} \to 1,$$

where G_ϕ is the kernel of the G-action on $\mathbb{P}^1$ and $G_{\mathbb{P}^1}$ is the image of G in $\mathrm{Aut}(\mathbb{P}^1)$. Conditions ($\diamondsuit$) and ($\heartsuit$) imply that G_ϕ is cyclic. Conversely, if G_ϕ is cyclic, the G-orbit of a general point in $\tilde{S}$ satisfies ($\diamondsuit$) and ($\heartsuit$).

Using this observation, we obtain:

Lemma 2.12. *If G is the group in Cases (1) or (2) in Proposition 2.7, then $\mathrm{Bir}^G(S)$ is generated by Bertini, Geiser and Yasinsky involutions, and*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

Moreover, if $|G| > 28$ then $\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S)$.

Proof. As explained, every $\chi \in \mathrm{Bir}^G(S)$ can be decomposed into a composition of G-Sarkisov links. Every G-Sarkisov link that starts at S gives either a Bertini involution, or a Geiser involution, or a Yasinsky involution, or a G-birational map described in (2.4). On the other hand, by Remark 2.11, the map in (2.4) does not exist in our case, because G_ϕ in Remark 2.11 is not cyclic. Indeed, if ϕ is

$$(x, y) \mapsto \frac{x}{y},$$

then, in Case (1), G_ϕ contains

$$(x, y) \mapsto (\zeta_n x, \zeta_n y), \quad (x, y) \mapsto (\frac{1}{y}, \frac{1}{x}),$$

and in Case (2), G_ϕ contains

$$(x, y) \mapsto (\zeta_n x, \zeta_n y), \quad (x, y) \mapsto (\frac{\zeta_{2n}}{y}, \frac{\zeta_{2n}}{x}).$$

Similarly, if ϕ is

$$(x, y) \mapsto xy,$$

then, in Case (1), G_ϕ contains

$$(x, y) \mapsto (\zeta_n x, \zeta_n^{-1} y), \quad (x, y) \mapsto (y, x),$$

and in Case (2), G_ϕ contains

$$(x, y) \mapsto (\zeta_n x, \zeta_n^{-1} y), \quad (x, y) \mapsto (\zeta_{2n}^{-1} y, \zeta_{2n} x).$$

We conclude that $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, Geiser involutions, and Yasinsky involutions. Moreover, by [CTZ25, Remark 2.3] and Lemma 2.1, these birational involutions lie in the kernel of $\bar{\beta}$, so

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)),$$

as claimed.

If $|G| > 28$, then S does not contain G-orbits of length 4, 6 and 7, and the classification of G-Sarkisov links implies that $\mathrm{Bir}^G(S)$ does not contain Bertini, Geiser or Yasinsky involutions, so

$$\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S).$$

□

We proceed to groups described in Case (3) in Proposition 2.7.

Lemma 2.13. *Suppose that G is in Case (3) in Proposition 2.7.*

- *If n is even or $r \neq n$, then $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, Geiser involutions, and Yasinsky involutions, and*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

- *If n is odd and $r = n$, then* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$*, Bertini involutions, Geiser involutions, Yasinsky involutions, and the following birational transformations:*

$$(x_1, x_2) \times (y_1, y_2) \mapsto (r_1, r_2) \times (t_1, t_2), \tag{2.5}$$

where

$$r_1 = x_2^{\frac{n+1}{2}} y_1 y_2^{\frac{n-1}{2}} - x_1^{\frac{n+1}{2}} y_1^{\frac{n+1}{2}}, \qquad r_2 = x_2^{\frac{n+1}{2}} y_2^{\frac{n+1}{2}} - x_1^{\frac{n-1}{2}} x_2 y_1^{\frac{n+1}{2}},$$
$$t_1 = x_1 x_2^{\frac{n-1}{2}} y_2^{\frac{n+1}{2}} - x_1^{\frac{n+1}{2}} y_1^{\frac{n+1}{2}}, \qquad t_2 = x_2^{\frac{n+1}{2}} y_2^{\frac{n+1}{2}} - x_1^{\frac{n+1}{2}} y_1^{\frac{n-1}{2}} y_2,$$

and

$$(x_1, x_2) \times (y_1, y_2) \mapsto (f_1, f_2) \times (g_1, g_2), \tag{2.6}$$

where

$$f_1 = -s^n x_1^{n+1} y_1^n + (s^{2n}+1) x_1^{\frac{n+1}{2}} x_2^{\frac{n+1}{2}} y_1^{\frac{n+1}{2}} y_2^{\frac{n-1}{2}} - s^n x_1 x_2^n y_2^n,$$
$$f_2 = (s^{2n}+1) x_1^{\frac{n+1}{2}} x_2^{\frac{n+1}{2}} y_1^{\frac{n-1}{2}} y_2^{\frac{n+1}{2}} - s^n x_2^{n+1} y_2^n - s^n x_1^n x_2 y_1^n,$$
$$g_1 = -s^n y_1^{n+1} x_1^n + (s^{2n}+1) y_1^{\frac{n+1}{2}} y_2^{\frac{n+1}{2}} x_1^{\frac{n+1}{2}} x_2^{\frac{n-1}{2}} - s^n y_1 y_2^n x_2^n,$$
$$g_2 = (s^{2n}+1) y_1^{\frac{n+1}{2}} y_2^{\frac{n+1}{2}} x_1^{\frac{n-1}{2}} x_2^{\frac{n+1}{2}} - s^n y_2^{n+1} x_2^n - s^n y_1^n y_2 x_1^n,$$

where $s \in k^\times$ and $s^{2n} \neq -1$. In this case, the image of the map (2.6) *under β is trivial, and the image of the map* (2.5) *under $\bar\beta$ is a nontrivial element in* $\mathrm{Out}(G)$ *which does not lie in* $\bar\beta(\mathrm{Aut}^G(S))$.

Proof. The first assertion is proved using the same argument as in the proof of Lemma 2.12. We consider the case when n is odd and $r = n$. It suffices to show that ψ in (2.4) is a G-birational map given by (2.5) or (2.6). If $\Sigma = (0,0) \cup (\infty,\infty)$, $\tilde S$ does not have G-orbits that satisfy both ($\diamondsuit$) and ($\heartsuit$), and we see that $\Sigma = (0,\infty) \cup (\infty, 0)$, and ϕ in (2.4) is given by $(x,y) \mapsto xy$. Similarly,

$$\tilde\Sigma \subset \tilde C \subset \tilde S,$$

where $\tilde C$ is the strict transform of $C \subset S$ given by $x = y$. The curve C is pointwise fixed by the involution $(x,y) \mapsto (y,x)$ in G. Note that $\tilde C$ is a 2-section of the conic bundle π. It follows that Σ is the G-orbit of the strict transform of the point $(s,s) \in \mathbb{P}^1 \times \mathbb{P}^1$ with $s \in k^\times$. We have two cases:

(1) $s^{2n} = 1$ and Σ has length n,
(2) $s^{2n} \neq 1$ and Σ has length $2n$.

In the second case, we also have $s^{2n}+1\neq 0$, since otherwise $\tilde{\Sigma}$ does not satisfy ($\diamondsuit$). In the first case, (2.5) either gives ψ in (2.4) or ψ conjugated by the involution $(x,y)\mapsto(-x,-y)$, which is contained in $\mathrm{Aut}^G(S)$. In the second case, ψ is given by (2.6).

The formulas (2.5) and (2.6) can be found as follows. Let E_1 and E_2 be ρ-exceptional curves, let L_1 and L_2 be general fibers of the projections to the first and the second factor of $S=\mathbb{P}^1\times\mathbb{P}^1$, respectively, let H_1 and H_2 be the strict transforms on $\tilde{S}$ of the curves L_1 and L_2, let E_1' and E_2' be the ρ'-exceptional curves, let L_1' and L_2' be general fibers of the projections to the first and the second factor of $S'=\mathbb{P}^1\times\mathbb{P}^1$, respectively, let H_1' and H_2' be the strict transforms on $\tilde{S}'$ of the curves L_1 and L_2, and let $\tilde{E}_1'$, $\tilde{E}_2'$, $\tilde{L}_1'$, $\tilde{L}_2'$ be the strict transforms on $\tilde{S}$ of the curves E_1', E_2', L_1', L_2', respectively. Then

$$\tilde{E}_1'\sim aH_1+bH_2-m_1E_1-m_2E_2$$

for some non-negative integers a, b, m_1, m_2. Moreover, since $\tilde{E}_1'$ and $\tilde{E}_2'$ are swapped by G-action, we see that

$$\tilde{E}_2'\sim bH_1+aH_2-m_2E_1-m_1E_2.$$

Furthermore, by construction,

$$a^2+b^2-2m_1m_2=\tilde{E}_1'\cdot\tilde{E}_2'=|\Sigma|$$

and

$$2ab-m_1^2-m_2^2=\big(\tilde{E}_1'\big)^2=\big(\tilde{E}_1'\big)^2=-1+|\Sigma|.$$

Since $\tilde{E}_1'$ and $\tilde{E}_1'$ are sections of π, we also have

$$a+b-m_1-m_2=\tilde{E}_1'\cdot(H_1+H_2-E_1-E_2)=\tilde{E}_2'\cdot(H_1+H_2-E_1-E_2)=1.$$

Recall that the conic bundle π has two reducible fibers (over the points 0 and ∞). Let ℓ_0, ℓ_0', ℓ_∞, ℓ_∞' be their irreducible components such that $\pi(\ell_0)=\pi(\ell_0')=0$, $\pi(\ell_\infty)=\pi(\ell_\infty')=\infty$. Then we may assume that $\ell_0\sim H_1-E_1$, $\ell_0'\sim H_2-E_2$, $\ell_\infty\sim H_2-E_1$, $\ell_\infty'\sim H_1-E_2$, so $\ell_0\cap E_1\neq\varnothing$, $\ell_0'\cap E_2\neq\varnothing$, $\ell_\infty\cap E_1\neq\varnothing$, $\ell_\infty'\cap E_2\neq\varnothing$. Without loss of generality, we may assume that $\tilde{E}_1'\cap\ell_0\neq\varnothing$ and $\tilde{E}_2'\cap\ell_0'\neq\varnothing$, so

$$\tilde{E}_1'\cdot\ell_0=1,\quad \tilde{E}_1'\cdot\ell_0'=0,\quad \tilde{E}_2'\cdot\ell_0=0,\quad \tilde{E}_2'\cdot\ell_0'=1.$$

This gives us

$$b-m_1=\tilde{E}_2'\cdot\ell_0'=\tilde{E}_1'\cdot\ell_0=1$$

and

$$a-m_2=\tilde{E}_2'\cdot\ell_0=\tilde{E}_1'\cdot\ell_0'=0.$$

Hence, we have one of the following possibilities:

(1) either $\tilde{E}_1' \cdot \ell_\infty = 1$, $\tilde{E}_1' \cdot \ell'_\infty = 0$, $\tilde{E}_2' \cdot \ell_\infty = 0$, $\tilde{E}_2' \cdot \ell'_\infty = 1$, or
(2) $\tilde{E}_1' \cdot \ell_\infty = 0$, $\tilde{E}_1' \cdot \ell'_\infty = 1$, $\tilde{E}_2' \cdot \ell_\infty = 1$, $\tilde{E}_2' \cdot \ell'_\infty = 0$.

In the first case,

$$a - m_1 = \tilde{E}_2' \cdot \ell'_\infty = \tilde{E}_1' \cdot \ell_\infty = 1$$

and

$$b - m_2 = \tilde{E}_2' \cdot \ell_\infty = \tilde{E}_1' \cdot \ell'_\infty = 0,$$

which gives $a = b = m_2 = \frac{|\tilde{\Sigma}|}{2}$ and $m_1 = \frac{|\tilde{\Sigma}|-2}{2}$, so, in particular, $|\tilde{\Sigma}|$ is even. This gives

$$\tilde{H}_1' \sim \frac{|\tilde{\Sigma}|+2}{2} H_1 + \frac{|\tilde{\Sigma}|}{2} H_2 - \frac{|\tilde{\Sigma}|}{2}(E_1 + E_2),$$

and

$$\tilde{H}_2' \sim \frac{|\tilde{\Sigma}|}{2} H_1 + \frac{|\tilde{\Sigma}|+2}{2} H_2 - \frac{|\Sigma|}{2}(E_1 + E_2),$$

because

$$H_1 + H_2 - E_1 - E_2 \sim \tilde{H}_1' + \tilde{H}_2' - \tilde{E}_1' - \tilde{E}_2'.$$

In the second case, we have

$$a - m_1 = \tilde{E}_2' \cdot \ell'_\infty = \tilde{E}_1' \cdot \ell_\infty = 0$$

and

$$b - m_2 = \tilde{E}_2' \cdot \ell_\infty = \tilde{E}_1' \cdot \ell'_\infty = 1,$$

which gives $b = m_1 = m_2 = \frac{|\Sigma|-1}{2}$ and $a = \frac{|\Sigma|+1}{2}$. In particular, $|\Sigma|$ is odd. This gives

$$\tilde{H}_1' \sim \frac{|\tilde{\Sigma}|+1}{2}(H_1 + H_2) - \frac{|\tilde{\Sigma}|+1}{2} E_1 - \frac{|\tilde{\Sigma}|-1}{2} E_2,$$

and

$$\tilde{H}_2' \sim \frac{|\tilde{\Sigma}|+1}{2}(H_1 + H_2) - \frac{|\tilde{\Sigma}|-1}{2} E_1 - \frac{|\tilde{\Sigma}|+1}{2} E_2.$$

Now, we can find pencils in the linear systems $|\tilde{H}_1'|$ and $|\tilde{H}_2'|$ that consist of curves passing through $\tilde{\Sigma}$. Choosing an appropriate basis in each of these pencils, we obtain an explicit equation for ψ in (2.4). In particular, if $s = 1$, this map is given by (2.5). If $s = -1$, this map is given by (2.5), conjugated by the involution $(x, y) \mapsto (-x, -y)$. Finally, if $s \neq \pm 1$, then ψ is given by (2.6).

Note that when $n = r$ is odd, $G \simeq C_2 \times \mathfrak{D}_n$. One can check that (2.6) commutes with the actions, and thus lies in the kernel of β. Let $\{a, b, c\}$ be a set of generators of G, where a is in the center of G, b is an order

2 element in $\mathfrak{D}_n$, and c is an order n element in $\mathfrak{D}_n$. The image of (2.5) under $\bar{\beta}$ is the outer automorphism of G given by

$$a \mapsto a, \quad b \mapsto ba, \quad c \mapsto c.$$

On the other hand, $\mathrm{Aut}^G(S)$ is generated by G and translation elements in the torus, which cannot give rise to the above outer automorphism. □

We proceed to groups described in Case (4) in Proposition 2.7.

Lemma 2.14. *If G is the group in Case (4) in Proposition 2.7 then*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

Moreover, the following assertions hold:

- *If r is odd, $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, Geiser involutions, and Yasinsky involutions.*
- *If r is even, $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, Geiser involutions, Yasinsky involutions, and the following birational transformations:*

$$(x_1, x_2) \times (y_1, y_2) \mapsto (r_1, r_2) \times (t_1, t_2) \tag{2.7}$$

where

$$r_1 = x_1(x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} + \zeta_4 x_2^{\frac{n}{2}} y_2^{\frac{n}{2}}), \qquad r_2 = x_2(x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} + \zeta_4 x_1^{\frac{n}{2}} y_1^{\frac{n}{2}}),$$
$$t_1 = y_1(x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} - \zeta_4 x_2^{\frac{n}{2}} y_2^{\frac{n}{2}}), \qquad t_2 = y_2(x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} - \zeta_4 x_1^{\frac{n}{2}} y_1^{\frac{n}{2}}),$$

and

$$(x_1, x_2) \times (y_1, y_2) \mapsto (f_1, f_2) \times (g_1, g_2) \tag{2.8}$$

where

$$f_1 = s^{\frac{n}{2}} y_2^n x_1 x_2^n + (1 - s^n) y_1^{\frac{n}{2}} y_2^{\frac{n}{2}} x_2^{\frac{n}{2}} x_1^{\frac{n}{2}+1} - s^{\frac{n}{2}} y_1^n x_1^{n+1},$$
$$f_2 = s^{\frac{n}{2}} y_1^n x_2 x_1^n + (1 - s^n) y_2^{\frac{n}{2}} y_1^{\frac{n}{2}} x_1^{\frac{n}{2}} x_2^{\frac{n}{2}+1} - s^{\frac{n}{2}} y_2^n x_2^{n+1},$$
$$g_1 = s^{\frac{n}{2}} x_2^n y_1 y_2^n - (1 - s^n) x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} y_1^{\frac{n}{2}+1} - s^{\frac{n}{2}} x_1^n y_1^{n+1},$$
$$g_2 = s^{\frac{n}{2}} x_1^n y_2 y_1^n - (1 - s^n) x_2^{\frac{n}{2}} x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} y_2^{\frac{n}{2}+1} - s^{\frac{n}{2}} x_2^n y_2^{n+1},$$

where $s \in k^\times$ such that $s^{2n} \neq 1$.

Proof. Arguing as in the proof of Lemma 2.12, we may assume that r is even. It is enough to show that ψ in (2.4) is given by (2.7) or (2.8), which can be done arguing as in the proof of Lemma 2.13, so we will use the notation introduced in the proof of that lemma. The only difference is that now $\tilde{\Sigma}$ is contained in the union of ρ-exceptional curves $E_1 \cup E_2$, so we have $\tilde{E}_1' = E_1$ and $\tilde{E}_2' = E_2$. This gives

$$\tilde{H}_1' \sim \frac{|\tilde{\Sigma}| + 2}{2} H_1 + \frac{|\tilde{\Sigma}|}{2} H_2 - \frac{|\tilde{\Sigma}|}{2}(E_1 + E_2),$$

and

$$\tilde{H}_2' \sim \frac{|\tilde{\Sigma}|}{2} H_1 + \frac{|\tilde{\Sigma}| + 2}{2} H_2 - \frac{|\tilde{\Sigma}|}{2}(E_1 + E_2),$$

Hence, as in the proof of Lemma 2.13, to find explicit equation of the map ψ in (2.4), we have to find pencils in $|\tilde{H}_1'|$ and $|\tilde{H}_2'|$ consisting of curves that pass through $\tilde{\Sigma}$.

The conic bundle π gives us isomorphisms $E_1 \simeq \mathbb{P}^1$ and $E_2 \simeq \mathbb{P}^1$. Let us use these (explicit) isomorphisms to identify $E_1 = \mathbb{P}^1$ and $E_2 = \mathbb{P}^1$. Fix $s \in E_1 \cap \tilde{\Sigma}$, and let F_s be the fiber of π that passes through s. Then $s \notin \{0, \infty\}$, and F_s is the strict transform of the curve in S given by $xy = s$. Set $m = \frac{n}{2}$. Then $\tilde{\Sigma} \cap E_1$ consists of the points

$$s, \zeta_m s, \zeta_m^2 s, \cdots, \zeta_m^{m-1} s, \frac{\zeta_n}{s}, \zeta_m \frac{\zeta_n}{s}, \zeta_m^2 \frac{\zeta_n}{s}, \cdots, \zeta_m^{m-1} \frac{\zeta_n}{s},$$

and the intersection $\tilde{\Sigma} \cap E_2$ consists of the points

$$\zeta_n s, \zeta_m \zeta_n s, \zeta_m^2 \zeta_n s, \cdots, \zeta_m^{m-1} \zeta_n s, \frac{1}{s}, \zeta_m \frac{1}{s}, \zeta_m^2 \frac{1}{s}, \cdots, \zeta_m^{m-1} \frac{1}{s}.$$

Since $\tilde{\Sigma}$ satisfies ($\diamondsuit$), we have $E_2 \cap F_s \notin \tilde{\Sigma}$, which gives $s^n \neq 1$. Moreover, $|\tilde{\Sigma}| = 2n$ unless $s^{2n} = 1$. If $s^{2n} = 1$, then $|\tilde{\Sigma}| = n$. As in the proof of Lemma 2.13, we find pencils in the linear systems $|\tilde{H}_1'|$ and $|\tilde{H}_2'|$ that consist of curves passing through $\tilde{\Sigma}$, and choose appropriate basis in each of these pencils. This gives us explicit equations for the map ψ in each case. □

Lemma 2.15. *If G is the group in Case (5) in Proposition 2.7 then*

$$\bar{\beta}(\mathrm{Bir}^G(S)) = \bar{\beta}(\mathrm{Aut}^G(S)).$$

Moreover, the following assertions hold:

- *If $r \equiv 0 \pmod 4$ then $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, Geiser involutions, Yasinsky involutions, the birational transformations*

$$(x_1, x_2) \times (y_1, y_2) \mapsto (r_1, r_2) \times (t_1, t_2), \tag{2.9}$$

where

$$r_1 = x_1(x_1^k y_2^k - \zeta_4 y_1^k x_2^k), \qquad r_2 = x_2(x_2^k y_1^k - \zeta_4 y_2^k x_1^k),$$
$$t_1 = y_1(y_1^k x_2^k + \zeta_4 x_1^k y_2^k), \qquad t_2 = y_2(-y_2^k x_1^k - \zeta_4 x_2^k y_1^k),$$

with $k = \frac{n}{r}$, *and the birational transformations*

$$(x_1, x_2) \times (y_1, y_2) \mapsto (f_1, f_2) \times (g_1, g_2) \tag{2.10}$$

where

$$f_1 = x_1(s^k x_1^{2k} y_2^{2k} + (1 - s^{2k}) x_1^k y_1^k x_2^k y_2^k - s^k y_1^{2k} x_2^{2k}),$$
$$f_2 = x_2(s^k x_2^{2k} y_1^{2k} + (1 - s^{2k}) x_2^k y_2^k x_1^k y_1^k - s^k y_2^{2k} x_1^{2k}),$$
$$g_1 = y_1(s^k y_1^{2k} x_2^{2k} - (1 - s^{2k}) y_1^k x_1^k y_2^k x_2^k - s^k x_1^{2k} y_2^{2k}),$$
$$g_2 = y_2(s^k y_2^{2k} x_1^{2k} - (1 - s^{2k}) y_2^k x_2^k y_1^k x_1^k - s^k x_2^{2k} y_1^{2k}),$$

with $k = \frac{n}{r}$, *and* $s \in k^\times$ *such that* $s^{4k} \neq 1$.

- *If* $r \equiv 2 \pmod 4$ *then* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$, *Bertini involutions, Geiser involutions, Yasinsky involutions, the birational transformations* (2.9) *and* (2.10), *and the birational transformations*

$$(x_1, x_2) \times (y_1, y_2) \mapsto (r_1, r_2) \times (t_1, t_2), \tag{2.11}$$

where

$$r_1 = x_1(x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} + \zeta_4 x_2^{\frac{n}{2}} y_2^{\frac{n}{2}}), \qquad r_2 = x_2(x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} + \zeta_4 x_1^{\frac{n}{2}} y_1^{\frac{n}{2}}),$$
$$t_1 = y_1(x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} - \zeta_4 x_2^{\frac{n}{2}} y_2^{\frac{n}{2}}), \qquad t_2 = y_2(x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} - \zeta_4 x_1^{\frac{n}{2}} y_1^{\frac{n}{2}}),$$

and

$$(x_1, x_2) \times (y_1, y_2) \mapsto (f_1, f_2) \times (g_1, g_2), \tag{2.12}$$

where

$$f_1 = s^{\frac{n}{2}} y_2^n x_1 x_2^n + (1 - s^n) y_1^{\frac{n}{2}} y_2^{\frac{n}{2}} x_2^{\frac{n}{2}} x_1^{\frac{n}{2}+1} - s^{\frac{n}{2}} y_1^n x_1^{n+1},$$
$$f_2 = s^{\frac{n}{2}} y_1^n x_2 x_1^n + (1 - s^n) y_2^{\frac{n}{2}} y_1^{\frac{n}{2}} x_1^{\frac{n}{2}} x_2^{\frac{n}{2}+1} - s^{\frac{n}{2}} y_2^n x_2^{n+1},$$
$$g_1 = s^{\frac{n}{2}} x_2^n y_1 y_2^n - (1 - s^n) x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} y_2^{\frac{n}{2}} y_1^{\frac{n}{2}+1} - s^{\frac{n}{2}} x_1^n y_1^{n+1},$$
$$g_2 = s^{\frac{n}{2}} x_1^n y_2 y_1^n - (1 - s^n) x_2^{\frac{n}{2}} x_1^{\frac{n}{2}} y_1^{\frac{n}{2}} y_2^{\frac{n}{2}+1} - s^{\frac{n}{2}} x_2^n y_2^{n+1},$$

for $s \in k^\times$ *such that* $s^{2n} \neq 1$.

Proof. We use notation introduced in the proof of Lemma 2.14. As in the proof of Lemma 2.14, it suffices to consider the case when r is even. The only difference is that now we have the following two possibilities:

(1) either ρ is the blow up of the G-orbit $(0,\infty)\cup(\infty,0)$ and ϕ in (2.4) is given by $(x,y)\mapsto xy$ (as in the proof of Lemmas 2.13 and 2.14),
(2) or ρ is the blow up of the G-orbit $(0,0)\cup(\infty,\infty)$ and ϕ is given by $(x,y)\mapsto\frac{x}{y}$.

In the first case, we proceed exactly as in the proof of Lemma 2.14 to obtain the explicit equation of the birational map ψ in (2.4), which gives us one of the birational maps (2.11) or (2.12). Let us deal with the second case.

As in the proof of Lemma 2.14, we have $\tilde{\Sigma}\subset E_1\cap E_2$, which gives $\tilde{E}_1'=E_1$ and $\tilde{E}_2'=E_2$. This gives

$$\tilde{H}_1'\sim\frac{|\tilde{\Sigma}|+2}{2}H_1+\frac{|\tilde{\Sigma}|}{2}H_2-\frac{|\tilde{\Sigma}|}{2}(E_1+E_2),$$

and

$$\tilde{H}_2'\sim\frac{|\tilde{\Sigma}|}{2}H_1+\frac{|\tilde{\Sigma}|+2}{2}H_2-\frac{|\tilde{\Sigma}|}{2}(E_1+E_2),$$

Similarly, we identify $E_1=\mathbb{P}^1$ and $E_2=\mathbb{P}^1$ using the isomorphisms $E_1\simeq\mathbb{P}^1$ and $E_2\simeq\mathbb{P}^1$ induced by the conic bundle π. Fix $s\in E_1\cap\tilde{\Sigma}$, and let F_s be the fiber of π that passes through s. Then $s\notin\{0,\infty\}$, and F_s is the strict transform of the curve in S given by $x=sy$. Set $m=\frac{n}{r}$. Then $\tilde{\Sigma}\cap E_1$ consists of the points

$$s,\zeta_m s,\zeta_m^2 s,\cdots,\zeta_m^{m-1}s,\frac{\zeta_{2m}}{s},\zeta_m\frac{\zeta_{2m}}{s},\zeta_m^2\frac{\zeta_{2m}}{s},\cdots,\zeta_m^{m-1}\frac{\zeta_{2m}}{s},$$

and the intersection $\tilde{\Sigma}\cap E_2$ consists of the points

$$\frac{1}{s},\zeta_m\frac{1}{s},\zeta_m^2\frac{1}{s},\cdots,\zeta_m^{m-1}\frac{1}{s},\zeta_{2m}s,\zeta_m\zeta_{2m}s,\zeta_m^2\zeta_{2m}s,\cdots,\zeta_m^{m-1}\zeta_{2m}s.$$

Hence, $s^{2m}\neq1$, because $E_2\cap F_s\notin\tilde{\Sigma}$. We have that $|\tilde{\Sigma}|=4m$ when $s^{4m}\neq1$, and $|\tilde{\Sigma}|=2m$ when $s^{4m}=1$. We can explicitly find pencils in the linear systems $|\tilde{H}_1'|$ and $|\tilde{H}_2'|$ that consist of curves passing through $\tilde{\Sigma}$, choose appropriate basis in each of these pencils, and obtain an explicit equation for the birational map ψ in (2.4) in each case. This gives the maps (2.9) and (2.10). □

Arguing as in the proofs of Lemmas 2.13, 2.14, 2.15, we obtain

Lemma 2.16. *If G is the group in Case (6) in Proposition 2.7 then*

$$\bar{\beta}(\mathrm{Bir}^G(S))=\bar{\beta}(\mathrm{Aut}^G(S)).$$

Moreover, the following assertions hold:

- *If* $r \equiv 2 \pmod 4$*, then* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$*, Bertini involutions, Geiser involutions, Yasinsky involutions, and birational transformations* (2.9) *and* (2.10)*.*
- *If* $r \equiv 0 \pmod 4$*, then* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$*, Bertini involutions, Geiser involutions, Yasinsky involutions, and birational transformations* (2.9)*,* (2.10)*,* (2.11)*, and* (2.12)*.*

3. Degree 6 Del Pezzo surfaces

Let S be the degree 6 del Pezzo surface, i.e., the blowup of $\mathbb{P}^2$ in

$$[1:0:0], \quad [0:1:0], \quad [0:0:1].$$

Its automorphism group fits into a split exact sequence

$$1 \to \mathbb{G}_m^2 \to \mathrm{Aut}(S) \xrightarrow{\nu} \mathfrak{S}_3 \times \mathfrak{C}_2 \to 1.$$

We identify $\mathrm{Aut}(S)$ with the group generated by the natural $\mathbb{G}_m^2$-action on $\mathbb{P}^2_{x_1,x_2,x_3}$, a permutation action of

$$\mathfrak{S}_3 = \langle \sigma_{123}, \sigma_{12} \rangle,$$

$$\sigma_{123} : (x_1,x_2,x_3) \mapsto (x_2,x_3,x_1), \quad \sigma_{12} : (x_1,x_2,x_3) \mapsto (x_2,x_1,x_3),$$

and the standard Cremona involution ι.

Classification of groups. We proceed with the description of finite subgroups $G \subset \mathrm{Aut}(S)$, following [DI09, Theorem 6.3]. We have an exact sequence

$$1 \to G_T \to G \xrightarrow{\nu} \mathfrak{S}_3 \times \mathfrak{C}_2 \simeq \mathfrak{D}_6.$$

Assuming that $\mathrm{rkPic}(S)^G = 1$, we have

$$\nu(G) = \mathfrak{S}_3 \times \mathfrak{C}_2, \quad \mathfrak{C}_3 \times \mathfrak{C}_2, \text{ or } \mathfrak{S}_3',$$

where $\mathfrak{S}_3'$ is the *twisted*, by the center, subgroup of $\mathfrak{D}_6$.

Proposition 3.1. *Up to conjugation in* $\mathrm{Aut}(S)$*, one of the following holds:*

(1) $G \simeq \mathfrak{C}_n^2 \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2)$ *is generated by*

$$\mathrm{diag}(\zeta_n,1,1), \quad \mathrm{diag}(1,\zeta_n,1), \quad \sigma_{123}, \quad \sigma_{12}, \quad \iota.$$

(2) $G \simeq (\mathfrak{C}_n \times \mathfrak{C}_{n/3}) \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2)$ *is generated by*

$$\mathrm{diag}(\zeta_n^3,1,1), \quad \mathrm{diag}(\zeta_n^2,\zeta_n,1), \quad \sigma_{123}, \quad \sigma_{12}, \quad \iota,$$

where $3 \mid n$*.*

(3) $G \simeq \mathfrak{C}_n^2 \rtimes \mathfrak{S}_3$ *is generated by*

$$\mathrm{diag}(\zeta_n,1,1), \quad \mathrm{diag}(1,\zeta_n,1), \quad \sigma_{123}, \quad \sigma_{12} \cdot \iota.$$

(4) $G \simeq (\mathfrak{C}_n \times \mathfrak{C}_{n/3}) \rtimes \mathfrak{S}_3$ *is generated by*

$$\mathrm{diag}(\zeta_n^3, 1, 1), \quad \mathrm{diag}(\zeta_n^2, \zeta_n, 1), \quad \sigma_{123}, \quad \sigma_{12} \cdot \iota,$$

where $3 \mid n$.

(5) $G \simeq \mathfrak{C}_n^2 \rtimes \mathfrak{C}_6$ *is generated by*

$$\mathrm{diag}(\zeta_n, 1, 1), \quad \mathrm{diag}(1, \zeta_n, 1), \quad \sigma_{123} \cdot \iota.$$

(6) $G \simeq (\mathfrak{C}_n \times \mathfrak{C}_{n/r}) \rtimes \mathfrak{C}_6$ *is generated by*

$$\mathrm{diag}(\zeta_n^r, 1, 1), \quad \mathrm{diag}(\zeta_n^s, \zeta_n, 1), \quad \sigma_{123} \cdot \iota,$$

where $r \geq 1$, $r \mid n$, *and* $s^2 - s + 1 \equiv 0 \pmod r$.

Proof. When $\nu(G) = \mathfrak{S}_3 \times \mathfrak{C}_2$, G is generated by a subgroup H, which acts birationally to a transitive but imprimitive action on $\mathbb{P}^2$, and a Cremona involution on $\mathbb{P}^2_{x_1,x_2,x_3}$ given by

$$\iota_{ab} : (x_1, x_2, x_3) \mapsto (\frac{1}{x_1}, \frac{a}{x_2}, \frac{b}{x_3}), \quad a, b \in k^\times.$$

By [DI09], H is generated by G_T, σ_{123} and σ_{12}, where $G_T \simeq \mathfrak{C}_n^2$ or $\mathfrak{C}_n \times \mathfrak{C}_{n/3}$. Observe that

$$(\sigma_{123}^2 \cdot \sigma_{12} \cdot \iota_{ab} \cdot \sigma_{12} \cdot \iota_{ab} \cdot \sigma_{123})^{-1} = \mathrm{diag}(a, 1, a^2). \tag{3.1}$$

Multiplying ι_{ab} by (3.1), we may assume that $a = 1$. Then

$$\sigma_{123}^2 \cdot \iota_{ab} \cdot \sigma_{123} \cdot \iota_{ab} = \mathrm{diag}(b^2, b, 1).$$

When $G_T = \mathfrak{C}_n^2$, for some n, it follows that $\mathrm{diag}(1, 1, b) \in G_T$. We may assume $\iota_{ab} = \iota$ is the standard Cremona transformation and we obtain Case (1).

When $G_T = \mathfrak{C}_n \times \mathfrak{C}_{n/3}$, for some n divisible by 3, we know that b is a power of ζ_n. Up to multiplying ι_{ab} by a power of $\mathrm{diag}(1, 1, \zeta_n^3) \in G_T$, we may assume that $b = 1$ or ζ_3. These two choices give conjugated subgroups. Indeed, the groups

$$\langle \mathrm{diag}(\zeta_n^3, 1, 1), \quad \mathrm{diag}(\zeta_n^2, \zeta_n, 1), \quad \sigma_{123}, \quad \sigma_{12}, \quad \iota \rangle,$$

and

$$\langle \mathrm{diag}(\zeta_n^3, 1, 1), \quad \mathrm{diag}(\zeta_n^2, \zeta_n, 1), \quad \sigma_{123}, \quad \sigma_{12}, \quad \iota \cdot \mathrm{diag}(1, 1, \zeta_3) \rangle$$

are conjugated in $\mathrm{Aut}(S)$ via

$$(x_1, x_2, x_3) \mapsto (\frac{1}{x_1}, \frac{1}{x_2}, \frac{\zeta_3^2}{x_3}).$$

We obtain Case (2).

When $\nu(G) = \mathfrak{S}_3'$, the same argument gives rise to Case (3) and (4). When $\nu(G) = \mathfrak{C}_6$, G is generated by G_T and

$$\tau : (x_1, x_2, x_3) \mapsto (\frac{a}{x_2}, \frac{b}{x_3}, \frac{1}{x_1}).$$

Up to conjugation by

$$(x_1, x_2, x_3) \mapsto (\frac{a}{b}x_1, ax_2, x_3),$$

we may assume that $a = b = 1$, i.e., $\tau = \sigma_{123} \cdot \iota$. By [DI09, Theorem 4.7], there are two possibilities for G_T, giving rise to Case (5) and (6) in the assertion. □

Sarkisov links. By [DI09, Section 7], every Sarkisov link starting from S is of type II, i.e., of the form

$$\begin{array}{ccc} & \tilde{S} & \\ {}^{\rho}\swarrow & & \searrow^{\rho'} \\ S & \overset{\chi}{\dashrightarrow} & S' \end{array}$$

where ρ and ρ' are blowups of $d \leq 5$ points in general position on S', and one of the following holds:

- $d = 5$, S' is G-biregular to S, χ is the Bertini involution,
- $d = 4$, S' is G-biregular to S, χ is the Geiser involution,
- $d = 3$, S' is also a dP_6,
- $d = 2$, S' is also a dP_6,
- $d = 1$, $S' = \mathbb{P}^1 \times \mathbb{P}^1$.

***G*-Rigidity.** We introduce special finite subgroups of $\mathrm{Aut}(S)$:

$$G_1 = \langle \mathrm{diag}(-1, 1, 1), \sigma_{123}, \sigma_{12}, \iota \rangle \simeq (\mathfrak{C}_2)^2 \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2) \simeq \mathfrak{C}_2 \times \mathfrak{S}_4,$$

$$G_2 = \langle \mathrm{diag}(-1, 1, 1), \sigma_{123}, \sigma_{12} \cdot \iota \rangle \simeq (\mathfrak{C}_2)^2 \rtimes \mathfrak{S}_3 \simeq \mathfrak{S}_4,$$

$$G_3 = \langle \mathrm{diag}(-1, 1, 1), \sigma_{123} \cdot \iota \rangle \simeq (\mathfrak{C}_2)^2 \rtimes \mathfrak{C}_6 \simeq \mathfrak{C}_2 \times \mathfrak{A}_4,$$

$$G_4 = \langle \mathrm{diag}(\zeta_3, \zeta_3^2, 1), \sigma_{123}, \sigma_{12}, \iota \rangle \simeq \mathfrak{C}_3 \rtimes (\mathfrak{S}_3 \times \mathfrak{C}_2) \simeq \mathfrak{S}_3^2,$$

$$G_5 = \langle \mathrm{diag}(\zeta_3, \zeta_3^2, 1), \sigma_{123}, \sigma_{12} \cdot \iota \rangle \simeq \mathfrak{C}_3 \times \mathfrak{S}_3 \simeq \mathfrak{C}_3 \rtimes \mathfrak{C}_6,$$

$$G_6 = \langle \mathrm{diag}(\zeta_3, \zeta_3^2, 1), \sigma_{123} \cdot \iota \rangle \simeq \mathfrak{C}_3 \rtimes \mathfrak{C}_6 \simeq \mathfrak{C}_3 \times \mathfrak{S}_3,$$

$$G_7 = \langle \sigma_{123}, \sigma_{12}, \iota \rangle \simeq \mathfrak{S}_3 \times \mathfrak{C}_2,$$

$$G_8 = \langle \sigma_{123}, \sigma_{12} \cdot \iota \rangle \simeq \mathfrak{S}_3,$$

$$G_9 = \langle \sigma_{123} \cdot \iota \rangle \simeq \mathfrak{C}_6.$$

Observe that G_1 contains G_2 and G_3; G_4 contains $G_5 \simeq G_6$; and G_7 contains G_8 and G_9.

If G is G_1, G_2, or G_3, then S contains a unique G-orbit of length 4. Blowing it up, we obtain the G-equivariant commutative diagram:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \varphi\ } & \widetilde{S} \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\varpi} \\ S & \overset{\eta}{\dashrightarrow} & S \end{array} \tag{3.2}$$

where ϖ is the blowup of the G-orbit of length 4, $\widetilde{S}$ is the Fermat del Pezzo surface of degree 2, φ is a biregular involution, and η is a Geiser birational involution that centralizes G.

Similarly, S has a unique G_4-orbit of length 3, whose points are in general position, which is also a G_5-orbit and G_6-orbit. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{S} & \xrightarrow{\ \phi\ } & \widehat{S} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ S & \overset{\tau}{\dashrightarrow} & S \end{array} \tag{3.3}$$

where ρ is the blowup of the orbit of length 3, $\widehat{S}$ is the Fermat cubic surface, ϕ is a biregular involution, and τ is a birational involution such that

$$\tau G_4 \tau = G_4,$$

so τ normalizes G_4, and $\langle G_4, \tau\rangle \simeq \mathfrak{S}_3 \wr \mathfrak{C}_2$ has $\texttt{GAPID}(72, 40)$. It was noticed in [Yas25], that τ does not normalize G_5 and G_6: we have

$$\tau G_5 \tau = G_6.$$

In particular, G_5 and G_6 are conjugate in Cr_2, so S is neither G_5-birationally rigid nor G_6-birationally rigid.

Recall from [Isk03, Isk08] that the action of G_7 on S is not linearizable, and S is G_7-birational to a conic bundle. Hence, S is not G_7-solid. One can show that $\mathrm{Aut}^{G_7}(S) = G_7$, so $\bar{\beta}(\mathrm{Aut}^{G_7}(S))$ is trivial. Note also that G_7 fixes a point in S, so blowing up the G_7-fixed point, we get a G_7-Sarkisov link that ends at $\mathbb{P}^1 \times \mathbb{P}^1$, and the G_7-action on $\mathbb{P}^1 \times \mathbb{P}^1$ has been studied in Lemma 2.9, which yields

$$\bar{\beta}(\mathrm{Bir}^{G_7}(S)) = \mathrm{Out}(G_7) \simeq \mathfrak{C}_2.$$

Finally, we recall from [PSY24] that the actions of G_8 and G_9 on S are linearizable. Indeed, if G is one of these groups, then G fixes a point in S, and blowing it up we obtain a G-birational map to $\mathbb{P}^1 \times \mathbb{P}^1$, with G fixing a point in $\mathbb{P}^1 \times \mathbb{P}^1$, so the G-action is linearizable.

Theorem 3.2. *If G is one of the groups in Proposition 3.1 then:*

- *The surface S is G-birationally superrigid if and only if G is not conjugated to one of $G_1, G_2, G_3, G_4, G_5, G_5, G_7, G_8, G_9$.*
- *If G is G_1, G_2, or G_3, then S is G-birationally rigid,* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$*, η given in* (3.2)*, and*
$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)).$$
- *If $G = G_4$, then S is G-birationally rigid,* $\mathrm{Bir}^G(S)$ *is generated by* $\mathrm{Aut}^G(S)$ *and τ given in* (3.3)*,* $\bar{\beta}(\mathrm{Aut}^G(S))$ *is trivial, and*
$$\bar{\beta}(\mathrm{Bir}^G(S)) = \mathrm{Out}(G) \simeq \mathfrak{C}_2.$$
- *If $G = G_5$ or $G = G_6$, then*
$$\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S),$$
the surface S is G-birationally solid, and the only G-Mori fibre spaces that are G-birational are S with G_5 and G_6-actions.

Proof. Using the description of G-Sarkisov links that start at S, we see that such links do not exists if $|G_T| \geq 5$. Note that $|G_T| \neq 2$,5, by Proposition 3.1. Hence,
$$|G_T| \in \{1, 3, 4\}.$$
Using Proposition 3.1 again, we see that G is conjugated to one of the groups $G_1, G_2, G_3, G_4, G_5, G_5, G_7, G_8, G_9$, so we may assume that G is one of these.

If G is G_1, G_2, or G_3, then S has a unique G-orbit of length 4, and this is the only orbit of length ≤ 5. Using the classification of G-Sarkisov links, we see that S is G-birationally rigid, and $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$ and the involution η.

Similarly, if G is G_4, G_5 or G_6, then S has a unique G-orbit of length 3, up to conjugation by $\mathrm{Aut}^G(S)$, whose points are in general position. Moreover, S does not have G-orbits of length 1, 2, 4 or 5. Arguing as above, we see that S is G_4-birationally rigid, and $\mathrm{Bir}^{G_4}(S)$ is generated by $\mathrm{Aut}^{G_4}(S)$ and the involution τ. If $G = G_5$ or $G = G_6$, then the only G-Sarkisov link that starts at S is given by τ, which is not G-birational. This implies that S is G-solid, and the only G-Mori fiber spaces that are G-birational to S are the surface S equipped with G_5 and G_6-actions. This also shows that $\mathrm{Bir}^G(S) = \mathrm{Aut}^G(S)$ as claimed. □

4. Del Pezzo surfaces of degree 4

We follow the presentation in [DI09, Section 6.4].

Classification of groups. There is a natural action of the Weyl group $W(\mathsf{D}_5)$ on $\mathbb{P}^4$, via the projectivization of a faithful 5-dimensional representation. A del Pezzo surface of degree 4 can be given as an intersection of two diagonal quadrics $S := Q_1 \cap Q_2 \subset \mathbb{P}^4$, see, e.g., [DI09, Lemma 6.5]. We have a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{C}_2^4 & \longrightarrow & W(\mathsf{D}_5) & \longrightarrow & \mathfrak{S}_5 & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathfrak{C}_2^4 & \longrightarrow & \mathrm{Aut}(S) & \longrightarrow & \overline{\mathrm{Aut}(S)} & \longrightarrow & 1 \end{array}$$

The possibilities for $\overline{\mathrm{Aut}(S)}$ are

$$\mathfrak{C}_1, \quad \mathfrak{C}_2, \quad \mathfrak{C}_4, \quad \mathfrak{C}_3, \quad \mathfrak{S}_3, \quad \mathfrak{C}_5, \quad \mathfrak{D}_5.$$

Up to projectivity, the corresponding surfaces S are given by

(I) $\mathfrak{C}_1$: $\sum_{j=0}^4 x_j^2 = \sum_{j=1}^4 a_j x_j^2 = 0$, a_j general;

(II) $\mathfrak{C}_2$: $\sum_{j=0}^4 x_j^2 = x_0^2 + ax_1^2 - x_2^2 - ax_3^2 = 0$, $a \neq 0, \pm 1$;

(III) $\mathfrak{C}_4$: $\sum_{j=0}^4 x_j^2 = x_0^2 + \zeta_4 x_1^2 - x_2^2 - \zeta_4 x_3^2 = 0$;

(IV) $\mathfrak{S}_3$: $x_0^2 + \zeta_3 x_1^2 + \zeta_3^2 x_2^2 + x_3^2 = x_0^2 + \zeta_3^2 x_1^2 + \zeta_3 x_2^2 + x_4^2 = 0$;

(V) $\mathfrak{D}_5$: $\sum_{j=0}^4 \zeta_5^j x_j^2 = \sum_{j=0}^4 \zeta_5^{4-j} x_j^2 = 0$.

All cases with $\mathrm{rkPic}(S)^G = 1$ are listed in [DI09, Theorem 6.9]; we recomputed all possibilities and present the actions, with supplementary information. In each type, the generators of $\mathrm{Aut}(S)$ are given by:

(I) ι_i, $i = 1, \ldots, 4$,
(II) $\sigma_{(02)(13)}$, ι_i, $i = 1, \ldots, 4$,
(III) $\sigma_{(0123)}$, ι_i, $i = 1, \ldots, 4$,
(IV) $\sigma_{(12)(34)}$, $\sigma_{(012)} \cdot \tau$, ι_i, $i = 1, \ldots, 4$,
(V) $\sigma_{(01234)}$, $\sigma_{(14)(23)}$, ι_i, $i = 1, \ldots, 4$,

where ι_i is the sign change of x_i, σ_s is the permutation of variables corresponding to $s \in \mathfrak{S}_5$, and

$$\tau = \mathrm{diag}(1, 1, \zeta_3, \zeta_3^2).$$

Type	G	GapID	Generators	G-rigid
I	$\mathfrak{C}_2^2$	(4,2)	ι_i, ι_j	no
I	$\mathfrak{C}_2^3$	(4,2)	$\iota_i, \iota_j, \iota_r$	yes
I	$\mathfrak{C}_2^4$	(4,2)	$\iota_1, \iota_2, \iota_3, \iota_4$	yes
II	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\iota_4, \sigma_{(02)(13)}\iota_i, i = 0, 1$	yes
II	$\mathfrak{D}_4$	(8,3)	$\sigma_{(02)(13)}, \iota_i, i = 0, 1$	no
II	$\mathfrak{C}_2^2 \rtimes \mathfrak{C}_4$	(16,3)	$\sigma_{(02)(13)}\iota_0, \iota_0\iota_1, \iota_0\iota_2$	yes
II	$\mathfrak{C}_2^2 \rtimes \mathfrak{C}_4$	(16,3)	$\sigma_{(02)(13)}\iota_0\iota_1, \iota_0\iota_4, \iota_i, i = 0, 1$	yes
II	$\mathfrak{C}_2 \times \mathfrak{D}_4$	(16,11)	$\sigma_{(02)(13)}, \iota_4, \iota_0\iota_2, \iota_i, i = 0, 1$	yes
II	$\mathfrak{C}_2^2 \wr \mathfrak{C}_2$	(32,27)	$\sigma_{(02)(13)}, \iota_1, \iota_2$	yes
III	$\mathfrak{C}_8$	(8,1)	$\sigma_{(0123)}\iota_2$	no
III	OD16	(16,6)	$\sigma_{(0123)}\iota_2, \iota_0\iota_2$	yes
III	$\mathfrak{C}_2^3.\mathfrak{C}_4$	(32,7)	$\sigma_{(0123)}\iota_2, \iota_0\iota_1$	yes
III	$\mathfrak{C}_2 \wr \mathfrak{C}_4$	(64,32)	$\sigma_{(0123)}, \iota_0$	yes
IV	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_{(012)} \cdot \tau \cdot \iota_4, \iota_3$	no
IV	$\mathfrak{C}_2 \times \mathfrak{A}_4$	(24,13)	$\iota_3\iota_4, \iota_0\iota_1, \sigma_{(012)}\tau$	yes
IV	$\mathfrak{C}_2^2 \times \mathfrak{A}_4$	(48,49)	$\iota_3\iota_4, \iota_0\iota_1, \iota_0\iota_3, \sigma_{(012)}\tau$	yes
IV	$\mathfrak{C}_3 \rtimes \mathfrak{C}_4$	(12,1)	$\sigma_{(012)}\tau, \sigma_{(12)(34)}\iota_4$	no
IV	$\mathfrak{C}_3 \rtimes \mathfrak{D}_4$	(24,8)	$\sigma_{(012)}\tau, \sigma_{(12)(34)}, \iota_4$	no
IV	$\mathfrak{C}_2 \times \mathfrak{S}_4$	(48,48)	$\sigma_{(012)}\tau, \sigma_{(12)(34)}, \iota_3\iota_4, \iota_0\iota_2$	yes
IV	$\mathfrak{A}_4 \rtimes \mathfrak{C}_4$	(48,30)	$\sigma_{(012)}\tau, \iota_0\iota_2, \sigma_{(12)(34)}\iota_0\iota_3$	yes
IV	$\mathsf{GL}_2(\mathbb{F}_4)$	(96,195)	$\sigma_{(012)}\tau, \sigma_{(12)(34)}, \iota_1, \iota_3$	yes
V	$\mathfrak{C}_2^4 \rtimes \mathfrak{C}_5$	(80,49)	$\sigma_{(01234)}\iota_2, \iota_2\iota_3, \iota_4$	yes
V	$\mathfrak{C}_2^4 \rtimes \mathfrak{D}_5$	(160,234)	$\sigma_{(01234)}, \sigma_{(14)(23)}, \iota_1$	yes

Recall from [PSY24] that S is G-birationally rigid if and only if G does not fix points in S, i.e., $S^G = \varnothing$. This explains the entries in the last column. When $S^G = \varnothing$, the classification of G-Sarkisov links implies that every G-birational map from S to a smooth del Pezzo surface S' can be decomposed into a composition of G-biregular maps, Bertini involutions, and Geiser involutions, so, in particular, S' is G-birational to S, and $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini involutions, and Geiser involutions, which implies

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)).$$

Similarly, if S is not G-birationally rigid, our classification of G-actions implies a recent result of Shramov and Trepalin [ST25, Corollary 7.5], cf. also [Ela26, Theorem 3.5]:

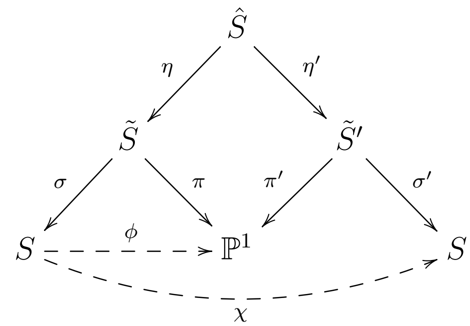
Theorem 4.1. *If there exists a G-birational map $S \dashrightarrow S'$ such that S' is a smooth del Pezzo surface of degree 4, then S and S' are G-biregular.*

However, if $S^G \neq \varnothing$, the group $\mathrm{Bir}^G(S)$ is not always generated by $\mathrm{Aut}^G(S)$, Bertini involutions, and Geiser involutions. The missing generators

$$\chi \in \mathrm{Bir}^G(S)$$

can be decomposed into a sequence of three G-Sarkisov links, given by the following G-equivariant commutative diagram:

(4.1)



where ϕ is the rational map induced by the linear projection from the embedded tangent space $T_P(S) \subset \mathbb{P}^4$ of a G-fixed point $P \in S^G$, and the remaining maps are:

- σ is the blow up of the point P,
- π is a conic bundle,
- η is a blow up of a G-orbit $\tilde{\Sigma}$ such that
  - ($\diamondsuit$) each smooth fiber of π contains at most one point in $\tilde{\Sigma}$,
  - ($\heartsuit$) no points of $\tilde{\Sigma}$ are contained in singular fibers of π,
- η' is the contraction of the strict transforms of the fibers of π that contain points of $\tilde{\Sigma}$, and
- π' is a conic bundle.
- σ' is the blow up of a G-fixed point.

The G-birational self-map χ is determined by a choice of a G-fixed point $P \in S^G$ and a G-orbit $\tilde{\Sigma} \subset \tilde{S}$ that satisfies both ($\diamondsuit$) and ($\heartsuit$). Note that $\tilde{S}$ and $\tilde{S}'$ in (4.1) are smooth cubic surfaces, that need not be G-biregular.

Definition 4.2. A map $\chi \in \mathrm{Bir}^G(S)$ as in (4.1) will be called *Iskovskikh self-map* of S.

Thus, the group $\mathrm{Bir}^G(S)$ is generated by $\mathrm{Aut}^G(S)$, Bertini and Geiser involutions, and Iskovskikh self-maps. Note that Iskovskikh self-maps of S may not exist even if $S^G \neq \varnothing$. However, arguing as in Remark 2.11, we obtain a simple criterion for their existence. Namely, if P is a point in S^G, and $\phi\colon S \dashrightarrow \mathbb{P}^1$ is the rational map induced by the linear projection from the embedded tangent space $T_P(S) \subset \mathbb{P}^4$ of P, then ϕ is G-equivariant, so it induces the exact sequence

$$1 \to G_\phi \to G \to G_{\mathbb{P}^1} \to 1,$$

where G_ϕ is the kernel of the G-action on $\mathbb{P}^1$, and $G_{\mathbb{P}^1}$ is the image of G in $\mathrm{Aut}(\mathbb{P}^1)$. Conditions ($\diamondsuit$) and ($\heartsuit$) imply that G_ϕ is cyclic, because otherwise each smooth fiber of π in (4.1) contains at least two points of $\tilde{\Sigma}$. Conversely, if G_ϕ is cyclic, then $\tilde{S}$ in (4.1) always contains G-orbits that satisfy both ($\diamondsuit$) and ($\heartsuit$).

Corollary 4.3. *If G is isomorphic to*

$$\mathfrak{C}_2^2, \quad \mathfrak{D}_4, \quad \mathfrak{C}_2 \times \mathfrak{C}_6, \quad \textit{or} \quad \mathfrak{C}_3 \rtimes \mathfrak{D}_4$$

then $\mathrm{Bir}^G(S)$ *does not have Iskovskikh self-maps, and is generated by* $\mathrm{Aut}^G(S)$*, Bertini and Geiser involutions. In particular,*

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)).$$

Proof. For every $P \in S^G$, the kernel of $G \to \mathrm{Aut}(\mathbb{P}^1)$ described above contains a subgroup isomorphic to $\mathfrak{C}_2^2$, so Iskovskikh self-maps of S do not exists. ☐

Lemma 4.4. *If $G \simeq \mathfrak{C}_8$ then*

$$\mathfrak{C}_2 \simeq \bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)) \neq \mathrm{Out}(\mathfrak{C}_8) \simeq \mathfrak{C}_2^2.$$

Proof. Recall that S is of type III, G is generated by $\sigma_{(0123)}\iota_2$. Then

$$\mathrm{Aut}^G(S) = \langle G, \iota_0\iota_2 \rangle \simeq \mathrm{OD16},$$

which implies that $\bar{\beta}(\mathrm{Aut}^G(S))$ is a subgroup of order 2 in $\mathrm{Out}(G)$. Recall that $\mathrm{Out}(\mathfrak{C}_8) \simeq \mathfrak{C}_2^2$. If $\phi \in \mathrm{Out}(G)$ sends $\sigma_{(0123)}\iota_2$ to $(\sigma_{(0123)}\iota_2)^3$ then $\phi \notin \bar{\beta}(\mathrm{Aut}^G(S))$. It suffices to show that $\phi \notin \bar{\beta}(\mathrm{Bir}^G(S))$. This can be done using the Burnside formalism: Let g be a generator of G and consider the actions

$$\varphi_1, \varphi_2 : G \to \mathrm{Aut}(X),$$

given by

$$\varphi_1(g) = \sigma_{(0123)}\iota_2 \quad \text{and} \quad \varphi_2(g) = (\sigma_{(0123)}\iota_2)^3.$$

Every G-action on X is given by either φ_1 or φ_2, up to isomorphism. In each case, the G-action is in standard form: it is free on X away from the elliptic curve

$$C := X \cap \{x_4 = 0\},$$

which has generic stabilizer $\mathfrak{C}_2 = \langle \iota_4 \rangle$ and residual $\mathfrak{C}_4$-action. We find incompressible symbols in the Burnside class of each action

$$[X \circlearrowleft \varphi_i(G)]^{\mathrm{inc}} = (\mathfrak{C}_2, \varphi_i'(\mathfrak{C}_4) \circlearrowright k(C), (1)) \in \mathrm{Burn}_2^{\mathrm{inc}}(G), \quad i = 1, 2,$$

where $\varphi_i' : \mathfrak{C}_4 \to \mathrm{Aut}(C)$ gives the corresponding residual action on X. Let g' be a generator of $\mathfrak{C}_4$, then $\varphi_1'(g') = \varphi_2'(g')^3$. The $\mathfrak{C}_4$-action on C fixes two points. We compute the Burnside classes of the $\mathfrak{C}_4$-action on C and obtain

$$[C \circlearrowleft \varphi_1'(\mathfrak{C}_4)] = 2(\mathfrak{C}_4, \mathrm{triv} \circlearrowright k, (1)) + (\mathfrak{C}_2, \mathrm{triv} \circlearrowright k, (1)) \in \mathrm{Burn}_1(\mathfrak{C}_4),$$

$$[C \circlearrowleft \varphi_2'(\mathfrak{C}_4)] = 2(\mathfrak{C}_4, \mathrm{triv} \circlearrowright k, (3)) + (\mathfrak{C}_2, \mathrm{triv} \circlearrowright k, (1)) \in \mathrm{Burn}_1(\mathfrak{C}_4).$$

Since there is no relation in $\mathrm{Burn}_1(\mathfrak{C}_4)$, we see that

$$[C \circlearrowleft \varphi_1'(\mathfrak{C}_4)] \neq [C \circlearrowleft \varphi_2'(\mathfrak{C}_4)],$$

which implies that the actions of $\mathfrak{C}_4$ on C given by φ_1' and φ_2' are not equivariantly birational. It follows that

$$[X \circlearrowleft \varphi_1(G)] \neq [X \circlearrowleft \varphi_2(G)].$$

□

Remark 4.5. One can also prove Lemma 4.4 by describing Iskovskikh self-maps in $\mathrm{Bir}^G(S)$ and their images in $\bar{\beta}(\mathrm{Bir}^G(S)) \subset \mathrm{Out}(G)$. In this case, G fixes two points in S, which are swapped by $\mathrm{Aut}^G(S) \simeq \mathrm{OD16}$, so we let P be one of these points, and Q the other. Consider the diagram (4.1) for σ being the blowup of P and some G-orbit $\tilde{\Sigma} \subset \tilde{S}$. Then $G_\phi \simeq \mathfrak{C}_2$, where G_ϕ is the kernel of the G-action on $\mathbb{P}^1$ induced by ϕ in (4.1) and (♢) is equivalent to

$$\tilde{\Sigma} \subset \widetilde{S}_\phi^G,$$

where $\widetilde{S}_\phi^G$ is the strict transform of the curve $\{x_4\} \subset S$. Then either $\tilde{\Sigma}$ is the G-fixed point that is mapped to Q, or $|\Sigma| = 4$ (general case). In the former case, the Iskovskikh self-map χ is the $\mathrm{Aut}^G(S)$-equivariant Geiser involution induced by blowing up S at P and Q, so it centralizes G, and lies in the kernel of $\bar{\beta} \colon \mathrm{Bir}^G(S) \to \mathrm{Out}(G)$. Similarly, we can describe χ in the case when $|\Sigma| = 4$ and show that its image in $\mathrm{Out}(G)$ is also trivial.

Lemma 4.6. *If* $G \simeq \mathfrak{C}_3 \rtimes \mathfrak{C}_4$ *then*

$$\bar{\beta}(\mathrm{Aut}^G(S)) = \bar{\beta}(\mathrm{Bir}^G(S)) = \mathrm{Out}(\mathfrak{C}_3 \rtimes \mathfrak{C}_4) \simeq \mathfrak{C}_2.$$

Proof. We have $\mathrm{Aut}^G(S) = \mathfrak{C}_3 \rtimes \mathfrak{D}_4$ and $\mathrm{Out}(\mathfrak{C}_3 \rtimes \mathfrak{C}_4) = \mathfrak{C}_2$. One can check that $\bar{\alpha} : \mathrm{Aut}^G(S) \to \mathrm{Out}(\mathfrak{C}_3 \rtimes \mathfrak{C}_4)$ is surjective, and so is $\bar{\beta}$. □

Appendix A. Tables

Cubic surfaces. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $G \subseteq \mathrm{Aut}(S)$ a subgroup such that $\mathrm{rkPic}(S)^G = 1$. Then S is one of the following surfaces:

(I) $\sum_{i=1}^4 x_1^3 = 0$;

(II) $(\sum_{i=1}^4 x_i^2)(\sum_{i=1}^4 x_i) + \sum_{1\le i<j<k\le 4} 2x_ix_jx_k = \sum_{i=1}^4 x_i^3$;

(III) $x_1^3 + x_2^3 + x_3^3 + x_4^3 + 6ax_2x_3x_4 = 0$, where $a = \frac{-1+\sqrt{3}}{2}$;

(IV) $x_1^3 + x_2^3 + x_3^3 + x_4^3 + 6ax_2x_3x_4 = 0$, where $a \in k$ is general;

(V) $x_1^3 + x_1(x_2^2 + x_3^2 + x_4^2) + ax_2x_3x_4 = 0$, where $a \in k$ is general;

(VI) $x_1^3 + x_2^3 + x_3^3 + x_4^3 + ax_3x_4(x_1 + x_2) = 0$, where $a \in k$ is general;

(VIII) $x_3^3 + x_4^3 + ax_3x_4(x_1 + bx_2) + x_1^3 + x_2^3 = 0$, where $a, b \in k$ are general.

We introduce the actions: σ_s is the permutation of coordinates via $s \in \mathfrak{S}_4$ and

$$\tau_1 : \mathrm{diag}(1, 1, \zeta_3^2, \zeta_3),$$
$$\tau_2 : \mathrm{diag}(1, -1, -1, 1),$$
$$\tau_3 : \mathrm{diag}(\zeta_3, 1, 1, 1),$$
$$\tau_4 : \mathrm{diag}(1, \zeta_3^2, \zeta_3, 1),$$
$$\tau_5 : \mathrm{diag}(\zeta_3, \zeta_3^2, 1, 1),$$
$$\tau_6 : \mathrm{diag}(1, 1, 1, \zeta_3),$$
$$\tau_7 : \mathrm{diag}(\zeta_3, 1, \zeta_3, 1),$$
$$\alpha_1 : (\mathsf{x}) \mapsto (-x_1 - x_2 - x_3 - x_4, x_1, x_2, x_3),$$
$$\alpha_2 : (\mathsf{x}) \mapsto (\mathsf{x}) \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \zeta_3 & \zeta_3^2 \\ 0 & 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}, \quad b^3 = 3\sqrt{3}.$$

The table below lists all cases when $\mathrm{rkPic}(S)^G = 1$.

Type	G	GapID	Generators
I	$\mathfrak{C}_3$	`(3,1)`	τ_3
I	$\mathfrak{S}_3$	`(6,1)`	$\sigma_{(23)}, \sigma_{(234)}$
I	$\mathfrak{C}_6$	`(6,2)`	$\tau_3\sigma_{(34)}$
I	$\mathfrak{C}_6$	`(6,2)`	$\tau_5\sigma_{(34)}$
I	$\mathfrak{S}_3$	`(6,1)`	$\tau_1, \sigma_{(34)}$
I	$\mathfrak{C}_3^2$	`(9,2)`	τ_1, τ_3
I	$\mathfrak{C}_3^2$	`(9,2)`	τ_1, τ_6
I	$\mathfrak{C}_3^2$	`(9,2)`	$\tau_1^2\sigma_{(234)}, \tau_1\tau_4\sigma_{(234)}$
I	$\mathfrak{C}_9$	`(9,1)`	$\tau_1\tau_3^2\tau_5\sigma_{(234)}$
I	$\mathfrak{D}_6$	`(12,4)`	$\sigma_{(34)}, \tau_1, \sigma_{(12)(34)}$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	`(18,3)`	$\sigma_{(34)}, \tau_1, \tau_3\tau_6$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	`(18,3)`	$\tau_1, \tau_6, \sigma_{(34)}$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	`(18,3)`	$\tau_1, \tau_3, \sigma_{(34)}$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	`(18,3)`	$\tau_6, \tau_1\tau_4, \sigma_{(13)(24)}$
I	$\mathfrak{C}_3 \times \mathfrak{C}_6$	`(18,5)`	$\tau_3\tau_5^2, \tau_3\sigma_{(34)}$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	`(18,3)`	$\tau_3, \sigma_{(34)}, \tau_1\sigma_{(234)}$
I	$\mathfrak{C}_3^3$	`(27,5)`	τ_1, τ_3, τ_6
I	He_3	`(27,3)`	$\tau_1, \tau_3, \sigma_{(234)}$
I	$\mathfrak{C}_9 : \mathfrak{C}_3$	`(27,4)`	$\tau_1, \tau_3\tau_4\tau_5\tau_6^2\sigma_{(234)}$
I	$\mathfrak{S}_4$	`(24,12)`	$\sigma_{(12)}, \sigma_{(1234)}$
I	$\mathfrak{S}_3^2$	`(36,10)`	$\tau_1, \tau_7, \sigma_{(12)}, \sigma_{(34)}$
I	$\mathfrak{C}_6 \times \mathfrak{S}_3$	`(36,12)`	$\tau_1, \tau_3\tau_5, \sigma_{(12)}, \sigma_{(34)}$
I	$\mathfrak{C}_3 \times \mathfrak{C}_3 : \mathfrak{S}_3$	`(54,13)`	$\tau_1, \tau_4, \tau_7, \sigma_{(13)(24)}$
I	$\mathfrak{C}_3^2 : \mathfrak{S}_3$	`(54,8)`	$\tau_1, \tau_3, \sigma_{(23)}, \sigma_{(234)}$
I	$\mathfrak{C}_3^2 \times \mathfrak{S}_3$	`(54,12)`	$\tau_1, \tau_3, \tau_6, \sigma_{(34)}$
I	$\mathfrak{C}_3 \wr \mathfrak{C}_3$	`(81,7)`	$\tau_1, \tau_3, \tau_6, \sigma_{(234)}$
I	$\mathfrak{S}_3 \wr \mathfrak{C}_2$	`(72,40)`	$\tau_1, \tau_7, \sigma_{(12)}, \sigma_{(34)}, \sigma_{(14)(23)}$
I	$\mathfrak{C}_3^3 : \mathfrak{C}_2^2$	`(108,40)`	$\tau_1, \tau_4, \tau_7, \sigma_{(12)(34)}, \sigma_{(14)(23)}$
I	$\mathfrak{C}_3^2 : (\mathfrak{C}_3 : \mathfrak{C}_4)$	`(108,37)`	$\tau_1, \tau_4, \tau_7, \sigma_{(12)(34)}, \sigma_{(1324)}$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3^2$	`(108,38)`	$\tau_1, \tau_4, \tau_7, \sigma_{(12)}, \sigma_{(34)}$
I	$\mathfrak{C}_3 \wr \mathfrak{S}_3$	`(162,10)`	$\tau_1, \tau_3^2\tau_6, \tau_1\tau_4, \sigma_{(3,4)}, \sigma_{(234)}$
I	$\mathfrak{S}_3^2 : \mathfrak{S}_3$	`(216,158)`	$\tau_3\tau_7\sigma_{(34)}, \sigma_{(14)(23)}$
I	$\mathfrak{C}_3^3 : \mathfrak{C}_2^2 : \mathfrak{C}_3$	`(324,160)`	$\tau_3\tau_7\sigma_{(234)}, \sigma_{(14)(23)}$
I	$\mathfrak{C}_3^3.\mathfrak{S}_4$	`(648,704)`	$\tau_3, \sigma_{(12)}, \sigma_{(1234)}$
II	$\mathfrak{S}_3$	`(6,1)`	$\sigma_{(23)}, \sigma_{(234)}$
II	$\mathfrak{C}_6$	`(6,2)`	$\sigma_{14}\alpha_1$

II	$\mathfrak{D}_6$	(12,4)	$\sigma_{(23)}, \sigma_{(234)}, \sigma_{14}\alpha_1$
II	$\mathfrak{S}_4$	(24,12)	$\sigma_{1234}, \sigma_{(12)}$
II	$\mathfrak{S}_5$	(120,34)	$\alpha_1, \sigma_{(12)}$
III	$\mathfrak{C}_3$	(3,1)	τ_3
III	$\mathfrak{S}_3$	(6,1)	$\sigma_{(34)}, \tau_1$
III	$\mathfrak{S}_3$	(6,1)	$\tau_1\sigma_{(34)}, \tau_1\sigma_{(234)}$
III	$\mathfrak{C}_6$	(6,2)	$\tau_1\tau_3\sigma_{(34)}$
III	$\mathfrak{C}_3^2$	(9,2)	τ_3, τ_1
III	$\mathfrak{C}_3^2$	(9,2)	$\tau_3, \tau_1\sigma_{(234)}$
III	$\mathfrak{C}_{12}$	(12,2)	α_2
III	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_1, \tau_3, \sigma_{(34)}$
III	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_3, \tau_1^2\sigma_{(234)}, \tau_1\sigma_{(34)}$
III	He_3	(27,3)	$\tau_1, \tau_3, \sigma_{(234)}$
III	$\mathfrak{C}_3^2 \rtimes \mathfrak{S}_3$	(54,8)	$\tau_1, \tau_3, \sigma_{(234)}, \sigma_{(23)}$
III	$\mathfrak{C}_3^2 \rtimes \mathfrak{S}_3.\mathfrak{C}_2$	(108,15)	$\tau_1, \tau_3, \sigma_{(234)}, \sigma_{(23)}, \alpha_2$
IV	$\mathfrak{C}_3$	(3,1)	τ_3
IV	$\mathfrak{S}_3$	(6,1)	$\sigma_{(34)}, \tau_1$
IV	$\mathfrak{S}_3$	(6,1)	$\sigma_{(34)}, \sigma_{(234)}$
IV	$\mathfrak{S}_3$	(6,1)	$\tau_1\sigma_{(34)}, \tau_1\sigma_{(234)}$
IV	$\mathfrak{S}_3$	(6,1)	$\tau_1^2\sigma_{(34)}, \tau_1^2\sigma_{(234)}$
IV	$\mathfrak{C}_6$	(6,2)	$\tau_1\tau_3\sigma_{(34)}$
IV	$\mathfrak{C}_3^2$	(9,2)	τ_3, τ_1
IV	$\mathfrak{C}_3^2$	(9,2)	$\tau_3, \tau_1\sigma_{(234)}$
IV	$\mathfrak{C}_3^2$	(9,2)	$\tau_4^2\sigma_{(234)}, \tau_1^2\sigma_{(234)}$
IV	$\mathfrak{C}_3^2$	(9,2)	$\tau_3, \sigma_{(234)}$
IV	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_1, \tau_3, \sigma_{(34)}$
IV	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_3, \sigma_{(234)}, \tau_1\sigma_{(34)}$
IV	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_3, \tau_1^2\sigma_{(234)}, \tau_1\sigma_{(34)}$
IV	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\tau_3, \tau_4\sigma_{(234)}, \tau_1\sigma_{(34)}$
IV	He_3	(27,3)	$\tau_1, \tau_3, \sigma_{(234)}$
IV	$\mathfrak{C}_3^2 \rtimes \mathfrak{S}_3$	(54,8)	$\tau_1, \tau_3, \sigma_{(234)}, \sigma_{(23)}$
V	$\mathfrak{S}_3$	(6,1)	$\sigma_{(234)}, \sigma_{(23)}$
V	$\mathfrak{S}_4$	(24,12)	$\sigma_{(234)}, \sigma_{(23)}, \tau_2$
VI	$\mathfrak{C}_6$	(6,2)	$\sigma_{(12)}\tau_1$
VI	$\mathfrak{S}_3$	(6,1)	$\sigma_{(34)}, \tau_1$
VI	$\mathfrak{D}_6$	(12,4)	$\sigma_{(12)}\tau_1, \sigma_{(23)}$
VIII	$\mathfrak{S}_3$	(6,1)	$\sigma_{(34)}, \tau_1$

Del Pezzo surfaces of degree 2. Let $S \subset \mathbb{P}(2_{x_0}, 1_{x_1}, 1_{x_2}, 1_{x_3})$ be a del Pezzo surface of degree 2 and $G \subseteq \mathrm{Aut}(S)$ a subgroup such that $\mathrm{rkPic}(S)^G = 1$. Then S is one of the following surfaces:

(I) $x_0^2 = x_1^3x_2 + x_1x_3^3 + x_2^3x_3$;
(II) $x_0^2 = x_1^4 + x_2^4 + x_3^4$;
(III) $x_0^2 = x_1^4 + x_2^4 + x_3^4 + (4\zeta_3 + 2)x_1^2x_2^2$;
(IV) $x_0^2 = x_1^4+x_2^4+x_3^4-a(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)$, where $a \in k$ is general;
(V) $x_0^2 = x_1^4 + x_2^4 + x_3^4 + ax_1^2x_2^2$, where $a \in k$ is general;
(VII) $x_0^2 = x_1^4+x_2^4+x_3^4+ax_1^2x_2^2+bx_3^2x_1x_2$, where $a, b \in k$ are general;
(VIII) $x_0^2 = x_3^3x_1 + x_1^4 + x_2^4 + ax_1^2x_2^2$, where $a \in k$ is general.

Put

$$\iota : (x_0, x_1, x_2, x_3) \mapsto (-x_0, x_1, x_2, x_3),$$

$$\sigma_1 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \\ 0 & a_7 & a_8 & a_9 \end{pmatrix},$$

$$\sigma_2 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 & b_6 \\ 0 & b_7 & b_8 & b_9 \end{pmatrix},$$

$$\sigma_3 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_2, x_3, x_1),$$

$$\sigma_4 : (x_0, x_1, x_2, x_3) \mapsto (x_0, -\zeta_4x_1, \zeta_4x_3, x_2),$$

$$\sigma_5 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3) \cdot \begin{pmatrix} \zeta_6 & 0 & 0 & 0 \\ 0 & \frac{1+\zeta_4}{2} & \frac{-1+\zeta_4}{2} & 0 \\ 0 & \frac{1+\zeta_4}{2} & \frac{1-\zeta_4}{2} & 0 \\ 0 & 0 & 0 & \zeta_3 \end{pmatrix},$$

$$\sigma_6 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3) \cdot \begin{pmatrix} \zeta_3 & 0 & 0 & 0 \\ 0 & \frac{1+\zeta_4}{2} & \frac{-1-\zeta_4}{2} & 0 \\ 0 & \frac{-1+\zeta_4}{2} & \frac{-1+\zeta_4}{2} & 0 \\ 0 & 0 & 0 & \zeta_3^2 \end{pmatrix},$$

$$\sigma_7 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_2, -x_1, x_3),$$

$$\sigma_8 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_3, x_2),$$

$$\sigma_9 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, \zeta_4x_2, \zeta_4^3x_3),$$

$$\sigma_{10} : (x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, x_2, x_3),$$

$$\sigma_{11} : (x_0, x_1, x_2, x_3) \mapsto (x_0, \zeta_4x_1, \zeta_4^3x_2, x_3),$$

$$\sigma_{12} : (-x_0, x_1, -x_2, \zeta_3 x_3) \mapsto (x_0, x_1, \zeta_3 x_2, -x_3),$$

where

$$a_1 = \frac{1}{7}(3\zeta_7^5 + \zeta_7^4 + \zeta_7^3 + 3\zeta_7^2 - 1), \quad a_2 = \frac{1}{7}(-\zeta_7^5 + 2\zeta_7^4 + 2\zeta_7^3 - \zeta_7^2 - 2),$$
$$a_3 = \frac{1}{7}(-2\zeta_7^5 - 3\zeta_7^4 - 3\zeta_7^3 - 2\zeta_7^2 - 4), \quad a_4 = \frac{1}{7}(-\zeta_7^5 + 2\zeta_7^4 + 2\zeta_7^3 - \zeta_7^2 - 2),$$
$$a_5 = \frac{1}{7}(-2\zeta_7^5 - 3\zeta_7^4 - 3\zeta_7^3 - 2\zeta_7^2 - 4), \quad a_6 = \frac{1}{7}(3\zeta_7^5 + \zeta_7^4 + \zeta_7^3 + 3\zeta_7^2 - 1),$$
$$a_7 = \frac{1}{7}(-2\zeta_7^5 - 3\zeta_7^4 - 3\zeta_7^3 - 2\zeta_7^2 - 4), \quad a_8 = \frac{1}{7}(3\zeta_7^5 + \zeta_7^4 + \zeta_7^3 + 3\zeta_7^2 - 1),$$
$$a_9 = \frac{1}{7}(-\zeta_7^5 + 2\zeta_7^4 + 2\zeta_7^3 - \zeta_7^2 - 2), \quad b_1 = \frac{1}{7}(2\zeta_7^4 - \zeta_7^3 - 2\zeta_7^2 - \zeta_7 + 2),$$
$$b_2 = \frac{1}{7}(-2\zeta_7^5 - \zeta_7^3 + 2\zeta_7^2 + 2\zeta_7 - 1), \quad b_3 = \frac{1}{7}(\zeta_7^5 + 3\zeta_7^4 - \zeta_7^3 + 3\zeta_7^2 + \zeta_7),$$
$$b_4 = \frac{1}{7}(-2\zeta_7^5 - \zeta_7^3 + 2\zeta_7^2 + 2\zeta_7 - 1), \quad b_5 = \frac{1}{7}(-\zeta_7^5 - \zeta_7^4 + 2\zeta_7^2 - 2\zeta_7 + 2),$$
$$b_6 = \frac{1}{7}(\zeta_7^5 + 4\zeta_7^4 + 2\zeta_7^3 + 2\zeta_7^2 + 4\zeta_7 + 1), \quad b_7 = \frac{1}{7}(\zeta_7^5 + 3\zeta_7^4 - \zeta_7^3 + 3\zeta_7^2 + \zeta_7),$$
$$b_8 = \frac{1}{7}(\zeta_7^5 + 4\zeta_7^4 + 2\zeta_7^3 + 2\zeta_7^2 + 4\zeta_7 + 1), \quad b_9 = \frac{1}{7}(\zeta_7^5 - \zeta_7^4 + \zeta_7^3 + 3\zeta_7 + 3).$$

The table below lists all cases when $\mathrm{rkPic}(S)^G = 1$.

Type	G	GapID	Generators
I	$\mathfrak{D}_4$	(8,3)	$\sigma_2^2\sigma_1\sigma_2\sigma_1\sigma_2^2, \sigma_2^3\sigma_1\sigma_2\sigma_1\sigma_2^3\sigma_1\sigma_2$
I	$\mathfrak{S}_4$	(24,12)	$\sigma_2^3\sigma_1\sigma_2, \sigma_2^2\sigma_1\sigma_2\sigma_1\sigma_2^2$
I	$\mathfrak{S}_4$	(24,12)	$\sigma_1\sigma_2\sigma_1, \sigma_2^3\sigma_1\sigma_2$
I	$\mathrm{PSL}_2(\mathbb{F}_7)$	(168,42)	σ_2, σ_1
II	$\mathfrak{C}_4$	(4,1)	$\iota\sigma_3\sigma_4^{-1}\sigma_3\sigma_4^{-1}$
II	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_4^4, \iota\sigma_3^{-1}\sigma_4^{-2}\sigma_3$
II	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\iota\sigma_3^{-1}\sigma_4^{-2}\sigma_3, \iota\sigma_4^4$
II	$Q8$	(8,4)	$\sigma_4^3\sigma_3^{-1}, \sigma_4\sigma_3^{-1}\sigma_4^2$
II	$\mathfrak{D}_4$	(8,3)	$\sigma_3\sigma_4, \sigma_4\sigma_3^{-1}\sigma_4^{-2}$
II	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_4^3\sigma_3^{-1}, \iota\sigma_3^{-1}\sigma_4^{-2}\sigma_3$
II	$\mathfrak{D}_4$	(8,3)	$\sigma_3\sigma_4, \sigma_3\sigma_4^{-3}$
II	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_3\sigma_4, \iota\sigma_3^{-1}\sigma_4^{-2}\sigma_3$
II	$OD16$	(16,6)	$\sigma_3\sigma_4^{-1}, \sigma_3\sigma_4^3$
II	$\mathfrak{C}_4^2$	(16,2)	$\iota\sigma_4^2, (\sigma_3^{-1}\sigma_4^{-1}\sigma_3)^2$
II	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_3\sigma_4, \sigma_4\sigma_3^{-1}\sigma_4^{-2}, \iota\sigma_4^3\sigma_3^{-1}$

II	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\iota\sigma_3\sigma_4, \sigma_4^3\sigma_3^{-1}, \sigma_4^{-2}\sigma_3\sigma_4^{-1}$
II	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_3\sigma_4, (\sigma_3\sigma_4^{-1})^2, \sigma_4\sigma_3^{-1}\sigma_4^{-2}$
II	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_3\sigma_4, \sigma_4^4, \iota\sigma_3^{-1}\sigma_4^{-2}\sigma_3$
II	$\mathfrak{S}_4$	(24,12)	$\sigma_3^{-1}\sigma_4^{-1}, \sigma_3^{-1}\sigma_4^2\sigma_3^{-1}$
II	$\mathfrak{C}_4 \wr \mathfrak{C}_2$	(32,11)	$\sigma_3\sigma_4, \iota\sigma_4^2$
II	$\mathfrak{C}_4 \wr \mathfrak{C}_2$	(32,11)	$\sigma_3\sigma_4, \sigma_3\sigma_4^{-1}$
II	$\mathfrak{C}_4 \wr \mathfrak{C}_2$	(32,11)	$\sigma_3\sigma_4^{-1}, \iota\sigma_3\sigma_4$
II	$\mathfrak{C}_4^2 \rtimes \mathfrak{C}_3 \rtimes \mathfrak{C}_2$	(96,64)	σ_3, σ_4^{-1}
III	$\mathfrak{C}_4$	(4,1)	$\iota\sigma_6^{-3}$
III	$\mathfrak{C}_6$	(6,2)	$\iota\sigma_6^{-2}$
III	$Q8$	(8,4)	$\iota\sigma_6^2\sigma_5^{-1}, \iota\sigma_5^{-1}\sigma_6^2$
III	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_5\sigma_6^{-1}\sigma_5, \sigma_6\sigma_5^{-1}\sigma_6$
III	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\iota\sigma_6^{-1}\sigma_5\sigma_6^{-1}, \sigma_5^2\sigma_6\sigma_5^{-1}$
III	$\mathfrak{D}_4$	(8,3)	$\sigma_5^2\sigma_6^{-1}, \sigma_5^{-2}\sigma_6$
III	$\mathfrak{C}_{12}$	(12,2)	$\iota\sigma_6^{-1}$
III	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_5^{-1}\sigma_6^{-1}, \sigma_5\sigma_6, \iota\sigma_6^2\sigma_5^{-1}$
III	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_6^{-3}, \sigma_5^2\sigma_6^{-1}, \sigma_5^{-2}\sigma_6$
III	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_6^{-2}\sigma_5, \iota\sigma_5^{-1}\sigma_6^{-1}, \sigma_6^2\sigma_5^{-1}$
III	$\mathsf{SL}_2(\mathbb{F}_3)$	(24,3)	$\iota\sigma_5^{-1}, \sigma_6^2$
III	$\mathsf{SL}_2(\mathbb{F}_3) \rtimes \mathfrak{C}_2$	(48,33)	$\sigma_6, \iota\sigma_5$
III	$\mathsf{SL}_2(\mathbb{F}_3) \rtimes \mathfrak{C}_2$	(48,33)	$\sigma_5^{-1}\sigma_6, \sigma_5^{-1}\sigma_6^{-1}$
IV	$\mathfrak{D}_4$	(8,3)	$\sigma_3\sigma_7, \sigma_3\sigma_7^{-1}$
IV	$\mathfrak{S}_4$	(24,12)	σ_3, σ_7
V	$\mathfrak{C}_4$	(4,1)	$\sigma_{10}\iota\sigma_{11}$
V	$\mathfrak{D}_4$	(8,3)	$\sigma_{11}^{-1}, \sigma_3^{-1}\sigma_{10}$
V	$Q8$	(8,4)	$\sigma_3^{-1}, \sigma_{11}^{-1}$
V	$\mathfrak{D}_4$	(8,3)	$\sigma_3^{-1}, \sigma_{10}$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_3\iota, \sigma_{10}\iota\sigma_{11}^{-1}$
V	$\mathfrak{D}_4$	(8,3)	$\sigma_{10}, \sigma_3\sigma_{11}$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_{11}^{-1}, \sigma_{10}\iota$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_3^{-1}, \sigma_{10}\iota\sigma_{11}^{-1}$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_3\sigma_{11}^{-1}, \sigma_{10}\iota\sigma_{11}^{-1}$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_3\sigma_{10}, \sigma_{10}\iota\sigma_{11}^{-1}$
V	$\mathfrak{C}_2 \times \mathfrak{C}_4$	(8,2)	$\sigma_{10}, \iota\sigma_{11}$
V	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_{11}^{-1}, \sigma_{10}, \sigma_3^{-1}$
V	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_{11}, \sigma_{10}\iota, \sigma_3^{-1}\sigma_{10}$
V	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_3, \sigma_{11}^{-1}, \sigma_{10}\iota$

V	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_{10}, \sigma_3\iota, \sigma_3\sigma_{11}^{-1}$
V	$\mathfrak{D}_4 \rtimes \mathfrak{C}_2$	(16,13)	$\sigma_{10}, \sigma_3^{-1}, \iota\sigma_{11}^{-1}$
VII	$\mathfrak{D}_4$	(8,3)	σ_8, σ_9
VIII	$\mathfrak{C}_6$	(6,2)	σ_{12}

Del Pezzo surfaces of degree 1. Let $S \subset \mathbb{P}(1_{x_0}, 1_{x_1}, 2_{x_2}, 3_{x_3})$ be a del Pezzo surface of degree 1. Then

$$S = \{x_3^2 + x_2^3 + F_4(x_0, x_1)x_2 + F_6(x_0, x_1) = 0\},$$

where F_4 and F_6 are homogeneous polynomials of degree 4 and 6, respectively. Put

$$\gamma : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, -x_3),$$
$$\delta : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, \zeta_3 x_2, x_3).$$

Lemma A.1. *Let S be a del Pezzo surface of degree 1, $G \subseteq \mathrm{Aut}(S)$, with* $\mathrm{rkPic}(S)^G = 1$. *Then one of the following holds:*

(1) *G contains γ, δ or $\gamma\delta$,*

(2) *$G \simeq C_5$ is generated by*

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, \zeta_5 x_1, x_2, x_3),$$

and

$$S = \{x_3^2 + x_2^3 + ax_0^4 x_2 + x_0(bx_0^5 + x_1^5) = 0\},$$

for some $a, b \in k$,

(3) *$G \simeq C_6$ is generated by*

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, \zeta_6 x_1, x_2, x_3),$$

and

$$S = \{x_3^2 + x_2^3 + ax_0^4 x_2 + x_0^6 + bx_1^6 = 0\},$$

for some $a, b \in k$.

Additionally, we list all cases when $\mathrm{rkPic}^G(S) = 1$, with generators, for S with maximal automorphism groups. Such S are of types:

(I) $x_3^2 + x_2^3 + x_0x_1(x_0^4 - x_1^4) = 0$;
(II) $x_3^2 + x_2^3 + x_0^6 + x_1^6 = 0$;
(IV) $x_3^2 + x_2^3 + x_0(x_0^5 + x_1^5) = 0$;
(VII) $x_3^2 + x_2^3 + x_2x_0^4 + x_1^6 = 0$;
(XV) $x_3^2 + x_2^3 + x_2(ax_0^4 + x_1^4) + x_0^2(bx_0^4 + cx_1^4) = 0$, where $a, b, c \in k$ are general.

Put

$$\sigma_1 : (x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, \zeta_3 x_2, -x_3),$$
$$\sigma_2 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_8^7(x_0 + x_1), \zeta_8^5 x_0 + \zeta_8 x_1, 2\zeta_3 x_2, 2\sqrt{2} x_3),$$
$$\sigma_3 : (x_0, x_1, x_2, x_3) \mapsto (-\zeta_8 x_1, \zeta_8^7 x_0, -x_2, \zeta_4 x_3),$$
$$\sigma_4 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_8 x_0, \zeta_8^7 x_1, -x_2, \zeta_4 x_3),$$
$$\sigma_5 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_{12} x_0, \zeta_{12}^{11} x_1, -x_2, \zeta_4 x_3),$$
$$\sigma_6 : (x_0, x_1, x_2, x_3) \mapsto (x_1, x_0, x_2, x_3),$$
$$\sigma_7 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_{10} x_0, \zeta_{10}^9 x_1, \zeta_5 x_2, \zeta_{10}^3 x_3),$$
$$\sigma_8 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_{24} x_0, \zeta_{24}^{23} x_1, \zeta_{24}^{14} x_2, \zeta_8^7 x_3),$$
$$\sigma_9 : (x_0, x_1, x_2, x_3) \mapsto (\zeta_8 x_0, \zeta_8^7 x_1, \zeta_4 x_2, \zeta_8^3 x_3).$$

The table below lists all cases when $\mathrm{rkPic}(S)^G = 1$.

Type	G	GapID	Generators
I	$\mathfrak{C}_2$	(2,1)	σ_1^3
I	$\mathfrak{C}_3$	(3,1)	σ_1^4
I	$\mathfrak{C}_6$	(6,2)	σ_1^5
I	$\mathfrak{C}_4$	(4,1)	σ_4^2
I	$\mathfrak{C}_{12}$	(12,2)	$\sigma_4^2\sigma_1^5$
I	$\mathfrak{C}_2^2$	(4,2)	σ_3, σ_1^3
I	$\mathfrak{C}_6$	(6,2)	$\sigma_3\sigma_1^5$
I	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_1^3, \sigma_3\sigma_1^4$
I	$\mathfrak{C}_6$	(6,2)	$\sigma_1\sigma_2^2$
I	$\mathfrak{C}_6$	(6,2)	σ_2^5
I	$\mathfrak{C}_3^2$	(9,2)	$\sigma_2^2, \sigma_1\sigma_2^5$
I	$\mathfrak{C}_3 \times \mathfrak{C}_6$	(18,5)	$\sigma_2^4, \sigma_1^2\sigma_2$
I	$Q8$	(8,4)	$\sigma_4\sigma_3, \sigma_4^2$
I	$\mathfrak{C}_3 \times Q8$	(24,11)	$\sigma_1\sigma_3\sigma_4, \sigma_3\sigma_4^7$
I	$\mathfrak{D}_4$	(8,3)	σ_4^2, σ_3
I	$\mathfrak{C}_3 \times \mathfrak{D}_4$	(24,10)	$\sigma_2\sigma_4\sigma_2, \sigma_3\sigma_1^5$
I	$\mathfrak{C}_8$	(8,1)	$\sigma_3\sigma_4\sigma_3$
I	$\mathfrak{C}_{24}$	(24,2)	$\sigma_1^4\sigma_4^7$
I	$\mathfrak{D}_6$	(12,4)	$\sigma_1\sigma_2^2, \sigma_1^2\sigma_2\sigma_3$
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	σ_3, σ_2^4
I	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\sigma_4\sigma_3\sigma_4, \sigma_2^4$
I	$\mathfrak{C}_6 \times \mathfrak{S}_3$	(36,12)	$\sigma_2^5, \sigma_3\sigma_1^4$

I	$SD16$	(16,8)	$\sigma_4\sigma_3, \sigma_3$
I	$\mathfrak{C}_3 \times SD16$	(48,26)	$\sigma_1^2\sigma_4^7, \sigma_1^2\sigma_3$
I	$\mathsf{SL}_2(\mathbb{F}_3)$	(24,3)	$\sigma_4^2, \sigma_1\sigma_2^5$
I	$\mathsf{SL}_2(\mathbb{F}_3)$	(24,3)	$\sigma_1\sigma_2, \sigma_4^2$
I	$\mathfrak{C}_3 \times \mathsf{SL}_2(\mathbb{F}_3)$	(72,25)	$\sigma_3\sigma_2^5\sigma_4^7, \sigma_2^4$
I	$\mathsf{GL}_2(\mathbb{F}_3)$	(48,29)	$\sigma_4\sigma_3\sigma_4, \sigma_1\sigma_4^7\sigma_2^5\sigma_3$
I	$\mathfrak{C}_3 \times \mathsf{GL}_2(\mathbb{F}_3)$	(144,122)	σ_2, σ_4
II	$\mathfrak{C}_2$	(2,1)	σ_1^3
II	$\mathfrak{C}_3$	(3,1)	σ_1^4
II	$\mathfrak{C}_6$	(6,2)	σ_1^5
II	$\mathfrak{C}_2^2$	(4,2)	$\sigma_1^3, \sigma_5^2\sigma_6\sigma_5^5\sigma_6$
II	$\mathfrak{C}_6$	(6,2)	$\sigma_5^3\sigma_1^4$
II	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_1^3, \sigma_5^3\sigma_1^5$
II	$\mathfrak{C}_2^2$	(4,2)	σ_1^3, σ_6
II	$\mathfrak{C}_6$	(6,2)	$\sigma_1\sigma_6$
II	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_1^3, \sigma_1^2\sigma_6$
II	$\mathfrak{C}_4$	(4,1)	$\sigma_5^2\sigma_6\sigma_5^5$
II	$\mathfrak{C}_{12}$	(12,2)	$\sigma_5^2\sigma_6\sigma_5^5\sigma_1^5$
II	$\mathfrak{C}_6$	(6,2)	$\sigma_5\sigma_6\sigma_5^5\sigma_6$
II	$\mathfrak{C}_6$	(6,2)	$\sigma_5^4\sigma_1^5$
II	$\mathfrak{C}_3^2$	(9,2)	$\sigma_5^2, \sigma_5^2\sigma_1^2$
II	$\mathfrak{C}_3 \times \mathfrak{C}_6$	(18,5)	$\sigma_5^4\sigma_1^4, \sigma_1$
II	$\mathfrak{D}_4$	(8,3)	$\sigma_5^2\sigma_6\sigma_5^5\sigma_6, \sigma_6$
II	$\mathfrak{C}_3 \times \mathfrak{D}_4$	(24,10)	$\sigma_6, \sigma_5^3\sigma_1$
II	$\mathfrak{C}_3 \rtimes \mathfrak{C}_4$	(12,1)	$\sigma_5^4, \sigma_5^2\sigma_6\sigma_5^5$
II	$\mathfrak{C}_3 \times \mathfrak{C}_3 \rtimes \mathfrak{C}_4$	(36,6)	$\sigma_5^2\sigma_1^2, \sigma_5\sigma_1\sigma_6$
II	$\mathfrak{C}_6$	(6,2)	$\sigma_5^5\sigma_1^5$
II	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_1^3, \sigma_6\sigma_5^5\sigma_6$
II	$\mathfrak{C}_2 \times \mathfrak{C}_6$	(12,5)	$\sigma_1^3, \sigma_1^2\sigma_5^5$
II	$\mathfrak{C}_3 \times \mathfrak{C}_6$	(18,5)	$\sigma_5^2\sigma_1^2, \sigma_5^5$
II	$\mathfrak{C}_6^2$	(36,14)	$\sigma_5\sigma_1, \sigma_6\sigma_5\sigma_6\sigma_5^5$
II	$\mathfrak{D}_6$	(12,4)	$\sigma_5^2\sigma_6, \sigma_5\sigma_6\sigma_5$
II	$\mathfrak{C}_3 \times \mathfrak{S}_3$	(18,3)	$\sigma_5^4\sigma_1^4, \sigma_6$
II	$\mathfrak{C}_6 \times \mathfrak{S}_3$	(36,12)	$\sigma_1^2\sigma_6, \sigma_5^5\sigma_6\sigma_5$
II	$\mathfrak{C}_3 \rtimes \mathfrak{D}_4$	(24,8)	$\sigma_5^3\sigma_6, \sigma_5^2\sigma_6$
II	$\mathfrak{C}_3 \times \mathfrak{C}_3 \rtimes \mathfrak{D}_4$	(72,30)	$\sigma_6\sigma_5^2, \sigma_5^5\sigma_1^4$
IV	C_2	(2,1)	σ_1^3
IV	C_3	(3,1)	σ_1^2

IV	C_6	`(6,2)`	σ_1^5
IV	C_5	`(5,1)`	σ_7^2
IV	C_{10}	`(10,2)`	$\sigma_1^3\sigma_7$
IV	C_{15}	`(15,1)`	$\sigma_7\sigma_1^4$
IV	C_{30}	`(30,4)`	$\sigma_7\sigma_1^5$
VII	$\mathfrak{C}_2$	`(2,1)`	σ_1^3
VII	$\mathfrak{C}_2^2$	`(4,2)`	σ_8^6, σ_1^3
VII	$\mathfrak{C}_6$	`(6,2)`	$\sigma_8^4\sigma_1^3$
VII	$\mathfrak{C}_2 \times \mathfrak{C}_4$	`(8,2)`	$\sigma_1^3, \sigma_8^3\sigma_1^3$
VII	$\mathfrak{C}_6$	`(6,2)`	$\sigma_1^3\sigma_8^{10}$
VII	$\mathfrak{C}_2 \times \mathfrak{C}_6$	`(12,5)`	σ_1^3, σ_8^2
VII	$\mathfrak{C}_2 \times \mathfrak{C}_{12}$	`(24,9)`	$\sigma_1^3, \sigma_1^3\sigma_8^{11}$
XV	$\mathfrak{C}_2$	`(2,1)`	σ_1^3
XV	$\mathfrak{C}_2^2$	`(4,2)`	σ_1^3, σ_9^2
XV	$\mathfrak{C}_2 \times \mathfrak{C}_4$	`(8,2)`	$\sigma_1^3, \sigma_1^3\sigma_9^3$

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