RATIONALITY OF FORMS OF $\overline{\mathcal{M}}_{0,n}$

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Abstract. We study equivariant geometry and rationality of moduli spaces of points on the projective line, for twists associated with permutations of the points.

1. Introduction

In this note, we strengthen a theorem of Florence–Reichstein [FR18] concerning rationality of moduli spaces. They consider *forms* of $\overline{\mathcal{M}}_{0,n}$, i.e., varieties over nonclosed fields F which are isomorphic to the moduli space of n points on $\mathbb{P}^1$ over an algebraic closure of F. These forms are obtained by twisting via Galois actions permuting the points over F. The main results of [FR18] are:

- if $n \geq 5$ is odd, and F is infinite of characteristic $\neq 2$, then every form over F is rational;
- if $n \geq 6$ is even, and F has nontrivial 2-torsion in its Brauer group and contains fourth roots of unity, then there exists a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that X is not retract rational over F.

These were inspired by a classical theorem of Enriques, Manin, and Swinnerton-Dyer concerning rationality of twists of $\overline{\mathcal{M}}_{0,5}$, a del Pezzo surface of degree 5, over any field F. The proof for $n \geq 5$ uses (a twisted form of) the Gelfand-MacPherson correspondence, and techniques developed in connection with Noether's problem for twisted forms of the groups in question.

By [FR18], every form over an infinite field F is unirational over F. It is known that every form of $\overline{\mathcal{M}}_{0,6}$ over $\mathbb{R}$ is rational [Avi20, Proposition 2.9]; see Corollary 21 for generalizations.

Here, we strengthen their conclusions in two directions: we prove rationality in several situations not addressed in [FR18]. On the other hand, we show failure of rationality via Galois cohomology in instances not covered by [FR18], e.g., where the Brauer group of F is trivial.

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An important ingredient throughout is a theorem of [BM13]:

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}) = \mathfrak{S}_n, \quad n \geq 5,$$

acting via permutations of the n points on $\mathbb{P}^1$. In particular, Galois twists of $\overline{\mathcal{M}}_{0,n}$ factor through subgroups of $\mathfrak{S}_n$, and there is a close link between rationality of twists and linearizability of G-actions on $\overline{\mathcal{M}}_{0,n}$; see [DR15] for a general discussion of such connections. In both situations, there is an action of a finite group on the geometric Picard group

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}),$$

via a subgroup of $\mathfrak{S}_n$.

We present several stable rationality and linearizability results, including Propositions 3 and 5 (based on the Kapranov construction) and Theorem 24 (using torsors and quotients). Section 3 focuses on geometric constructions. One rationality construction uses Schubert calculus and the geometry of Grassmannians; Theorem 14 extends results of [FR18] to small fields (Corollary 16) and some point configurations in higher-dimensional projective spaces (Corollary 17). Another relies on fibration structures; see Theorem 20. We close with a comprehensive discussion of the $n = 6$ case (Theorem 34).

For nonrationality/nonlinearizability, we focus on situations where the twisted moduli spaces are toric via the Losev-Manin construction [LM00]. We utilize cohomological **(H1)** and **(SP)**-obstructions (see Section 5): In the arithmetic context, the group is replaced by the absolute Galois group of the ground field F and the Picard module by the geometric Picard module. We focus on *even* n:

Theorem 1 (Corollary 29 and Theorem 30)**.** *For every even $n \geq 6$ there exists a subgroup $G = C_2^2 \subset \mathfrak{S}_n$ such that*

$$\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2.$$

In particular,

- *for all subgroups of $\mathfrak{S}_n$ containing G, the corresponding action is not stably linearizable,*
- *for all fields F admitting a Galois extension L/F with Galois group $\mathrm{Gal}(L/F) \simeq G$ there exists a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that X is not retract rational over F.*

Indeed, nonvanishing group cohomology is an obstruction to (stable) linearizability, see, e.g., [BP13, Corollary 2.5.2.]. In the context of nonclosed fields, one can find a twist X of $\overline{\mathcal{M}}_{0,n}$ over F so that the corresponding Galois action on the geometric Picard group of X factors through the prescribed action of G. This yields nontrivial Galois cohomology, which in turn obstructs retract rationality of X over F. In particular, our result applies to fields F with *trivial* Brauer group, e.g., $F = \mathbb{C}(t)$.

Remark 2. Florence and Reichstein have pointed out that the proof of [FR18, Theorem 1.2(b)] – giving forms of $\overline{\mathcal{M}}_{0,n}$ that are not retract rational – implicitly assumes that the base field contains fourth roots of unity. These are needed to harmonize sign choices in the quaternion algebras constructed in [FR18, Section 7]. Indeed, the field $\mathbb{R}$ has Brauer group $\mathbb{Z}/2\mathbb{Z}$ but real forms of $\overline{\mathcal{M}}_{0,n}$ are rational (see Corollary 21).

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2. $\mathfrak{S}_n$-equivariant geometry

We recall some terminology: Let G be a finite group acting regularly on a projective variety X. Assume the action is generically free. The action is *linearizable* if X is equivariantly birational to the projectivization $\mathbb{P}(V)$ of a linear representation V of G on a vector space. It is *stably linearizable* if $X \times \mathbb{P}^r$ – where G acts trivially on the second factor – is linearizable. By the No-Name Lemma, this is equivalent to saying that $X \times V$ is linearizable for some linear representation V of G, or that the total space of a G-equivariant vector bundle $E \to X$ is linearizable.

Stable linearizability and stable rationality of twisted forms are tightly linked [DR15, Theorem 1.1(d)]: A G-action on X is stably linearizable over F iff for every infinite field K/F and every form of X over K obtained via twisting by the G-action, the resulting variety is stably rational.

Kapranov blowup. We make use of the Kapranov blowup realization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}, \quad n \geq 4,$$

where β_n is an iterated blowup of $n-1$ general points on $\mathbb{P}^{n-3}$, lines through pairs of points, etc., see, e.g., [HT02, Section 3.1]. Precisely, we regard

$$\mathbb{P}^{n-3} = \mathbb{P}(k[\mathfrak{S}_{n-1}]/(1,\dots,1)),$$

so that the $\mathfrak{S}_{n-1}$-action is linear. Boundary divisors D_I are labeled by partitions

$$[1,\dots,n] = I \sqcup I^c, \quad |I|, |I^c| \geq 2.$$

Recall that the Picard group $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ has rank $2^{n-1} - \binom{n}{2} - 1$, and an explicit basis is given by

$$\{H, E_{i_1}, E_{i_1,i_2}, \dots, E_{i_1,\dots,i_{n-4}}\},$$

where H is the (pullback of the) hyperplane class on $\mathbb{P}^{n-3}$, and the other elements are (classes of) exceptional divisors from blowups of points, lines, etc. The boundary divisors D_I expressed in this basis are

$$D_{i_1,\dots,i_k,n} = E_{i_1,\dots,i_k}, \quad \{i_1,\dots,i_k\} \subset \{1,\dots,n-1\}, \quad k \leq n-4,$$

and

$$[D_{i_1,\dots,i_{n-3},n}] = L - E_{i_1} - E_{i_2} - \dots - E_{i_1,\dots,i_{n-4}} - E_{i_2,\dots,i_{n-3}}.$$

The $\mathfrak{S}_n$-action on $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ is best understood in terms of the natural $\mathfrak{S}_n$-action on the boundary divisors via permutations of indices of D_I. In particular, there is a distinguished $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ acting via permutation of indices on E_i, for $i \in \{1,\dots,n-1\}$.

The Kapranov construction has applications to linearizability:

Proposition 3. *Suppose that $G \subseteq \mathfrak{S}_{n-1}$ acts on $\overline{\mathcal{M}}_{0,n}$ leaving the nth point invariant. Then the action of G is linearizable.*

For $n = 2m+1$ and $G \subseteq \mathfrak{S}_{2m+1}$, the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.

More generally, for $G \subseteq \mathfrak{S}_n$ leaving an odd cycle invariant, the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.

Proof. The first assertion reflects the fact that the Kapranov morphism β_n is $\mathfrak{S}_{n-1}$ invariant and the $\mathfrak{S}_{n-1}$-action on $\mathbb{P}^{n-3}$ is linear. The second assertion is a special case of the third. For the third statement, consider the universal curve

$$\overline{\mathcal{C}}_{0,n} \to \overline{\mathcal{M}}_{0,n}.$$

Lemma 4. *Let $G \subset \mathfrak{S}_n$ act on $\overline{\mathcal{M}}_{0,n}$ by permutation of the marked points. Then there is a canonical lift of the action to the universal curve*

$$\phi : \overline{\mathcal{C}}_{0,n} \to \overline{\mathcal{M}}_{0,n}.$$

We prove the lemma. Interpreting $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$, we have

$$\mathrm{Aut}(\overline{\mathcal{C}}_{0,n}) = \mathfrak{S}_{n+1} \supset \mathfrak{S}_n \hookrightarrow \mathrm{Aut}(\overline{\mathcal{M}}_{0,n}),$$

with the last inclusion an equality when $n \geq 5$. The induced action on $\mathrm{Aut}(\overline{\mathcal{C}}_{0,n})$ is equivariant under forgetting the $(n+1)$st point.

Returning to the Proposition, we assume that G leaves an odd cycle invariant. Then the forgetting morphism ϕ – an étale $\mathbb{P}^1$-bundle over $\mathcal{M}_{0,n}$ – admits a multisection of odd degree. It must therefore be the projectivization of a rank-two G-equivariant vector bundle over $\mathcal{M}_{0,n}$. However, we have already seen that the G-action on $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$ is linearizable. We conclude then that $\overline{\mathcal{M}}_{0,n}$ is stably linearizable. □

A similar argument yields dividends for the Galois-theoretic question:

Proposition 5. *Let L/F be a Galois extension with Galois group Γ. Fix a representation*

$$\rho : \Gamma \to \mathfrak{S}_n$$

and let ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ denote the corresponding twist of $\overline{\mathcal{M}}_{0,n}$ defined over F.

- *If ρ factors through an $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ then ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ is rational over F.*
- *If n is odd then $\mathbb{P}^1 \times {}^{\rho}\overline{\mathcal{M}}_{0,n}$ is rational. The same holds if ρ leaves an odd cycle invariant.*

This gives a weaker version of [FR18, Theorem 1.2]; however, our statement is valid over a finite field as well. See Remark 22 below for a related result.

Proof. The Kapranov morphism $\beta : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}$ is equivariant for $\mathfrak{S}_{n-1}$, which acts linearly on the target. Thus it descends to

$${}^{\rho}\overline{\mathcal{M}}_{0,n} \stackrel{\sim}{\dashrightarrow} \mathbb{P}^{n-3}$$

over F, proving rationality. For the second assertion, the Kapranov construction yields

$${}^{\rho}\overline{\mathcal{C}}_{0,2m+1} \stackrel{\sim}{\dashrightarrow} \mathbb{P}^{2m-1};$$

moreover

$${}^{\rho}\overline{\mathcal{C}}_{0,2m+1} \to {}^{\rho}\overline{\mathcal{M}}_{0,2m+1}$$

is a $\mathbb{P}^1$-bundle over a Zariski open subspace of the base. (The generic fiber is a smooth genus zero curve with a cycle of odd degree.) In particular, $\mathbb{P}^1 \times {}^{\rho}\overline{\mathcal{M}}_{0,2m+1}$ is rational over F. □

Example 6. Let $\mathfrak{S}_n$ act on $\overline{\mathcal{M}}_{0,n}$, for $n \geq 5$. This action is not linearizable since $\mathfrak{S}_n$ does not act linearly and generically freely on $\mathbb{P}^{n-3}$. Indeed, the smallest faithful representation of $\mathfrak{S}_n$ has dimension $n-1$. When $n = p$ is a prime, then even the action of the Frobenius subgroup $\mathfrak{F}_p = \mathrm{Aff}_1(\mathbb{F}_p) \subset \mathfrak{S}_p$ is not linearizable, for the same reason.

The Losev-Manin construction. This construction [LM00], [Has03, Section 6.4] is a distinguished factorization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \to \overline{L}_n \to \mathbb{P}^{n-3},$$

where we blow up linear subspaces spanned by just $(n-2)$ points in linear general position. (Note that our indexing of $\overline{L}_n$ differs from [LM00].) The first arrow contracts the boundary divisors

$$D_{i_1,\dots,i_k,(n-1),n}, \{i_1, \dots, i_k\} \subset \{1, \dots, n-2\}, \quad k \leq n-5,$$

by allowing points indexed by

$$\{1, \dots, n-2\} \setminus \{i_1, \dots, i_k\}$$

to coincide.

We record some properties:

- $\overline{L}_n$ is toric [LM00, Section 2.6];
- the Losev-Manin construction is equivariant under $\mathfrak{S}_{n-2} \times \mathfrak{S}_2 \subset \mathfrak{S}_n$, realized as permutations of $\{1, \dots, n-2\}$ and $\{n-1, n\}$ [LM00, Theorem 2.5(b)].

The constructions of Losev-Manin give an explicit realization of the torus T and its character module $\mathfrak{X}^*(\mathsf{T})$. Let P denote the permutation module for $\mathfrak{S}_{n-2}$ associated with the first $n-2$ letters and L the non-trivial rank-one module for $\mathfrak{S}_2$ corresponding to $n-1$ and n. We regard these as modules for $\mathfrak{S}_{n-2} \times \mathfrak{S}_2$. Consider the short exact sequence

$$0 \to P_0 \to P \to \mathbb{Z} \to 0$$

associated with summing over the $n-2$ letters. Then we have

$$\mathfrak{X}^*(\mathsf{T}) = L \otimes P_0. \tag{2.1}$$

Indeed, we may describe the open torus orbit in $\overline{L}_n$ in geometric terms: We identify the points $n-1$ and n as 0 and ∞ and the first $n-2$ points as elements of

$$\mathrm{Hom}(P, \mathbb{P}^1 \setminus \{0, \infty\}) = \mathrm{Hom}(P, \mathsf{T}_L),$$

where T_L is the rank-one torus associated with L. To get moduli, we quotient out by the diagonal action of T_L.

We record one last observation: Consider the Kapranov blowups associated with points $n-1$ and n:

$$\beta_n[n-1], \beta_n[n] : \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}.$$

These two maps are related by an elementary Cremona transformation

$$\mathrm{Cr} : \mathbb{P}^{n-3} \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-3}$$

associated with the points indexed by $\{1, \dots, n-2\}$. This is equivariant for the T-actions and we obtain a birational contraction

$$\overline{L}_n \to \mathrm{Graph}(\mathrm{Cr}).$$

We summarize this as follows:

Proposition 7. *Consider a twist of $\overline{\mathcal{M}}_{0,n}$ associated with a subgroup of $\mathfrak{S}_n$ leaving a pair of points invariant. This variety is necessarily toric, realized as a twist of the Losev-Manin space.*

This applies in both equivariant and Galois-theoretic situations.

The Gelfand-MacPherson correspondence. Our main source is Kapranov [Kap93].

Let $\mathrm{Mat}(2, n)$ denote the $2 \times n$ matrices. The group GL_2 acts via multiplication from the left

$$\mathrm{A} \cdot \mathrm{M} \mapsto \mathrm{AM}$$

and the torus $\mathsf{T} = \mathbb{G}_m^n$ acts via multiplication from the right

$$\mathrm{M} \cdot \mathsf{T} \mapsto \mathrm{M}\mathsf{T}, \quad \mathsf{T} = \mathrm{diag}(t_1, \dots, t_n).$$

Considering the action by the product $\mathrm{GL}_2 \times \mathbb{G}_m^n$, with the elements

$$\left(t^{-1}\,\mathrm{I}_2, \mathrm{diag}(t, t, \dots, t)\right)$$

in the kernel, we obtain a faithful action of the quotient group

$$(\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m.$$

We have an exact sequence

$$1 \to \mu_2 \to \mathrm{SL}_2 \times \mathbb{G}_m^n \to (\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m \to 1,$$

where

$$\mu_2 = \left(-\mathrm{I}_2, \mathrm{diag}(-1, -1, \dots, -1)\right).$$

The invariant theory quotient is

$$\mathrm{SL}_2 \backslash \mathrm{Mat}(2, n) = \mathrm{CGr}(2, n),$$

the cone over the Grassmannian $\mathrm{Gr}(2, n)$ in its Plücker imbedding. The residual action of $\mathbb{G}_m^n$ on this cone has generic stabilizer μ_2; the action on the Grassmannian has generic stabilizer $\mathbb{G}_m = \mathrm{diag}(t, t, \ldots, t)$. On the other hand, the geometric invariant theory quotient

$$\mathrm{Mat}(2, n)//\mathbb{G}_m, \quad \mathbb{G}_m = \mathrm{diag}(t, t, \ldots, t)$$

yields $(\mathbb{P}^1)^n$ with factors induced by the columns of the matrix. The residual SL_2 acts on this product with the distinguished linearization introduced above, which is $\mathfrak{S}_n$-symmetric. Again, this action fails to be faithful, as $\mu_2 \subset \mathrm{SL}_2$ acts trivially.

The Gelfand-MacPherson construction yields isomorphisms

$$(\mathrm{CGr}(2, n) \setminus \{0\}) /\mathbb{G}_m^n \xrightarrow{\sim} \mathrm{SL}_2 \backslash\backslash (\mathbb{P}^1)^n, \tag{2.2}$$

where both sides are interpreted as GIT quotients [Kap93, 2.4.7]. Note that we have numerous choices for how to linearize the actions on the left- and right-hand sides, reflecting linearizations of the torus action and ample line bundles on the product; Kapranov's result makes clear how to identify these choices. Let X_n denote the quotient arising from the $\mathfrak{S}_n$-symmetric linearization.

Recall that the stable and strictly semistable loci on $(\mathbb{P}^1)^n$ are easily identified

$$(p_1, \ldots, p_n) \text{ stable if there is no point with multiplicity } \geq \frac{n}{2}. \tag{2.3}$$

It is semistable if all points have multiplicity $\leq \frac{n}{2}$. For odd n, stable and semistable coincide; for even $n = 2m$, collections of points where m indices coincide are strictly stable, with closed orbits consisting of collections where

$$p_{i_1} = \cdots = p_{i_m}, \quad p_{i_{m+1}} = \cdots = p_{i_{2m}}, \quad \{i_1, \ldots, i_{2m}\} = \{1, \ldots, 2m\}.$$

In particular, $X_{2m}, m \geq 3$ has $\frac{1}{2}\binom{2m}{m}$ distinguished singular points over which the orbits are identified.

The stable loci on the Grassmannian $\mathrm{Gr}(2, n)$ for the action of $\mathbb{G}_m^n \cap \mathrm{SL}_n$ may be described as well: Choose a basis diagonalizing the torus action and let $(A_{ij}), 1 \leq i < j \leq n$ denote the associated Plücker coordinates. The point (A_{ij}) is stable if there are

(1) no index i with $A_{ij} = 0$ for every j; and
(2) no subset $I \subset \{1, \ldots, n\}$ with $|I| \geq \frac{n}{2}$ and $A_{ij} = 0$ for all $i, j \in I$.

These descriptions yield an $\mathfrak{S}_n$-equivariant stratified blowup [Kap93, 0.4.3,4.1.8]

$$\beta : \overline{\mathcal{M}}_{0,n} \to X_n.$$

This blows down all the boundary divisors D_I except those where $|I|$ or $|I^c| = 2$. The divisors D_I with $2|I| = n$ are collapsed to the distinguished singular points $\Sigma \subset X_{2m}$ where $m = |I|$ and $n = 2m$.

The Gelfand-MacPherson construction is a powerful tool for computing class groups. The induced homomorphism

$$\beta_* : \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}) = \mathrm{Cl}(\overline{\mathcal{M}}_{0,n}) \to \mathrm{Cl}(X_n) \tag{2.4}$$

is surjective because β is a fibration away from the distinguished singular points. Thus we get an exact sequence

$$0 \to N \to M \to Q \to 0, \tag{2.5}$$

where

$$N = \ker(\beta_*), \quad M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}).$$

In particular, N is generated by the D_I where $|I|, |I^c| \neq 2$. We can easily compute Q is well. Write

$$\mathfrak{X}^*(\mathbb{G}_m^n) = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n,$$

so the quotient acting faithfully on the $\mathrm{CGr}(2,n)$ has characters

$$\{\sum a_i g_i : a_i \in \mathbb{Z}, \sum a_i \equiv 0 \pmod 2\}.$$

These give rise to line bundles on $X_n \setminus \Sigma$ and divisor classes on the full space. Thus we deduce that

$$Q \subset \mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$$

as an index-two subgroup. Note that the element $g_{i_1} + g_{i_2}, i_1 \neq i_2$ corresponds to the boundary divisor $D_{i_1 i_2}$; indeed, this locus is cut out by the 2×2 determinant on $\mathbb{P}^1_{i_1} \times \mathbb{P}^1_{i_2}$. Since Q is an index-two subgroup of a permutation module, we have

$$\mathrm{H}^1(G, Q) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{H}^1(G, M) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z}. \tag{2.6}$$

When n is odd, i.e., $n = 2m+1$, then X_{2n+1} is nonsingular,

$$\mathrm{Pic}(X_{2m+1}) = \mathrm{Cl}(X_{2m+1}),$$

and β is the iteration of a sequence of blowups along smooth disjoint centers. Precisely, we blow up the strata where m points coincide, then where $m-1$ points coincide, etc. (see [Has03, §8]); this is naturally equivariant under the $\mathfrak{S}_{2m+1}$-action. By the blowup formula [Ful98, Prop. 6.7], we have

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,2m+1}) = \mathrm{Pic}(X_{2m+1}) \oplus \{\text{free group on the exceptional divisors}\}.$$

We summarize this in algebraic terms:

Proposition 8. *For odd $n = 2m + 1$, the exact sequence (2.5) splits $\mathfrak{S}_{2m+1}$-equivariantly:*

$$M \simeq N \oplus Q.$$

On the other hand, for n even, e.g., $n = 6$, there are examples of $G \subset \mathfrak{S}_n$ such that the sequence does not split equivariantly, since in those cases $\mathrm{H}^1(G, Q) \neq 0$ while $\mathrm{H}^1(G, M) = 0$ (see Example 27).

We return to the isomorphism (2.2) over nonclosed fields. Up to this point, we have been working with schemes but this is compatible with the μ_2-gerbe structure over the dense open subset where this is the full stabilizer. When $n = 2m$ the stabilizers may be larger, e.g., where the sequence in $(\mathbb{P}^1)^{2m}$ consists of m copies of a pair of points conjugate over a quadratic extension. In the cone over the Grassmannian, $2\binom{m}{2} = m^2 - m$ coordinates vanish and the m^2 remaining coordinates are equal to the determinant of the conjugate pair.

We can apply the same analysis to nonsplit actions. This includes working over nonclosed fields, where the n points are a Galois orbit, or in the equivariant context, where the n points are invariant under the action of a finite group. In the former situation, over a ground field F of characteristic zero, let E/F be an étale algebra of degree n classified by a representation of the Galois group $\Gamma_F \to \mathfrak{S}_n$. We replace the group $(\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$ with $(\mathrm{GL}_2 \times R_{E/F}\mathbb{G}_m)/\mathbb{G}_m$ and $(\mathbb{P}^1)^n$ with $R_{E/F}\mathbb{P}^1$ (see [FR18, §4]). Note however that twisting $\mathrm{Mat}(2, n) = \mathbb{A}^{2n}$ yields a variety isomorphic to $\mathbb{A}^{2n}$, albeit with an action of a nonsplit torus.

The μ_2-gerbe has an explicit geometric interpretation along $\mathcal{M}_{0,n}$: It is encoded by the universal family

$$\phi : \mathcal{C}_{0,n} \to \mathcal{M}_{0,n},$$

a conic fibration, in general.

3. Rationality constructions

In this section, we work over an arbitrary field F, and we let Γ be the absolute Galois group of F.

Schubert calculus background. Our reference is [Kly85].

Consider the Grassmannian $\mathrm{Gr} = \mathrm{Gr}(p, p + q)$ of p-dimensional subspaces of a vector space of dimension $p + q$. The maximal torus $\mathsf{T} = \mathbb{G}_m^{p+q}$ acts diagonally on the vector space. Let X be a generic orbit in Gr.

We set combinatorial notation: Consider shuffles of $\{1,\dots,p+q\}$

$$I=\{i_1<\cdots<i_p\},\quad J=\{j_1<\cdots<j_q\}.$$

For each such shuffle, record the pairs $(k,\ell), k=1,\dots,p, \ell=1,\dots,q$, such that $i_k>j_\ell$. Write

$$\lambda_{p+1-k}=\#\{\ell : j_\ell<i_k\}$$

and note that

$$q\geq\lambda_1\geq\cdots\geq\lambda_p.$$

Write $\lambda=(\lambda_1,\dots,\lambda_p)$ and use the same notation for the associated Young diagram, which fits into a $p\times q$ rectangle. The *height* $\mathrm{ht}(\lambda)$ is the number of indices i with $\lambda_i>0$. Set $|\lambda|=\lambda_1+\cdots+\lambda_p$ and let σ_λ denote the associated Schubert cycle on Gr, a class in $\mathrm{H}^{2|\lambda|}(\mathrm{Gr},\mathbb{Z})$.

We recall dimension formulae for representations. Let V be an n-dimensional vector space and $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition of $|\lambda|$ as above; in particular, $n\geq\mathrm{ht}(\lambda)$. The Schur functor $\mathbb{S}_\lambda(V)$ is a representation of $\mathrm{SL}(V)$ with dimension [FH91, Theorem 6.3, Exercise 6.4]:

$$\begin{aligned}d_n(\lambda):=\dim\mathbb{S}_\lambda(V)&=\prod_{1\leq i<j\leq n}\frac{\lambda_i-\lambda_j+j-i}{j-i}\\&=\prod_{(a,b)}\frac{n-a+b}{h_{ab}},\end{aligned}$$

where $a=1,\dots,n$ labels the rows of λ (from top to bottom), b labels the columns (from left to right), and h_{ab} labels the "hook length". This is defined as the number of boxes immediately below and to the right of a given box, including the box. For $n<\mathrm{ht}(\lambda)$ we set $d_n(\lambda)=0$.

For example, when $\lambda=(\lambda_1,\lambda_2,0,\dots)$ and $n\geq 2$,

$$d_n(\lambda_1,\lambda_2)=$$
$$\frac{(n-1+1)\cdots(n-1+\lambda_1)}{1\cdots(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+2)\cdots(\lambda_1+1)}\frac{(n-2+1)\cdots(n-2+\lambda_2)}{1\cdots\lambda_2}$$

$$=\binom{n-1+\lambda_1}{\lambda_1}\binom{n-2+\lambda_2}{\lambda_2}\frac{\lambda_1-\lambda_2+1}{\lambda_1+1}.$$

For instance,

$$d_n(2,1)=\frac{(n+1)n(n-1)}{3},\quad n\geq 1.$$

Another combinatorial quantity is

$$m_k(\lambda) := \sum_{i=0}^{k} (-1)^i \binom{|\lambda|+1}{i} d_{k-i}(\lambda).$$

If λ has height k then $m_k(\lambda) = d_k(\lambda)$, as the terms in the sum with $i > 0$ are zero.

We record a fact that we will use repeatedly in examples:

Proposition 9. *Fix an integer $d \geq 0$. If $f(x)$ is a polynomial of degree $\leq d$ then the $(d+1)$th iterated difference*

$$\sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} f(x-i) = 0.$$

When $\lambda = (\lambda_1, \lambda_2, 0, \dots)$ we have:

$$m_k(\lambda_1, \lambda_2) =$$

$$\sum_{i=0}^{k} (-1)^i \binom{\lambda_1+\lambda_2+1}{i} \binom{k-i-1+\lambda_1}{\lambda_1} \binom{k-i-2+\lambda_2}{\lambda_2} \frac{\lambda_1-\lambda_2+1}{\lambda_1+1}.$$

For instance, when $\lambda_1 = 2$ and $\lambda_2 = 1$ we have

$$\begin{aligned} m_k(2,1) &= \sum_{i=0}^{k} (-1)^i \binom{4}{i} \frac{(k-i+1)(k-i)(k-i-1)}{3} \\ &= 2\left(\binom{k+1}{3} - 4\binom{k}{3} + 6\binom{k-1}{3} - 4\binom{k-2}{3} + \binom{k-3}{3}\right) \\ &= \begin{cases} 2 & \text{if } k = 2, \\ 0 & \text{if } k \geq 3. \end{cases} \end{aligned}$$

For general λ_1 and λ_2,

$$m_2(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2 + 1$$

and

$$m_3(\lambda_1, \lambda_2) = \frac{\lambda_1(\lambda_2-1)(\lambda_1-\lambda_2+1)}{2}.$$

Theorem 10. [Kly85, Theorem 5] *If X is the generic torus orbit in $\mathrm{Gr} = \mathrm{Gr}(p, p+q)$ and λ is a partition with $|\lambda| = p+q-1$ then*

$$[X] \cdot \sigma_\lambda = m_p(\lambda).$$

For example, take $p = 2$. For $q = 2$

$$[X] \cdot \sigma_{21} = 2$$

and when $q = 3$ we have

$$[X] \cdot \sigma_{22} = 1, \quad [X] \cdot \sigma_{31} = 3.$$

For general q, we have $\lambda_1 \geq \lambda_2 = q + 1 - \lambda_1 \geq 0$, i.e.,

$$\frac{q+1}{2} \leq \lambda_1 \leq q + 1.$$

Here we have

$$[X] \cdot \sigma_{\lambda_1\, q+1-\lambda_1} = 2\lambda_1 - q;$$

in particular, when $q = 2m - 1$ and $\lambda_1 = m$ we find

$$[X] \cdot \sigma_{m\,m} = 1.$$

Remark 11. The signs in the formula for $m_k(\lambda)$ obscure the positivity of the result. An alternate formula [BF17, Theorem 5.1] makes this clearer:

$$[X] = \sum_{\lambda \subset (q-1)^{p-1}} \sigma_\lambda \sigma_{\widetilde{\lambda}},$$

where $\widetilde{\lambda}$ is the complement to λ in the rectangle $(q-1)^{p-1}$:

$$\lambda = (\lambda_1, \ldots, \lambda_{p-1}), \quad \widetilde{\lambda} = (q - 1 - \lambda_{p-1}, \ldots, q - 1 - \lambda_1).$$

We refer the reader to [Lia23] for the combinatorics directly relating these formulas.

This extends to general $p \in \mathbb{N}$:

Proposition 12. *Let V be a vector space with $\dim(V) = mp + 1$ so that*

$$q = (m-1)p + 1 \quad \text{and} \quad (p-1)(q-1) = (m-1)(p-1)p.$$

Consider the coefficient of

$$\sigma_{\underbrace{(m-1)(p-1)\ldots(m-1)(p-1)}_{p\ times}}$$

in the expansion of $[X]$ in $\mathrm{H}^{2(p-1)(q-1)}(\mathrm{Gr}(p, p+q))$. This equals 1, i.e.,

$$[X] \cdot \sigma_{\underbrace{m \ldots m}_{p\ times}} = 1.$$

Indeed, this follows from Klyachko's formula (Theorem 10) and

$$m_p(\underbrace{m,\ldots,m}_{p \text{ times}}) = d_p(\underbrace{m,\ldots,m}_{p \text{ times}}) = 1.$$

Example 13. When $\dim(V) = 3m+1$ the generic orbit X for the action of T on $\mathrm{Gr}(3,V)$ has codimension $3(3m-2)-3m = 6(m-1)$ and

$$[X]\cdot\sigma_{m\,m\,m} = m_3(m,m,m) = d_3(m,m,m) = 1.$$

This is not the case when $\dim(V) = 3m+2, m > 1$, e.g.,

$$[X] = 10\sigma_{5,3} + 8\sigma_{5,2,1} + 15\sigma_{4,4} + 15\sigma_{4,3,1} + 6\sigma_{4,2,2} + 3\sigma_{3,3,2}.$$

Grassmann geometry and rationality.

Theorem 14. *Let T be a maximal torus – possibly nonsplit - for SL_{pm+1} over a field F. Take $\mathrm{Gr}(p,V)$ for $\dim_F(V) = pm+1$ with the resulting T-action. Choose a subspace $W \subset V$ with*

$$\dim_F(W) = (p-1)m + 1$$

and transverse to T in the sense that $\mathrm{Gr}(p,W) \subset \mathrm{Gr}(p,V)$ meets some stable T-orbit properly. Then $\mathrm{Gr}(p,W)$ is a rational section of the quotient

$$\mathrm{Gr}(p,V) \stackrel{\sim}{\dashrightarrow} \mathrm{Gr}(p,V)/\mathsf{T}.$$

Thus if $\mathrm{Gr}(p,W)$ is rational, linearizable, or stably linearizable then the same holds true of the quotient.

Florence [Flo13, §3] has obtained similar results when V carries a suitable F-algebra structure.

Proof. The stability assumption guarantees that the quotient map is defined over a non-empty open subset of $\mathrm{Gr}(p,W)$. Properness of the intersection – which has degree one by Proposition 12 – implies $\mathrm{Gr}(p,W)$ is mapped birationally to the quotient. □

Proposition 15. *Retain the notation of Theorem 14.*

If F is infinite then $\mathrm{Gr}(p,V)$ admits a codimension-m subspace $W \subset V$ satisfying the transversality condition.

If F is finite and $p = 2$ then $\mathrm{Gr}(2,V)$ admits a stable F-rational point.

If F is arbitrary and $p = 2$ then for each stable point there exists a subspace W satisfying the transversality assumption.

Combining with Theorem 14 gives a generalization of the results of [FR18]:

Corollary 16. *Let F be a finite field and ρ a representation of its Galois group in $\mathfrak{S}_{2m+1}$. Then ${}^{\rho}\overline{\mathcal{M}}_{0,2m+1}$ is rational over F.*

We also obtain analogs in higher dimensions:

Corollary 17. *Let $m \geq 1$ and $p \geq 2$ be integers. Consider the moduli space of $pm+1$ points in $\mathbb{P}^{p-1}$ up to projective equivalence. Let X be a variety obtained by twisting via permutations of the points, over an infinite field F. Then X is rational.*

Proof of Proposition 15. Assume F is infinite; here we use [Kap93, §1.2]. While Kapranov assumes the ground field has characteristic zero, the toric constructions and interpretation of $\overline{\mathcal{M}}_{0,n}$ as a Chow quotient for the PGL_2-action are valid in positive characteristic [GG14].

The Grassmannian is rational over F so its F-rational points are Zariski dense. We note that the torus action determines a collection of $\overline{F}$-subspaces

$$V_I \subset V, \quad \emptyset \neq I = \{i_1, \dots, i_r\} \subset \{0, \dots, mp\},$$

spanned by eigenvectors of the torus. Consider the

$$W \in \mathrm{Gr}(mp+1-m, mp+1)$$

meeting some of these improperly, i.e.,

$$\dim(W \cap V_I) > \dim(W) + \dim(V) - \dim(V_I).$$

This is a Zariski closed proper subset of the Grassmannian, defined over F; its complement has F-rational points. Given such a subspace $W \subset V$, choose

$$w \in \Lambda \subset W, \quad \dim(\Lambda) = p,$$

defined over F, with w not contained in any of the $V_I \subsetneq V$ and Λ meeting all the V_I properly. Thus Λ is stable for the torus action and the torus orbit of Λ meets $\mathrm{Gr}(p, W)$ transversally there.

Now assume that F is finite and $p = 2$. We use the stability criterion (2.3) for points on $\mathbb{P}^1$ and Kapranov's analysis of the Gelfand-MacPherson correspondence. Here the Galois action ρ on the $2m+1$ points is encoded by a single element $\sigma \in \mathfrak{S}_{2m+1}$. Express σ as a product of r disjoint cycles of lengths ℓ_i with

$$\ell_1 + \cdots + \ell_r = 2m+1, \quad \ell_1 \geq \ell_2 \geq \cdots \geq \ell_r.$$

Only ℓ_1 can possibly be greater than m; if $\ell_1 \leq m$ then we have $r \geq 3$. When $\ell_1 > m$, choose a configuration of ℓ_1 points defined over a degree-ℓ_1 extension of F. Allow the remaining points to all coincide. We turn to the situation where $\ell_1 \leq m$. If $r = 3$ then we allow ℓ_1 points to coincide with $[0,1]$, ℓ_2 points to coincide with $[1,0]$, and ℓ_3 points to coincide with $[1,1]$. We may therefore assume that $r \geq 4$ and work inductively on r. There exists two indices, say ℓ_3 and ℓ_4, whose sum is less than m. Use this to "degenerate" to a new partition of $2m+1$, refined by $(\ell_1, \dots, \ell_r)$ but of length $r-1$, all of whose entries are less than m. For example, we could take

$$(\ell_1, \ell_2, \ell_3 + \ell_4, \ell_5, \dots, \ell_r).$$

Continuing in this way, we generate a partition

$$\{1, 2, \dots, r\} = A \sqcup B \sqcup C$$

such that

$$\sum_{a \in A} \ell_a, \sum_{b \in B} \ell_b, \sum_{c \in C} \ell_c \leq m.$$

Let points coincide in three groups according to this coarsening of our original partition, the first group to $[0,1]$, the second to $[1,0]$, and the third to $[1,1]$.

Assume $p = 2$ and F is arbitrary. We continue to assume that $\Lambda \subset V$ is a two-dimensional subspace that is stable in the sense of Geometric Invariant Theory. Let $\mathbf{T}_{2m}$ denote the tangent space to the torus orbit at Λ

$$\mathbf{T}_{2m} \subset \mathrm{Hom}(\Lambda, V/\Lambda),$$

an $2m$-dimensional subspace of the tangent space to $\mathrm{Gr}(2,V)$ at Λ. We claim there exists a subspace

$$\Lambda \subset W \subset V,$$

where W has codimension m in V, such that the composition

$$\mathbf{T}_{2m} \subset \mathrm{Hom}(\Lambda, V/\Lambda) \twoheadrightarrow \mathrm{Hom}(\Lambda, V/W)$$

has full rank $2m$. Since the latter space is the normal directions to $\mathrm{Gr}(2,W)$ at Λ, this will yield transversality.

We record some basic geometry:

Lemma 18. *There is a distinguished orbit*

$$\mathbb{P}^1 \times \mathbb{P}^{m-2} \simeq \mathbb{P}(\Lambda^*) \times \mathbb{P}(V/\Lambda) \subset \mathbb{P}(\mathrm{Hom}(\Lambda, V/\Lambda))$$

invariant under automorphisms of $\mathrm{Gr}(2,V)$ *fixing* $[\Lambda]$.

The subspace $\mathbb{P}^{2m-1} \simeq \mathbb{P}(\mathbf{T}_{2m})$ *cuts out the graph of a rational normal curve*

$$\begin{aligned} \varrho : \mathbb{P}^1_{s_0,s_1} &\hookrightarrow \mathbb{P}^{2m-2}_{x_0,\dots,x_{2m-2}} \\ [s_0, s_1] &\mapsto [s_0^{2m-2}, \dots, s_1^{2m-2}]. \end{aligned}$$

In these coordinates, the rational normal curve has equations

$$s_0 x_{i+1} = s_1 x_i, \quad i = 0, \dots, 2m-1.$$

Let $\Gamma \subset \mathbb{P}^1$ denote the length-$(2m+1)$ subscheme that is the image of the eigenvectors for $\mathbf{T}_{2m}$ under $V^* \twoheadrightarrow \Lambda^*$. Then ϱ realizes the Gale transform for $\Gamma \subset \mathbb{P}^1$ as a subscheme of $\mathbb{P}^{2m-2}$ contained in a rational normal curve.

The first assertion reflects the fact that the parabolic subgroup of PGL_{2m+1} fixing $[\Lambda]$ has semisimple part $(\mathrm{GL}_2 \times \mathrm{GL}_{2m-1})/\mathbb{G}_m$. Note that the unipotent part acts trivially on the tangent space. The second assertion is true for the generic codimension-$(2m-2)$ linear slice of $\mathbb{P}^1 \times \mathbb{P}^{2m-1}$. Of course, one has to show that this applies in our siutation! This follows from the third assertion, a special case of [EP00, Corollary 3.2] – the first application following the statement. This completes the proof of the lemma.

Returning to the proof of the Proposition, we may take W as the subspace given by

$$\{x_{2j} = 0, j = 0, \dots, m-1\},$$

where we interpret $x_j \in (V/\Lambda)^* a$. It is clear that the products

$$\{s_i x_{2j}, i = 0, 1, j = 0, \dots, m-1\}$$

have the desired spanning property; the elements

$$s_0^{2m-1}, \dots, s_1^{2m-1}$$

are a basis for bilinear forms of degree $2m-1$. □

Partitioning the points. We start with a general construction: Let $n \geq 3$ be an integer and $n = \ell m$ a factorization in integers $\ell, m > 1$. Suppose that $H \subset \mathfrak{S}_\ell, A \subset \mathfrak{S}_m$ are subgroups. The *wreath product*

$$A \wr H = A \wr_{1,\dots,\ell} H$$

is the semidirect product $A^\ell \rtimes H$ where

$$(a_1, \dots, a_\ell) \cdot h = (a_{h^{-1}(1)}, \dots, a_{h^{-1}(\ell)}).$$

This comes with a natural embedding

$$\rho : A \wr H \hookrightarrow \mathfrak{S}_{\ell m}$$

as permutations of pairs

$$(i, j), \quad i \in \{1, \dots, m\}, j \in \{1, \dots, \ell\}.$$

Now assume that $m \geq 3$. Forgetting maps yield an equivariant morphism

$$\phi : {}^{\rho}\overline{\mathcal{M}}_{0,\ell m} \to \prod_H {}^{\alpha}\overline{\mathcal{M}}_{0,m},$$

where $\alpha : A \hookrightarrow \mathfrak{S}_m$ and the twisted product denotes ℓ copies of the moduli space with the associated H-action. The generic fiber of this morphism is irreducible of dimension

$$(\ell m - 3) - \ell(m-3) = 3\ell - 3.$$

It is birational to the Hilbert scheme of multidegree-$(1, \ldots, 1)$ curves in the H-twisted product $\prod_H C_j$ of ℓ genus-zero curves. Geometrically, this is a compactification of the homogeneous space

$$\underbrace{\mathrm{PGL}_2 \times \cdots \times \mathrm{PGL}_2}_{\ell \text{ times}} / \mathrm{PGL}_2$$

with the last PGL_2 embedded diagonally.

We record some observations on the generic fiber of ϕ:

- Suppose $\ell = 2$. Geometrically, $(1,1)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are parametrized by $\mathbb{P}^3$ – the dual to the projective space containing the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. Over an arbitrary field the fiber is a Brauer-Severi threefold.
- Suppose that m is odd. Then the genus-zero curves C_j appearing in the twisted product are split and – over the extension/-subgroup associated with $A^\ell \subset A \wr H$ – isomorphic to $\mathbb{P}^1$'s. Here the twisted product $\prod_H C_j$ is rational, as it is isomorphic to the restriction of scalars of $\mathbb{P}^1$.
- Now assume $\ell = 2$ and m odd. Here the generic fiber of ϕ is isomorphic to $\mathbb{P}^3$ over the function field/linearizable for the full wreath product.

Example 19. Suppose $n = 6$ and consider $G = \mathfrak{S}_3 \wr \mathfrak{S}_2 \subset \mathfrak{S}_6$, a subgroup of index 10 preserving an unordered partition

$$\{1,2,3,4,5,6\} = \{i,j,k\} \sqcup \{a,b,c\}.$$

Then the associated ${}^{\rho}\overline{\mathcal{M}}_{0,6}$ is rational/linearizable. These actions correspond to situations where the associated Segre threefold admits an invariant node (cf. Theorem 34 below).

Theorem 20. *Let $n = 2m$, with $m \geq 3$ odd. Fix a subgroup $A \subset \mathfrak{S}_m$ and the diagonal subgroup*

$$G := A \times \mathfrak{S}_2 \subset A \wr \mathfrak{S}_2 \subset \mathfrak{S}_{2m}.$$

- *For each Galois representation $\rho : \Gamma \to G$ the twist ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ is rational over F.*
- *The G action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.*

Proof. We assume $\mathfrak{S}_m$ permutes the points with odd and even indices respectively.

We focus first on the arithmetic case. Let L/F be the quadratic extension associated with A. Over L, the generic point of the twisted moduli space corresponds to $\mathbb{P}^1$ equipped with reduced and disjoint zero-cycles $Z_{odd}, Z_{even} \subset \mathbb{P}^1$ of length m. The parity of m ensures that the underlying curve is $\mathbb{P}^1$.

Note that the variety ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ is already stably rational over L by Proposition 5.

Consider forgetting the even and odd points

$$(\pi_{odd}, \pi_{even}) : ({}^{\rho}\overline{\mathcal{M}}_{0,n})_L \to {}^{\varpi_{odd}}\overline{\mathcal{M}}_{0,m} \times {}^{\varpi_{even}}\overline{\mathcal{M}}_{0,m}$$

where the Galois actions come via restriction to the even and odd points. These actions are conjugate for the quadratic extension L/F. Descent therefore gives a morphism over F

$$\phi : {}^{\rho}\overline{\mathcal{M}}_{0,n} \to R_{L/F}({}^{\varpi_{odd}}\overline{\mathcal{M}}_{0,m}),$$

where the target is the restriction of scalars. The twists of $\overline{\mathcal{M}}_{0,m}$ are rational over L by [FR18] and Corollary 16. The restriction of scalars of a rational variety is rational.

We claim that the generic fiber of ϕ is rational over the function field of the base, which implies rationality for ${}^{\rho}\overline{\mathcal{M}}_{0,n}$ over F. This follows from the analysis above for $\ell = 2$ and odd m.

For the equivariant case, our geometric argument shows that the G-variety $\overline{\mathcal{M}}_{0,n}$ is birationally the projectivization of an equivariant vector bundle over a stably linearizable variety (by Proposition 5). Note that restriction of scalars in the arithmetic situation corresponds to passing to an induced representation in the equivariant context; thus stable linearizability is clearly preserved. We conclude then that $\overline{\mathcal{M}}_{0,n}$ is stably linearizable. □

Corollary 21. *Let C_{2m}, with m odd, be a cyclic group. Then twists of $\overline{\mathcal{M}}_{0,n}$ by this group are rational (in the Galois case) and stably linearizable (in the equivariant situation).*

Proof. If the action has an odd orbit then this follows from Propositions 3 and 5. Otherwise, all the orbits are even and we may apply Theorem 20. □

Remark 22. Similar reasoning applies for a Galois action

$$\rho : \Gamma \to \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2} \subset \mathfrak{S}_{m_1+m_2}, \quad m_1, m_2 \geq 3 \text{ odd},$$

with restricted actions ϖ_1 and ϖ_2 on the first m_1 points and last m_2 points respectively. Proposition 5 already gives stable rationality in this case. The forgetting morphism

$$\phi : {}^{\rho}\overline{\mathcal{M}}_{0,m_1+m_2} \to {}^{\varpi_1}\overline{\mathcal{M}}_{0,m_1} \times {}^{\varpi_2}\overline{\mathcal{M}}_{0,m_2}$$

has generic fiber birational to $\mathbb{P}^3$ by the reasoning above. Since the factors ${}^{\varpi_i}\overline{\mathcal{M}}_{0,m_i}$ are rational, ${}^{\rho}\overline{\mathcal{M}}_{0,m_1+m_2}$ is rational as well.

4. Stable linearizability via torsors

Let G be a finite group and T a G-torus, i.e., a torus equipped with a representation of G on its character module $\mathfrak{X}^*(\mathsf{T})$. Recall that T is stably linearizable if $\mathfrak{X}^*(\mathsf{T})$ is stably permutation, see, e.g., [HT23, Proposition 2].

Proposition 23. *Let U be a smooth quasi-projective variety with G-action. Assume that we have a T-torsor*

$$\mathcal{P} \to U,$$

i.e., a T-principal homogeneous space over U, in the category of G-varieties. Assume that

- *the G-action on U is generically free,*
- *the characters $\mathfrak{X}^*(\mathsf{T})$ are a stably permutation G-module,*
- *the G-action on $\mathcal{P}$ is stably linearizable.*

Then the G-action on U is stably linearizable.

Proof. We claim there is a G-equivariant birational map,

$$\begin{array}{ccc} \mathcal{P} & \overset{\sim}{\dashrightarrow} & \mathsf{T} \times U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

which would follow if $\mathcal{P} \to U$ admits a G-equivariant rational section. We clearly have such a section after discarding the G-action, by Hilbert's Theorem 90.

Since T is stably permutation, a product $\mathsf{T} \times \mathsf{T}_1$, where T_1 is a permutation torus, is isomorphic to a permutation torus and may be realized as a dense open subset of affine space. It follows that we have an open embedding

$$\begin{array}{ccc} \mathcal{P} \times_U \mathsf{T}_1 & \hookrightarrow & \mathcal{V} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

where $\mathcal{V} \to U$ is a vector bundle with G-action. The vector bundle admits a rational section (by the No-Name Lemma) thus $\mathcal{P}$ does as well.

We assumed that $\mathcal{P}$ is stably linearizable, i.e. $\mathcal{P} \times \mathbb{G}_m^r$ is linearizable for some r. Thus $U \times \mathsf{T} \times \mathbb{G}_m^r$ is as well. We observed that T is stably linearizable because its character module is stably permutation, i.e. $\mathsf{T} \times \mathsf{T}_1$ is a permutation torus. Another application of the No-Name Lemma, using the assumption that the action on U is generically free, gives that U is stably linearizable. □

We recall the exact sequence (2.5)

$$0 \to N \to M \to Q \to 0,$$

with $M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$, N an $\mathfrak{S}_n$-permutation module, and Q is an index-2 submodule of the permutation module $\mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$. We record:

- if $\mathrm{H}^1(G, Q) = 0$ for some $G \subset \mathfrak{S}_n$, then also $\mathrm{H}^1(G, M) = 0$, by the long exact sequence in cohomology,
- if Q is a stably permutation G-module, then the sequence splits and $\mathrm{Pic}(\overline{\mathcal{M}}_{n,0})$ is a stably permutation module, by [CTS77, Lemma 1].

Theorem 24. *Let $G \subseteq \mathfrak{S}_n$ be a subgroup such that Q is a stably permutation module. Then the G-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable.*

Let X be a form of $\overline{\mathcal{M}}_{0,n}$ over F such that the action of the absolute Galois group on Q gives rise to a stable permutation module. Then X is stably rational over F.

Proof. For the equivariant statement, we apply Proposition 23. Here T, with character module Q acts on $\mathrm{CGr}(2, n)$ (see Section 2). Let $V \subset \mathrm{CGr}(2, n)$ the open subset over which T acts freely and $U \subset X_n$ the corresponding locus in the quotient, i.e., remove all the strictly semistable points. We have a torsor

$$V \xrightarrow{\mathsf{T}} U.$$

By [HT23, Proposition 19], the $\mathfrak{S}_n$-action on $\mathrm{Gr}(2, n)$ (and its cone) is stably linearizable. Assuming that $Q = \mathfrak{X}^*(\mathsf{T})$ is a stable permutation module for $G \subset \mathfrak{S}_n$, and applying Proposition 23, we conclude that the G-action on U, and thus $\overline{\mathcal{M}}_{0,n}$, is stably linearizable as well.

The Galois-theoretic result is proven analogously, with [BCTSSD85, Prop. 3] playing the role of Proposition 23. This is an application of the torsor formalism of [CTS87]. □

Remark 25. There exist linearizable G-actions on $\overline{\mathcal{M}}_{0,n}$ such that the induced action on Q is not stably permutation. Consider n even and $G = C_2$ generated by $\sigma := (1,2)\cdots(n-1,n)$; we have $\mathrm{H}^1(C_2, Q) \neq 0$ (see Remark 32) so Q is not stably permutation. This action is equivariantly birational – by Proposition 7 – to an action on a torus $\mathsf{T} = \mathbb{G}_m^{n-3}$. Its character module consists of the elements of $\mathbb{Z}^{n-2}$ – the twisted permutation module on $\{1,\ldots,n-2\}$ – whose coordinates sum to zero (see Equation 2.1). The action of C_2 on the twisted permutation module consists of $(n-2)/2$ copies of $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Hence $\mathfrak{X}^*(\mathsf{T})$ decomposes as a sum of $\frac{n}{2} - 2$ permutation modules and one invariant, a permutation module. We conclude T is linearizable.

Remark 26. By [FR18, Remark 5.5], for *odd* n, every form of $\overline{\mathcal{M}}_{0,n}$ over a nonclosed field F is an F-rational variety. *A priori*, this does *not* imply that $\overline{\mathcal{M}}_{0,n}$ is (stably) linearizable for $\mathfrak{S}_n$. However, this does imply that M is a stable permutation module, for the $\mathfrak{S}_n$-action.

For n *odd*, we have

$$M \simeq N \oplus Q, \tag{4.1}$$

as $\mathfrak{S}_n$-modules, by Proposition 8. Since N is a permutation module, for all n, and M a stably permutation module, for odd n, we see that Q is also stably permutation, for odd n. Thus, the $\mathfrak{S}_n$-action on $\overline{\mathcal{M}}_{0,n}$ is stably linearizable, by Theorem 24.

The splitting (4.1) can also be seen explicitly: Recall that under the Kapranov basis, $Q = M/N$ is generated by the image of the classes

$$H, \quad E_i, \quad i = 1, \ldots, n-1$$

in M under the projection modulo N. The $\mathbb{Z}$-linear map

$$s : Q \to M,$$

given on these generators by

$$H \mapsto H + \sum_{\substack{I \subset \{1,\ldots,n-1\},\\ |I| = \frac{n-1}{2},\ldots,n-4.}} (|I|-1)\cdot E_I, \qquad E_i \mapsto E_i + \sum_{\substack{I \subset \{1,\ldots,n-1\}, i\in I,\\ |I| = \frac{n-1}{2},\ldots,n-4.}} E_I.$$

is a section of the exact sequence (2.5). We check that it is $\mathfrak{S}_n$-equivariant. Let $\tau = (1,2)$ and $\sigma = (1,\dots,n)$. In Q, one has

$$H = D_{12} + \sum_{i=3}^{n} E_i$$

and $\tau(H) = H$, $\tau(E_1) = E_2$, $\tau(E_2) = E_1$ and $\tau(E_i) = E_i$. Note that s is τ-equivariant by construction. Next, observe

$$\begin{aligned} s\sigma(H) &= s\left(\sigma\left(D_{12} + \sum_{i=3}^{n} E_i\right)\right) = s\left((n-3)H - (n-4)\sum_{i=2}^{n-1} E_i\right) \\ &= (n-3)H - (n-4)\sum_{i=2}^{n-1} E_i - \sum_{\substack{I\subset\{1,\dots,n-1\},1\notin I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} (n-|I|-3)\cdot E_I \\ &\quad + \sum_{\substack{I\subset\{1,\dots,n-1\},1\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} (|I|-1)\cdot E_I. \end{aligned}$$

$$\begin{aligned} \sigma s(H) &= \sigma\left(H + \sum_{|I|=\frac{n-1}{2},\dots,n-4} (|I|-1)\cdot E_I\right) \\ &= \sigma\left(D_{n-2,n-1} + \sum_{\substack{I\subset\{1,\dots,n-3\},\\ |I|=1,\dots,n-4.}} E_I + \sum_{|I|=\frac{n-1}{2},\dots,n-4.} (|I|-1)\cdot E_I\right) \\ &= E_{n-1} + \sum_{\substack{I\subset\{2,\dots,n-2\}\\ |I|=1,\dots,n-5}} D_{I\cup\{n-1,n\}} + \sum_{i=2}^{n-2} D_{1,i} \\ &\quad + \underbrace{\sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=\frac{n-1}{2}-1,\dots,n-5.}} E_{\{1\}\cup I} + \sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=2,\dots,\frac{n-1}{2}-1.}} (n-3-|I|)E_I}_{:=A} \\ &= (n-3)H - (n-4)\sum_{i=2}^{n-1} E_i - \sum_{\substack{I\subset\{2,\dots,n-1\},i\notin I,\\ |I|=1,\dots,n-4.}} E_I \end{aligned}$$

$$+ \sum_{\substack{I\subset\{2,\dots,n-2\},\\ |I|=1,\dots,n-5.}} D_{I\cup\{n-1,n\}} + A.$$

One can then verify $\sigma s(H) = s\sigma(H)$ by comparing the coefficients of each generator E_I. To check actions on E_i, for $i = 1, \dots, n-2$, one has

$$\begin{aligned} s\sigma(E_i) &= s(H - \sum_{k=2,k\neq i+1}^{n-1} E_k) \\ &= H - \sum_{k=2,k\neq i+1}^{n-1} E_k - \sum_{\substack{I\subset\{1,\dots,n-1\},\\ 1,i+1\notin I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I \quad + \sum_{\substack{I\subset\{1,\dots,n-1\},\\ 1,i+1\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma s(E_i) &= \sigma(E_i + \sum_{\substack{I\subset\{1,\dots,n-1\},i\in I,\\ |I|=\frac{n-1}{2},\dots,n-4.}} E_I) \\ &= H - \sum_{\substack{I\subset\{2,\dots,n-1\},i+1\notin I,\\ |I|=1,\dots,n-4}} D_{I\cup\{n\}} + \sum_{\substack{I\subset\{2,\dots,n-1\},\\ |I|=\frac{n-1}{2}-1,\dots,n-5.}} D_{I\cup\{1,n\}} \\ &\quad + \sum_{\substack{I\subset\{2,\dots,n-1\},i+1,\notin I\\ |I|=2,\dots,\frac{n-1}{2}-1.}} E_I. \end{aligned}$$

Similarly, one can check $\sigma s(E_i) = s\sigma(E_i)$ for $i \neq n-1$ by comparing the coefficients. Finally, one can verify

$$s(\sigma(E_{n-1})) = s(E_1) = \sigma(s(E_{n-1})).$$

5. Computing cohomology

In this section, we study the G-module

$$M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n}),$$

and the quotient $Q = M/N$, from (2.5), for various $G \subset \mathfrak{S}_n$.

Cohomological criteria. We focus on two properties, which are necessary for linearizability of a regular G-action on a smooth projective rational variety X, see, e.g., [BP13, Proposition 2.5]:

(H1) For all subgroups $G' \subset G$ one has

$$\mathrm{H}^1(G', \mathrm{Pic}(X)) = \mathrm{H}^1(G', \mathrm{Pic}(X)^*) = 0.$$

(SP) The G-module $\mathrm{Pic}(X)$ is stably permutation.

Since H^1 vanishes on permutation modules, (**SP**) implies **(H1)**, but the converse does not hold, in general. Computationally, it is easier to check **(H1)**.

Example 27. For $n = 6$ and $G \subseteq \mathfrak{S}_6$, property **(H1)** for the action on $M = \mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ does not imply **(SP)**, e.g., for the action of

$$G \simeq C_2 \times C_4 := \langle (3,4), (1,2,5,6) \rangle,$$

and

$$G \simeq (C_2)^3 := \langle (1,5)(2,6), (3,4), (1,2)(5,6) \rangle,$$

see the analysis in [CTZ23, Section 6], as well as [Kun87, Section 4]. Furthermore, there are $G \subset \mathfrak{S}_6$ such that

- Q fails **(H1)** but M satisfies it, e.g., for $G = \langle (1,2)(3,4)(5,6) \rangle$, one has

  $$\mathrm{H}^1(G, M) = 0, \quad \mathrm{H}^1(G, Q) = \mathbb{Z}/2.$$

  Actually, M is a permutation module while Q is not. Indeed, under appropriate choices of basis, M is of the form

  $$\mathbb{Z}^4 \oplus \mathbb{Z}[C_2]^6,$$

  and Q is of the form

  $$\mathbb{Z} \oplus \mathbb{Z}[C_2]^2 \oplus \mathbb{Z}[e],$$

  where G acts on e via -1.
- Both Q and M fail **(H1)**: all groups containing $G = C_2^2$ from Proposition 28, in these cases we have

  $$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, Q) = \mathbb{Z}/2.$$

Statement of results.

Proposition 28. *For $n_1, n_2, n_3 \in \mathbb{N}$ with $2(n_1 + n_2 + n_3) = n$ let*

$$\iota_1 = (1,2)\dots(2n_1 - 1, 2n_1)(2(n_1+n_2)+1, 2(n_1+n_2)+2)\dots(n-1,n),$$

$$\iota_2 = (2n_1+1, 2n_1+2), \dots, (2(n_1+n_2)-1, 2(n_1+n_2))\dots(n-1,n),$$

and put $G := \langle \iota_1, \iota_2 \rangle$. Then

$$\mathrm{H}^1(G, M) = \mathbb{Z}/2.$$

The first part of Theorem 1 follows:

Corollary 29. *For every even $n > 5$ and every subgroup of $\mathfrak{S}_n$ containing G, the induced action on $\overline{\mathcal{M}}_{0,n}$ is not stably linearizable.*

For example, when $n_1 = n_2 = n_3 = 1$

$$\iota_1 = (12)(56), \quad \iota_2 = (34)(56),$$

and the corresponding action on $\overline{\mathcal{M}}_{0,6}$, which is $\mathfrak{S}_6$-equivariantly birational to the Segre cubic, is not stably linearizable.

We apply the results above to rationality questions over nonclosed fields, completing the proof of Theorem 1:

Theorem 30. *Let F be a field admitting a biquadratic extension. Then, for all even $n \geq 6$ there exist forms of $\overline{\mathcal{M}}_{0,n}$ over F that are not retract rational, and thus not stably rational, over F.*

In particular, this yields nonrational forms over $F = \mathbb{C}(t)$, a field with trivial Brauer group.

Proof. Indeed, let $G \simeq C_2^2$ be the group identified in Proposition 28, with $\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2$. Let $\Gamma = \mathrm{Gal}(F'/F)$ be the Galois group of the biquadratic extension F'/F. We construct a form X of $\overline{\mathcal{M}}_{0,n}$ over F such that Γ acts on $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ via G. This gives an **(H1)**-obstruction to retract rationality. □

Proof of Proposition 28. Put

$$\sigma := \iota_1\iota_2 = (1,2)\cdots(2(n_1+n_2)-1, 2(n_1+n_2)),$$

$$\tau := \iota_2 = (2n_1+1, 2n_1+2)\cdots(n-1, n),$$

so that $G = \langle \sigma, \tau \rangle$. We will repeatedly use the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\langle\tau\rangle, A^\sigma) \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(\langle\sigma\rangle, A)^\tau, \tag{5.1}$$

with the usual notation for invariants under the actions of σ, τ.

Step 1. Observe that M admits a decomposition, as a G-module,

$$M = L \oplus P,$$

where L consists of $\mathbb{Z}$-linear combinations of H and E_I, with $n-1 \notin I$, and P is generated, over $\mathbb{Z}$, by E_I with $n-1 \in I$. We have

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, L) \oplus \mathrm{H}^1(G, P).$$

Step 2. The involution σ is contained in $\mathfrak{S}_{n-1}$, permuting $(n-1)$ points and therefore linearizable. Thus

$$\mathrm{H}^1(\langle\sigma\rangle, M) = \mathrm{H}^1(\langle\sigma\rangle, L) = \mathrm{H}^1(\langle\sigma\rangle, P) = 0.$$

Moreover, P is a G-permutation module. Indeed, for I with $n-1 \in I$, $\sigma E_I = E_{\sigma(I)} \in P$, and $\tau E_I = E_{(\tau\cdot(n-1,n))(I)} \in P$. It follows that

$$\mathrm{H}^1(G, P) = 0,$$

and

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(G, L) = \mathrm{H}^1(\langle\tau\rangle, L^\sigma).$$

Remark 31. Geometrically, cohomology is already contributed on the *toric model* $\overline{L}_n$, obtained by blowing up $(n-2)$ general points on $\mathbb{P}^{n-3}$.

Step 3. Let $N \subset L$ be the submodule of $\mathbb{Z}$-linear combinations of E_I with $|I| \geq 2$ and $n-1 \notin I$. We have a short exact sequence

$$0 \to N \to L \to Q \to 0,$$

of G-modules, with Q generated by $H, E_1, \ldots, E_{n-2}$, modulo N, and the corresponding long exact sequence of $\langle\tau\rangle$-modules:

$$0 \to N^\sigma \to L^\sigma \to Q^\sigma \to \mathrm{H}^1(\langle\sigma\rangle, N) \to \ldots$$

Since $\sigma(E_I) = E_{\sigma(I)}$, the σ-action on N yields naturally a permutation module, realized via permutation of indices of E_I. So

$$\mathrm{H}^1(\langle\sigma\rangle, N) = 0.$$

The short exact sequence

$$0 \to N^\sigma \to L^\sigma \to Q^\sigma \to 0$$

gives rise to the long exact sequence

$$\mathrm{H}^1(\langle\tau\rangle, N^\sigma) \to \mathrm{H}^1(\langle\tau\rangle, L^\sigma) \to \mathrm{H}^1(\langle\tau\rangle, Q^\sigma) \to \mathrm{H}^2(\langle\tau\rangle, N^\sigma). \tag{5.2}$$

Step 4. The $\langle\tau\rangle$-module N^σ has the form:

$$N^\sigma = \mathbb{Z}[\langle\tau\rangle] \oplus \cdots \oplus \mathbb{Z}[\langle\tau\rangle].$$

In particular,

$$\mathrm{H}^1(\langle\tau\rangle, N^\sigma) = \mathrm{H}^2(\langle\tau\rangle, N^\sigma) = 0.$$

Indeed, a $\mathbb{Z}$-basis of N^σ is given by

$$e_I := \begin{cases} E_I + E_{\sigma(I)} & \text{if } \ \sigma(I) \neq I, \\ E_I & \text{if } \ \sigma(I) = I, \end{cases}$$

for

$$I \subset \{1, 2, \dots, n-2\}, \quad 2 \le |I| \le n-4.$$

To show that N^σ is a direct sum of copies of $\mathbb{Z}[\langle \tau \rangle]$, it suffices to show that $\tau(e_I) = e_{I'}$, for some $I' \ne I$ and $e_I \ne e_{I'}$. Observe that

$$\sigma(I)^c = \sigma(I^c), \quad I^c := \{1, \dots, n-2\} \setminus I.$$

There are three cases:

- If $\sigma(I) = \tau(I) = I$, then
$$\tau(e_I) = \tau(E_I) = D_{I \cup \{n-1\}} = E_{I^c} = e_{I^c}$$
and thus $e_I \ne e_{I^c}$.
- If $\sigma(I) \ne I$ and $\tau(I) = I$, then
$$\begin{aligned}\tau(e_I) &= \tau(E_I) + \tau(E_{\sigma(I)}) = D_{I \cup \{n-1\}} + D_{\sigma(I) \cup \{n-1\}} \\ &= E_{I^c} + E_{\sigma(I)^c} = E_{I^c} + E_{\sigma(I^c)} = e_{I^c}.\end{aligned}$$
Since $I^c \ne I$ and $I^c \ne \sigma(I^c)$, we know that $e_I \ne e_{I^c}$.
- If $\tau(I) \ne I$, then $\sigma(I) \ne I$, and
$$\begin{aligned}\tau(e_I) &= E_{\tau(I)^c} + E_{(\tau\sigma(I))^c} = E_{\tau(I)^c} + E_{(\sigma\tau(I))^c} \\ &= E_{\tau(I)^c} + E_{\sigma(\tau(I)^c)} = e_{\tau(I)^c}.\end{aligned}$$
To be concrete, assume that $1 \in I$ and $2 \notin I$. Then $1 \in \tau(I)^c$ and $1 \notin \sigma(I)$, so that $\tau(I)^c \ne \sigma(I)$. Since $|I| \ge 2$, one can see that $\tau(I)^c \ne I$ and thus $e_{\tau(I)^c} \ne e_I$.

In conclusion, $\tau(e_I) \ne e_I$, in all cases, and N^σ is as claimed, and thus has vanishing first and second cohomology. It follows that

$$\mathrm{H}^1(\langle \tau \rangle, M^\sigma) = \mathrm{H}^1(\langle \tau \rangle, L^\sigma) = \mathrm{H}^1(\langle \tau \rangle, Q^\sigma).$$

Step 5. To show that $\mathrm{H}^1(\langle \tau \rangle, Q^\sigma) = \mathbb{Z}/2$, let

$$\Sigma_i := \sum_{|I|=i} E_I,$$

where the sum is over $I \subseteq \{1, 2, \dots, n-2\}$ with $|I| = i$. Put $\Sigma := \Sigma_1$ and set

$$\begin{aligned}
e_0 &:= H - \Sigma, && \\
e_i &:= H - \Sigma + (E_{2i-1} + E_{2i}), && 1 \le i \le n_1 + n_2, \\
w_j &:= E_{2j-1}, && n_1 + n_2 + 1 \le j \le \frac{n-2}{2}, \\
v_j &:= H - \Sigma + E_{2j}, && n_1 + n_2 + 1 \le j \le \frac{n-2}{2}.
\end{aligned}$$

Then

$$\{e_i, w_j, v_j\}$$

for $0 \leq i \leq n_1 + n_2$ and $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$ gives a $\mathbb{Z}$-basis of Q^σ. Moreover, for $1 \leq i \leq n_1 + n_2$ and $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$, one has

$$\tau(e_0) = -e_0, \quad \tau(e_i) = e_i, \quad \text{and} \quad \tau(w_j) = v_j.$$

Indeed, Q^σ is generated, over $\mathbb{Z}$, by

$$H, (E_1 + E_2), \ldots, (E_{2(n_1+n_2)-1} + E_{2(n_1+n_2)}), E_{2(n_1+n_2)+1}, \ldots, E_{n-2}.$$

We now show that $\{e_i, w_j, v_j\}$ gives another basis. First, observe that

$$H - \Sigma = D_{34\ldots n} - (E_1 + E_2) + \underbrace{\sum_{1,2 \notin I, E_I \in N} E_I}_{\in N^\sigma}.$$

Indeed, if $1, 2 \notin I$ and $E_I \in N$, $1, 2 \notin \sigma(I)$ and $E_{\sigma(I)}$ will also appear in the summand. Then $\sigma(H - \Sigma) = H - \Sigma \pmod{N^\sigma}$ and

$$e_j, w_j, v_j \in Q^\sigma.$$

Moreover, $\{e_j, w_j, v_j\}$ generates Q^σ since

$$E_{2i-1} + E_{2i} = e_i - e_0, \quad E_{2j} = v_j - e_0$$

and

$$H = (\frac{4-n}{2})e_0 + \sum_{i=1}^{n_1+n_2} e_i + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} (w_j + v_j).$$

To compute the τ-action on this basis, one can first compute

$$\begin{aligned} H - \Sigma =& D_{34\ldots n} - (E_1 + E_2) \pmod{N^\sigma} \\ \stackrel{\tau}{\longmapsto}& D_{34\ldots n} - D_{1,n-1} - D_{2,n-1} \\ =& D_{34\ldots n} - 2H + 2\Sigma - (E_1 + E_2) \pmod{N^\sigma} \\ =& H - \Sigma + (E_1 + E_2) - 2H + 2\Sigma + (E_1 + E_2) \pmod{N^\sigma} \\ =& -H + \Sigma, \end{aligned}$$

i.e.,

$$\tau(e_0) = -e_0.$$

Then we have

$$\begin{aligned} H - \Sigma + E_{2i-1} + E_{2i} &\stackrel{\tau}{\longmapsto} -H + \Sigma + D_{2i-1,n-1} + D_{2i,n-1} \\ &= -H + \Sigma + 2H - 2\Sigma + (E_{2i-1} + E_{2i}) \pmod{N^\sigma} \\ &= \quad H - \Sigma + (E_{2i-1} + E_{2i}) \pmod{N^\sigma}. \end{aligned}$$

Note that the equalities hold for all $1 \le i \le \frac{n}{2}$. In particular,

$$\tau(e_i) = e_i, \quad \text{for} \quad 1 \le i \le n_1 + n_2.$$

Finally,

$$\begin{aligned} \tau(w_j) = D_{2_j, n-1} &= H - \Sigma + E_{2j} - \sum_{\substack{2j \notin I \\ E_I \in N}} E_I \\ &= H - \Sigma + E_{2j} \pmod{N^\sigma}, \end{aligned}$$

i.e.,

$$\tau(w_j) = v_j, \quad \text{for} \quad n_1 + n_2 + 1 \le j \le \frac{n-2}{2}.$$

In conclusion,

$$Q^\sigma = \mathbb{Z}[e_0] + \sum_{i=1}^{n_1+n_2} \mathbb{Z}[e_i] + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} \mathbb{Z}[w_j, v_j],$$

where τ acts trivially on e_i, permutes w_j and v_j, and the unique (-1)-eigenvector e_0 contributes to

$$\mathrm{H}^1(\langle \tau \rangle, Q^\sigma) = \mathbb{Z}/2.$$

This completes the proof of Proposition 28.

Remark 32. Notice that when $n_1 = n_2 = 0$, the argument above shows

$$\mathrm{H}^1(C_2, Q) = \mathbb{Z}/2,$$

where the C_2 is generated by $(1,2)(3,4)\dots(n-1,n)$. Computational experiments suggest that

$$\mathrm{H}^1(H, M) = 0,$$

for all cyclic subgroups $H \subset \mathfrak{S}_n$.

Small dimensional examples.
$\mathbf{n = 6}$: By Theorem 1 and the analysis in Section 6 of [CTZ23], we know that the G-action on $\mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ satisfies **(SP)** iff the G-action is linearizable, thus, nonlinearizable actions are not stably linearizable, as they fail **(SP)**.

Remark 33. This indicates an error in the application in [HT23, p. 295]: Proposition 21 there asserts that the standard and non-standard actions of $\mathfrak{A}_5$ are stably birational, contradicting our cohomology computation. The gap occurs in the sentence: "However, for any finite group G and automorphism $a : G \to G$, precomposing by a yields an action on G-modules; this respects permutation and stably permutation modules."

$\mathbf{n=8}$: There is a unique (conjugacy class of) $G' = C_2^2 \subset \mathfrak{S}_8$ such that

$$\mathrm{H}^1(G', \mathrm{Pic}(\overline{\mathcal{M}}_{0,8})) = \mathbb{Z}/2,$$

and all $G \subseteq \mathfrak{S}_8$ failing **(H1)** on M contain G'. With `magma`, we find:

- There are 66 (conjugacy classes of) groups containing this G'.
- Of the remaining 230 classes, 96 are contained in the (unique) $\mathfrak{S}_7 \subset \mathfrak{S}_8$, the corresponding actions are linearizable.
- After that, there are 56 contained in the (unique) $\mathfrak{S}_6 \times C_2$ – these actions are birational to an action on a 5-dimensional torus; such actions have been analyzed, over nonclosed fields, in [HY17].
- We are left with 78 classes. Applying [HY17, Algorithm F4] to these classes, we found at least 37 classes of groups $G \subset \mathfrak{S}_8$ having vanishing cohomology but with $\mathrm{Pic}(\overline{\mathcal{M}}_{0,8})$ failing the **(SP)** condition.
- Among the 41 remaining classes, 13 leave invariant an odd cycle. These actions are stably linearizable by Proposition 3.
- There are 28 remaining classes, including a minimal

  $$C_2^2 = \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8) \rangle,$$

  which (up to conjugation) is contained in every remaining class. The action of this C_2^2 on M yields a permutation module:

  $$\mathbb{Z}[C_2^2]^{19} \oplus \mathbb{Z}[C_2^2/C_2]^3 \oplus \mathbb{Z}[C_2^2/C_2']^3 \oplus \mathbb{Z}[C_2^2/C_2'']^3 \oplus \mathbb{Z}^5.$$

  However, on Q, this action fails **(H1)**, and Theorem 24 is not applicable to any of these cases.

$\mathbf{n=10}$: We find more minimal groups contributing cohomology:

$$\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,10})) = \mathbb{Z}/2$$

when

- $G = C_2^2 = \langle (1,2)(3,4)(5,6)(7,8), (1,2)(9,10) \rangle$,
- $G = C_2^2 = \langle (1,2)(3,4)(5,6), (5,6)(7,8)(9,10) \rangle$,
- $G = C_2 \times C_4 = \langle (3,6)(8,10), (1,2)(5,9), (1,2)(3,10,6,8)(4,7) \rangle$,
- $G = \mathfrak{D}_4 = \langle (3,6)(8,10), (1,2)(5,9)(8,10), (1,2)(3,10,6,8)(4,7) \rangle$.

6. Three-dimensional case

Next, we give a criterion for rationality of the Segre cubic, exhibit forms failing stable rationality over arbitrary fields admitting a biquadratic extension, and establish stable rationality, provided Q is stably permutation, for the action of the absolute Galois group.

Recall that X_6 denotes the symmetrically linearized GIT quotient with equivalent presentations:

- $(\mathbb{P}^1)^6$ under the diagonal action of SL_2; or
- $\mathrm{Gr}(2,6)$ under the diagonal action of the torus $\mathsf{T} \simeq \mathbb{G}_m^5$.

These have ten isolated nodes, the images of the $D_I, |I| = 3$ under the blow down $\beta : \overline{\mathcal{M}}_{0,6} \to X_6$. These are classically embedded $X_6 \subset \mathbb{P}^4$ as cubic threefolds, known as Segre cubic threefolds [CTZ23]. The remaining boundary divisors $D_I, |I| = 2$ correspond to planes passing through four nodes.

Theorem 34. *Let X be a form of the Segre cubic threefold over a nonclosed field F of characteristic zero, and $\tilde{X}$ its standard resolution of singularities, a form of $\overline{\mathcal{M}}_{0,6}$. Then X is rational over F if and only if the Galois-module* $\mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$ *satisfies* (**SP**).

Proof. This is closely related to the linearizability result [CTZ23, Theorem 1]. The group-theoretic analysis there shows that the only cases where the Galois action on the Picard group is stably permutation are:

- when one of the ten nodes is Galois invariant;
- the Galois action is contained in an $\mathfrak{S}_5$-action associated with permutations of *five* of the marked points;
- the Galois group acts via C_2^2, leaving three planes invariant, and the set of nodes splits into a union of five Galois orbits of length two.

Note that the first two cases are easily shown to be rational: Projecting from a node gives a birational map to $\mathbb{P}^3$, cf. Example 19. And when the action factors through $\mathfrak{S}_5$, the moduli space arises via the Kapranov construction, i.e., is a blow-up of $\mathbb{P}^3$.

Recall that in the third case, the Galois action factors through $\mathfrak{S}_2 \times \mathfrak{S}_4 \subset \mathfrak{S}_6$ corresponding to a partition of the six points conjugate to

$$\{1,2,3,4,5,6\} = \{3,4\} \cup \{1,2,5,6\}.$$

Our $C_2 \times C_2$ action is conjugate to

$$\langle (34), (15)(26) \rangle \subset \mathfrak{S}_6$$

This leaves the boundary divisors D_{34}, D_{15}, and D_{26} invariant. Identifying singular points with the boundary divisors in $\overline{\mathcal{M}}_{0,6}$, the orbits are

$$\begin{aligned}&\{D_{123} = D_{456}, D_{124} = D_{356}\}, \quad \{D_{125} = D_{346}, D_{156} = D_{234}\},\\ &\{D_{126} = D_{345}, D_{256} = D_{134}\}, \quad \{D_{135} = D_{246}, D_{145} = D_{236}\},\\ &\{D_{136} = D_{245}, D_{146} = D_{235}\}.\end{aligned}$$

We emphasize that the invariant divisor classes reflect boundary divisors defined over F. Indeed, our moduli space has F-rational smooth points so there is no obstruction to descending Galois-invariant divisors.

We claim this moduli space is birational over F to a toric threefold, i.e., an equivariant compactification of a nonsplit torus over F.

Consider the Losev-Manin moduli space associated to the partition above. Specifically, points 3 and 4 are not permitted to collide with other points but points from $\{1, 2, 5, 6\}$ may collide with one another. This is toric by Proposition 7, i.e., the orbits of the homogeneous quartic forms vanishing along $\{1, 2, 5, 6\}$ modulo the torus fixing $\{3, 4\}$. This geometric description is compatible with the Galois action.

Rationality of three-dimensional toric varieties has been settled in [Kun87, Theorem 2]: The variety is rational over F iff the Picard module is stably permutation for the Galois action.

Here is an alternative rationality construction: Pick one of the boundary divisors $D_I, |I| = 2$ invariant under the Galois action. With our choice of indexing this could be D_{34}, D_{15}, or D_{26}; we take D_{34}. This corresponds to a plane $P \subset X$ containing four ordinary singularities, i.e., the images of $D_{34j}, j = 1, 2, 5, 6$. We blow this plane up – inducing a small resolution of the four singularities – and then blow down the proper transform of the plane. This yields a complete intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$ with six singularities, the images of the singularities of X *not* contained in P. Under the $C_2 \times C_2$ action, we have three orbits each with two singular points. For each orbit, the line joining the singularities is contained in $X_{2,2}$. Projecting from that line gives

$$X_{2,2} \stackrel{\sim}{\dashrightarrow} \mathbb{P}^3;$$

the birationality is classical cf. [CTSSD87, Proposition 2.2]. □

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