UNRAMIFIED BRAUER GROUP OF QUOTIENT SPACES BY FINITE GROUPS

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Abstract. We provide a general algorithm for the computation of the unramified Brauer group of quotients of rational varieties by finite groups.

1. Introduction

Let V be a variety over an algebraically closed field k of characteristic zero and G a finite group acting generically freely on V. For example, V could be a finite-dimensional faithful representation of G. The rationality problem for the field of invariants

$$K = k(V)^G = k(V/G)$$

has attracted the attention of many mathematicians, e.g., in connection with Noether's problem (see [15] for a survey and further references).

One of the obstructions is the *unramified Brauer group*

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \mathrm{Br}(X) = \mathrm{H}^2(X, \mathbb{G}_m),$$

which coincides with the Brauer group of a smooth projective model X of K. By a result of Bogomolov [7] (see also [15, Thm. 6.1]), this group can be computed in terms of the set $\mathcal{B}_G$ of *bicyclic* subgroups of G:

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) = \{\alpha \in \mathrm{Br}(k(V)^G) \,|\, \alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A),\ \forall A \in \mathcal{B}_G\}. \quad (1.1)$$

This yields explicit formulas in special cases.

(1) If V is a faithful representation of G then (cf. [15, Thm. 7.1])

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \ker\Big(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{A \in \mathcal{B}_G} \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})\Big).$$

(2) If $V = T$ is an algebraic torus over k, with G-action arising from an injective homomorphism $G \to \mathrm{Aut}(M)$, where $M = \mathfrak{X}^*(T)$, then (cf. [15, Thm. 8.7])

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \ker\Big(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z} \oplus M) \to \bigoplus_{A \in \mathcal{B}_G} \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z} \oplus M)\Big).$$

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(3) The case $V = \mathrm{SL}_n$ with $G \subset \mathrm{SL}_n$ acting by translations, is treated in [13] and, by means of a stable equivariant birational equivalence to a linear action, leads to the same outcome as case (1).

After some preliminary material (Sections 2 and 3), we highlight the role of the Brauer group of the quotient *stack*

$$[V/G]$$

(Section 4) and give a uniform treatment of some known (Section 5) and new cases (V a projective space in Section 5, a Grassmannian variety in Section 6, a flag variety in Section 7). The main result (Section 8) is a general procedure to determine the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k(V)^G)$ for a G-action on a rational variety V.

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2. Generalities

We work over an algebraically closed field k of characteristic zero.

Group cohomology. As recalled in [23, §2.1], there is a natural identification

$$\mathrm{H}^i(G, k^\times) \cong \mathrm{H}^i(G, \mu_\infty) \qquad (i \geq 1)$$

of group cohomology for any finite group G with trivial action on $k^\times$, respectively μ_∞. We identify μ_∞ with $\mathbb{Q}/\mathbb{Z}$ and write

$$\mathrm{H}^i(G) = \mathrm{H}^i(G, \mathbb{Q}/\mathbb{Z}).$$

For $i = 1$ we have $\mathrm{H}^1(G) := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, and for $i = 2$, an interpretation of $\mathrm{H}^2(G)$ in terms of central extensions of G; see [8, §IV.3].

For any subgroup $A \subseteq G$ we denote by

$$\mathrm{res}^i_A \colon \mathrm{H}^i(G) \to \mathrm{H}^i(A)$$

the restriction homomorphism. For a normal subgroup with $Q = G/A$, the Hochschild-Serre spectal sequence yields the long exact sequence

$$0 \to \mathrm{H}^1(Q) \to \mathrm{H}^1(G) \to \mathrm{H}^1(A)^Q \to \mathrm{H}^2(Q) \to \ker(\mathrm{res}^2_A) \to \mathrm{H}^1(Q, \mathrm{H}^1(A)).$$

This gives two split short exact sequences when $G = A \rtimes Q$.

For G cyclic with generator g and a G-module M the group cohomology $\mathrm{H}^i(G, M)$ can be identified with the cohomology of the complex

$$M \stackrel{\Delta}{\to} M \stackrel{N}{\to} M \stackrel{\Delta}{\to} M \dots,$$

where $\Delta = g - 1$ and $N = 1 + g + \cdots + g^{n-1}$ ($n = |G|$), cf. [8, Exa. III.1.2]. The case G is abelian, expressed as a product of cyclic groups, may be treated via tensor product of resolutions corresponding to the factors as described in [8, Prop. V.1.1], e.g., for bicyclic $G \cong G_1 \times G_2$ with correspnding Δ_i and N_i, $i = 1, 2$:

$$M \xrightarrow{\left(\begin{smallmatrix}\Delta_1\\ \Delta_2\end{smallmatrix}\right)} M^2 \xrightarrow{\left(\begin{smallmatrix}N_1 & 0\\ -\Delta_2 & \Delta_1\\ 0 & N_2\end{smallmatrix}\right)} M^3 \dots.$$

We see easily, this way, that $\mathrm{H}^2(G) = 0$ when G is cyclic, and

$$\mathrm{H}^2(G_1 \times G_2) \cong \mathbb{Z}/d\mathbb{Z}, \qquad d = \gcd(n_1, n_2),$$

for cyclic G_i of order n_i for $i = 1, 2$ (cf. [23, §2.1]).

Fields. Throughout, $K = k(V)$ is the function field of an algebraic variety V over k. We write $\mathcal{D}\mathrm{Val}_K$ for the set of divisorial valuations of K. Every $\nu \in \mathcal{D}\mathrm{Val}_K$ can be realized as a valuation corresponding to a divisor on some smooth projective model of K.

Unramified cohomology. Let $\nu \in \mathcal{D}\mathrm{Val}_K$ with residue field κ and absolute Galois group $\mathcal{G}_\kappa$ of κ. There is a residue homomorphism

$$\partial_\nu \colon \mathrm{Br}(K) \to \mathrm{H}^1_{\mathrm{cont}}(\mathcal{G}_\kappa) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{G}_\kappa, \mathbb{Q}/\mathbb{Z})$$

with values in the continuous group cohomology. We have

$$\mathrm{Br}_{\mathrm{nr}}(K) \subset \mathrm{Br}(K), \qquad \mathrm{Br}_{\mathrm{nr}}(K) = \bigcap_{\nu \in \mathcal{D}\mathrm{Val}_K} \mathrm{Ker}(\partial_\nu),$$

with $\mathrm{Br}_{\mathrm{nr}}(K) \cong \mathrm{Br}(X)$ for any smooth projective model X of K. The group $\mathrm{Br}_{\mathrm{nr}}$ is invariant under purely transcendental extensions. In particular, a rational variety V has $\mathrm{Br}_{\mathrm{nr}}(k(V)) = 0$.

An important result, Fischer's theorem [17], asserts the rationality of V/A for a linear action of an abelian group A. Then $\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$.

Basic exact sequence. Let V be a smooth projective G-variety over k. Assume that V is rational. The Leray spectral sequence, applied to the morphism from the Deligne-Mumford stack (DM stack) $[V/G]$, associated with the G-action on V, to the stack BG of G-torsors, yields the long exact sequence

$$\begin{aligned} &0 \to \mathrm{Hom}(G, k^\times) \to \mathrm{Pic}(V, G) \to \mathrm{Pic}(V)^G \xrightarrow{\delta_2} \mathrm{H}^2(G, k^\times) \\ &\to \mathrm{Br}([V/G]) \to \mathrm{H}^1(G, \mathrm{Pic}(V)) \xrightarrow{\delta_3} \mathrm{H}^3(G, k^\times) \to \mathrm{H}^3([V/G], \mathbb{G}_m), \end{aligned} \tag{2.1}$$

where $\mathrm{Pic}(V, G)$ denotes the group of isomorphism classes of G-linearized line bundles. In [23] this is used to exhibit G-actions on rational surfaces with obstructions to (stable) linearizability of the G-action, e.g., nonvanishing of

- the Amitsur group $\mathrm{Am}(V,G) := \mathrm{im}(\delta_2)$ (see [6, Sect. 6]),
- the image $\mathrm{im}(\delta_3)$,
- the cohomology $\mathrm{H}^1(G,\mathrm{Pic}(V))$.

If V has a G-fixed point, then by basic functoriality the map from $\mathrm{H}^2(G,k^\times)=\mathrm{Br}(BG)$ to $\mathrm{Br}([V/G])$ is injective, thus $\delta_2=0$, and similarly, $\delta_3=0$.

If V is quasiprojective then the Leray spectral sequence leads to a basic exact sequence with first term $\mathrm{H}^1(G,\mathbb{G}_m(V))$ and $\mathrm{H}^i(G,k^\times)$ ($i=2$, 3) replaced by $\mathrm{H}^i(G,\mathbb{G}_m(V))$ and $\mathrm{Br}([V/G])$ by $\ker(\mathrm{Br}([V/G])\to\mathrm{Br}(V))$.

We will use the following observation, which appears in [28].

Lemma 2.1. *Suppose $V\to W$ is a G-equivariant morphism of smooth projective G-varieties, such that the induced homomorphism*

$$\mathrm{Pic}(W)\to\mathrm{Pic}(V)$$

is injective (resp., an isomorphism). Then $\mathrm{Pic}(W,G)\to\mathrm{Pic}(V,G)$ *is injective (resp., an isomorphism), and* $\mathrm{Am}(W,G)$ *is contained in (resp., is equal to)* $\mathrm{Am}(V,G)$.

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(G,k^\times) & \longrightarrow & \mathrm{Pic}(W,G) & \longrightarrow & \mathrm{Pic}(W)^G & \xrightarrow{\delta_2} & \mathrm{H}^2(G,k^\times) \\
 & & \| & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathrm{Hom}(G,k^\times) & \longrightarrow & \mathrm{Pic}(V,G) & \longrightarrow & \mathrm{Pic}(V)^G & \xrightarrow{\delta_2} & \mathrm{H}^2(G,k^\times)
\end{array}$$

with exact rows. The result follows. □

Linearized bundles. Let V be a smooth projective G-variety over k and E a vector bundle over V. We suppose that the projectivization $\mathbb{P}(E)$ is endowed with a G-action, so that the projection to V is G-equivariant, and we have a central cyclic extension

$$1\to Z\to\widetilde{G}\to G\to 1 \tag{2.2}$$

and a compatible $\widetilde{G}$-linearization of E, with scalar action of Z. We may suppose the latter, by replacing Z and $\widetilde{G}$ by suitable quotients, to be by the identity character of $Z=\mu_\ell$, $\ell=|Z|$. Then:

- A splitting of (2.2) leads to a G-linearization of E.
- Generally, (2.2) determines a class $\gamma_E\in\mathrm{H}^2(G)$, obstruction to existence of a splitting (for sufficiently divisible ℓ).
- We have $\gamma_{E\otimes E'}=\gamma_E+\gamma_{E'}$.
- A line bundle L with $[L]\in\mathrm{Pic}(V)^G$ leads to $\gamma_L=\delta_2([L])$.

If the G-action on V is generically free and E admits a G-linearization, then $k(E)^G$ is a purely transcendental extension of $k(V)^G$; this is known as the No-Name Lemma, see [11, Sect. 4.3].

Example 2.2. Let V° be a k-vector space of dimension n with projectivization $V = \mathbb{P}(V^\circ)$, and let G act on V. We adopt the convention that this is a right action. So it is given by a homomorphism $G \to \mathrm{PGL}(V^{\circ\vee})$. We have, canonically, a central cyclic extension (2.2) and compatible $\widetilde{G} \to \mathrm{SL}(V^{\circ\vee})$, with $Z = \mu_n$. Then (2.2) determines an n-torsion class

$$\gamma = \delta_2([\mathcal{O}_V(-1)]) \in \mathrm{H}^2(G),$$

with

$$\mathrm{Am}(V, G) = \langle\gamma\rangle.$$

For the trivial bundle $\underline{V}^\circ$ associated with the given vector space we have the given G-action on the projectivization and as above a $\widetilde{G}$-linearization, thus $\gamma_{\underline{V}^\circ} = \gamma$. The corresponding $\widetilde{G}$-linearization of $E = \underline{V}^\circ \otimes \mathcal{O}_V(1)$ has trivial Z-character, and we get a canonical G-linearization of E.

3. Bogomolov multiplier

The description of $\mathrm{Br}_{\mathrm{nr}}(k(V)^G)$ for a faithful representation of G from special case (1) of the Introduction involves a subgroup of $\mathrm{H}^2(G)$, known as the *Bogomolov multiplier*:

$$\mathrm{B}_0(G) := \ker\Big(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{A \in \mathcal{B}_G} \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})\Big).$$

Here, $\mathcal{B}_G$ denotes the set of bicyclic subgroups of G. In this section we recall some facts about $\mathrm{B}_0(G)$, including its vanishing for some classes of groups G. All groups G, A, etc., considered in this section, are finite.

The following facts follow from the long exact sequence coming from the Hochschild-Serre spectral sequence, recalled in Section 2:

- If $G \to A$ is a surjective homomorphism of abelian groups, then the induced homomorphism $\mathrm{H}^2(A) \to \mathrm{H}^2(G)$ is injective.
- If G is abelian, $G = G_1 \times \cdots \times G_r$ with cyclic factors G_i, then

$$\mathrm{H}^2(G) \cong \bigoplus_{i<j} \mathrm{H}^2(G_i \times G_j).$$

By the second fact, the Bogomolov multiplier of a group G may be defined equivalently with direct sum over all abelian subgroups A of G (as in [7]).

Lemma 3.1. *Assume that there is a short exact sequence of groups*

$$1 \to A \to G \to C \to 1,$$

where A is abelian and $C = \langle c \rangle$ is cyclic, and let $0 \neq \alpha \in \mathrm{H}^2(G)$ be given, with $\mathrm{res}^2_A(\alpha) = 0$. Then there exists an element $a \in A$, in the center of G, such that for any lift $b \in G$ of c we have $\mathrm{res}^2_{\langle a,b \rangle}(\alpha) \neq 0$. In particular, $\mathrm{B}_0(G) = 0$.

Proofs of this and similar statements make use of the long exact sequence coming from the Hochschild-Serre spectral sequence and the descriptions of group cohomology of abelian groups, given in Section 2.

Proof. The class $\alpha \in \ker(\mathrm{res}^2_A)$ determines a class $0 \neq \tilde{\alpha} \in \mathrm{H}^1(C, A^\vee)$, where $A^\vee$ denotes $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We employ the notation Δ and N for A as C-module, and equally well for $A^\vee$. Under the identification of $\mathrm{H}^1(C, A^\vee) \cong \ker(N)/\Delta(A^\vee)$, a representative $\tilde{\chi} \in A^\vee$, $N(\chi) = 0$, may be chosen so that $\ker(\tilde{\chi})$ contains $\Delta^i(A)$ (the image of the ith iterate of Δ) for some positive integer i. We suppose this is done, with i as small as possible. Then $\tilde{\chi}|_{\Delta^{i-1}(A)}$ does not lie in the image of the map

$$(\Delta^i(A)/\Delta^{i+1}(A))^\vee \to (\Delta^{i-1}(A)/\Delta^i(A))^\vee$$

induced by Δ. (The existence of $\chi \in A^\vee$ with $\Delta^{i+1}(A) \subset \ker(\chi)$ and $\Delta(\chi)|_{\Delta^{i-1}(A)} = \tilde{\chi}|_{\Delta^{i-1}(A)}$ would contradict the minimality of i.) Consequently, there exists

$$\bar{a} \in \ker\big(\Delta^{i-1}(A)/\Delta^i(A) \to \Delta^i(A)/\Delta^{i+1}(A)\big), \qquad \bar{a} \notin \ker(\tilde{\chi}).$$

There is then a lift $a \in \Delta^{i-1}(A)$, belonging to the center of G, and this satisfies the desired property. □

The conclusion $\mathrm{B}_0(G) = 0$ is known [7, Lemma 4.9]. We use the description of the indicated bicyclic subgroups of G in Lemma 3.1 to give a direct proof of the next lemma, established using different methods (group homology of certain universal semidirect products) in [2].

Lemma 3.2. *Suppose that $G = A \rtimes B$ is a semidirect product of abelian groups A and B, with B bicyclic. Then $\mathrm{B}_0(G) = 0$.*

Proof. Suppose $0 \neq \alpha \in \mathrm{H}^2(G)$ with $\mathrm{res}^2_A(\alpha) = 0 = \mathrm{res}^2_B(\alpha)$. Then the class $\tilde{\alpha} \in \mathrm{H}^1(B, A^\vee)$, determined by α, is nonzero.

We represent B as a product of a pair of cyclic subgroups and employ corresponding notation Δ_1, N_1, Δ_2, N_2. Then $\tilde{\alpha}$ may be represented by

$$(\tilde{\chi}, \tilde{\chi}') \in A^\vee \times A^\vee,$$

satisfying $N_1(\tilde{\chi}) = 0 = N_2(\tilde{\chi}')$ and $\Delta_2(\tilde{\chi}) = \Delta_1(\tilde{\chi}')$. This is unique up to coboundaries of the form $(\Delta_1(\chi), \Delta_2(\chi))$ for $\chi \in A^\vee$.

The product representation $B = C_1 \times C_2$ determines subgroups $G_i = A \rtimes C_i$ ($i = 1$, 2) of G. If $\mathrm{res}^2_{G_2}(\alpha) \neq 0$, then Lemma 3.1 supplies a bicyclic subgroup $\langle a, b \rangle$ of G_2 with $\mathrm{res}^2_{\langle a,b \rangle}(\alpha) \neq 0$, so we suppose, instead,

$\mathrm{res}^2_{G_2}(\alpha) = 0$. Then $\tilde{\chi}' = \Delta_2(\chi')$, for some $\chi' \in A^\vee$, and, modifying the cocycle representative by a coboundary, we are reduced to the case

$$\tilde{\chi}' = 0.$$

So $\Delta_2(\tilde{\chi}) = 0$, i.e., $\tilde{\chi} \in (A/\Delta_2(A))^\vee$, and $\tilde{\chi}$ determines

$$\beta \in \ker\big(\mathrm{H}^2(A/\Delta_2(A) \rtimes C_1) \to \mathrm{H}^2(A/\Delta_2(A))\big),$$

mapping to $\alpha \in \mathrm{H}^2(G)$.

We apply Lemma 3.1 to β to obtain $\bar{a} \in A/\Delta_2(A)$ in the center of $A/\Delta_2(A) \rtimes C_1$ and a set $\mathcal{B}_{\bar{a}}$ of bicyclic subgroups, to which β restricts nontrivially. Let a be a lift to A. Then $\Delta_1(a) = \Delta_2(b)$ for some $b \in A$. Now the elements of G, obtained by pairing a with chosen generator of C_2, and b with chosen generator of C_1, generate an abelian subgroup of G whose image in $A/\Delta_2(A) \rtimes C_1$ is in $\mathcal{B}_{\bar{a}}$. This concludes the proof. □

Lemma 3.3. *Suppose that G is a central extension of a bicyclic group. Then* $\mathrm{B}_0(G) = 0$.

Proof. We write a central exact sequence of groups

$$1 \to A \to G \to B \to 1,$$

with B bicyclic. The proof will use the easy observation that G is abelian if and only if $\mathrm{H}^1(G)$ maps surjectively to $A^\vee$ (cf. the long exact sequence coming from the Hochschild-Serre spectral sequence).

Let a given $0 \neq \alpha \in \mathrm{H}^2(G)$, with $\mathrm{res}^2_A(\alpha) = 0$, determine a class $\tilde{\alpha} \in \mathrm{H}^1(B, A^\vee) = \mathrm{Hom}(B, A^\vee)$. If $\tilde{\alpha} \neq 0$, then α remains nonzero upon restriction to the pre-image in G of a suitable cyclic subgroup of B, and we may conclude by Lemma 3.1. We suppose $\tilde{\alpha} = 0$, thus $\alpha \in \mathrm{H}^2(G)$ is the image under

$$\mathrm{H}^2(B) \to \mathrm{H}^2(G)$$

of some $\alpha_0 \in \mathrm{H}^2(B)$. We write $B = C_1 \times C_2$, cyclic subgroups of orders $|C_1| = n_1$ and $|c_2| = n_2$, so $\mathrm{H}^2(B) \cong \mathbb{Z}/d\mathbb{Z}$ with $d = \gcd(n_1, n_2)$.

Let e denote the order of the image of $A^\vee \to \mathrm{H}^2(B)$ (the transgression map, coming from the Hochschild-Serre spectral sequence) and f the order of $\alpha_0 \in \mathrm{H}^2(B)$. We have $f \nmid e$, since $\alpha \neq 0$. Restriction from B to the subgroup eB leads to the class $0 \neq \bar{\alpha}_0 \in \mathrm{H}^2(eB) \cong \mathbb{Z}/(d/e)\mathbb{Z}$. Letting G' denote the pre-image of eB in G, the corresponding Hochschild-Serre spectral sequence gives a trivial transgression map, hence surjective $\mathrm{H}^1(G') \to A^\vee$. Therefore G' is abelian, and $\mathrm{res}^2_{G'}(\alpha) \neq 0$. □

Remark 3.4. Lemmas 3.1 through 3.3 are somewhat sharp. There exist groups G, extensions by abelian groups of bicyclic groups with $\mathrm{B}_0(G) \neq 0$; an example is given in [7, Sect. 4]. For p prime, [7, Sect. 5] investigates and exhibits p-groups G with $[G,[G,G]]=0$ and $\mathrm{B}_0(G)\neq 0$; subject to a minimality condition it is shown that $G/[G,G]\cong(\mathbb{Z}/p\mathbb{Z})^{2m}$, $m\geq 2$.

4. Brauer group of the quotient stack

In [23], we explained the computation of $\mathrm{Br}([V/G])$ in case V is a rational surface. Now, V is a smooth projective rational variety of arbitrary dimension, and we give a description of $\mathrm{Br}([V/G])$ as a subgroup of

$$\mathrm{H}^2(G,k(V)^\times)\cong\ker\big(\mathrm{Br}(k(V)^G)\to\mathrm{Br}(k(V))\big). \tag{4.1}$$

We refer to the basic exact sequence of Section 2. A subgroup, isomorphic to $\mathrm{H}^2(G,k^\times)/\mathrm{Am}(V,G)$, gives rise directly, via $k^\times\hookrightarrow k(V)^\times$, to elements of $\mathrm{H}^2(G,k(V)^\times)$. To complete the description, we need to explain how to lift elements of $\ker(\delta_3)$ to the group (4.1). For this, we take a G-invariant collection of divisors D_i, generating $\mathrm{Pic}(V)$, introduce the exact sequences of G-modules

$$0\to R\to\bigoplus_i\mathbb{Z}\cdot[D_i]\to\mathrm{Pic}(V)\to 0$$

and, with complement U in V of $D=\bigcup_i D_i$ and corresponding exact sequence

$$0\to k^\times\to\mathbb{G}_m(U)\to R\to 0$$

of G-modules, consider the diagram (see [21, Sect. 6]):

$$\begin{array}{ccccc}
 & & \mathrm{H}^2(G,\mathbb{G}_m(U)) & & \\
 & & \downarrow & & \\
0\to\mathrm{H}^1(G,\mathrm{Pic}(V)) & \longrightarrow & \mathrm{H}^2(G,R) & \longrightarrow & \mathrm{H}^2(G,\bigoplus_i\mathbb{Z}\cdot[D_i]) \\
 & \searrow^{\delta_3} & \downarrow & & \\
 & & \mathrm{H}^3(G,k^\times) & &
\end{array}$$

Given an element of $\ker(\delta_3)$, its image in $\mathrm{H}^2(G,R)$ may be lifted to $\mathrm{H}^2(G,\mathbb{G}_m(U))$. We obtain a representative in $\mathrm{H}^2(G,k(V)^\times)$ of a corresponding Brauer class on $[V/G]$.

We also recall the formulation of purity. Here, V need not be projective or rational, but we suppose that G acts generically freely on V. An element $\alpha\in\mathrm{Br}(k(V)^G)$ comes from $\mathrm{Br}([V/G])$ if and only if it has vanishing residue along the divisors of $[V/G]$ [22, Prop. 2.2]. The residues along divisors of $[V/G]$ are related to the classical residues (Section 2) as follows. We fix an irreducible divisor on $[V/G]$, corresponding to a G-orbit $D=D_1\cup\cdots\cup D_m$ of components on V, and suppose that each D_i has generic stabilizer of order n. Then [23, Lemma 4.1] the residue of α along the divisor $[D/G]$ of $[V/G]$ is equal to $n\delta_\nu(\alpha)$, where $\nu\in\mathcal{D}\mathrm{Val}_{k(V)^G}$ is the associated divisorial valuation of the function field $k(V)^G$ of V/G.

For G acting generically freely on smooth projective rational V we have inclusions

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \subset \mathrm{Br}([V/G]) \subset \mathrm{Br}(k(V)^G).$$

Indeed, the defining conditions for $\mathrm{Br}_{\mathrm{nr}}(k(V)^G)$ are vanishing δ_ν for all $\nu \in \mathcal{D}\mathrm{Val}_{k(V)^G}$, while for the purity characterization of $\mathrm{Br}([V/G])$ only the ν associated with divisors on $[V/G]$ are involved, and then only the vanishing of $n_\nu \delta_\nu$ is required, for some positive integer n_ν. Since $\mathrm{Br}([V/G])$ is contained in the kernel of $\mathrm{Br}(k(V)^G) \to \mathrm{Br}(k(V))$, using (4.1) we have

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \subset \mathrm{Br}([V/G]) \subset \mathrm{H}^2(G, k(V)^\times). \tag{4.2}$$

Lemma 4.1. *Let A be an abelian group, acting generically freely on a smooth projective variety V, and let $\alpha \in \mathrm{Br}([V/A])$. For $v \in V^A$ we denote by*

$$i_v^*\colon\ \mathrm{Br}([V/A]) \to \mathrm{H}^2(A, k^\times)$$

the corresponding splitting in the basic exact sequence. If $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A)$, then $i_v^*(\alpha) = 0$, for all $v \in V^A$.

Proof. Replacing V by $V \times \mathbb{P}^1$ if needed (with trivial A-action on $\mathbb{P}^1$), we may suppose that V^A has no isolated points. Let $v \in V^A$. We blow up the point v to obtain $\widetilde{V}$ and note that A has a faithful linear action on the exceptional divisor E. By Fischer's theorem, $\mathrm{Br}_{\mathrm{nr}}(k(E)^A) = 0$, thus α restricts to $0 \in \mathrm{Br}([E/A])$. We conclude by functoriality. □

Example 4.2. We consider the action from [23, Rem. 4.3], the projectivization of the regular representation of the Klein 4-group $\mathfrak{K}_4$, and determine $\mathrm{Br}([\mathbb{P}^3/\mathfrak{K}_4])$. The action has fixed points, so δ_2 is trivial. We have $\mathrm{H}^2(\mathfrak{K}_4, k^\times) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}^1(G, \mathrm{Pic}(\mathbb{P}^3)) = 0$, so

$$\mathrm{Br}([\mathbb{P}^3/\mathfrak{K}_4]) \cong \mathbb{Z}/2\mathbb{Z}.$$

The generator α is not in $\mathrm{Br}_{\mathrm{nr}}(K) = 0$, $K = k(\mathbb{P}^3)^{\mathfrak{K}_r}$, so there exists $\nu \in \mathcal{D}\mathrm{Val}_K$ with $\partial_\nu(\alpha) \neq 0$. Since the $\mathfrak{K}_4$-action is free outside a subset of codimension 2, we have to blow up $\mathbb{P}^3$ to find a divisor giving such a ν. See Section 8 for a systematic approach to testing for ramification.

5. Basic cases

Our formalism permits a uniform treatment of several cases.

Linear actions. The main result of Bogomolov [7] tells us that for a faithful linear representation V° of a finite group G, the field of invariants $K = k(V^\circ)^G$ has unramified Brauer group

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \mathrm{B}_0(G). \tag{5.1}$$

We apply our formalism to the standard equivariant compactification $V = \mathbb{P}(1 \oplus V^\circ)$ of V°. The G-action on V has a fixed point, thus $\delta_2 = 0$. Moreover, $\mathrm{H}^1(G, \mathrm{Pic}(V)) = 0$. It follows that $\mathrm{Br}([V/G])$ is identified with $\mathrm{H}^2(G, k^\times)$, which we have already identified with $\mathrm{H}^2(G) = \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$. The middle term in the chain of inclusions (4.2) is

$$\mathrm{Br}([V/G]) \cong \mathrm{H}^2(G).$$

Here, subgroups of each side are identified by Bogomolov's result (5.1).

For the containment $\mathrm{Br}_{\mathrm{nr}}(K) \subset \mathrm{B}_0(G)$ we use Fischer's theorem (Section 2). If $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K)$, then $\alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$ for $A \in \mathcal{B}_G$. Thus the class in $\mathrm{H}^2(G)$, corresponding to α, lies in $\ker(\mathrm{res}_A)$.

For the reverse containment we use the equality (1.1), recalled in the Introduction. Suppose $\alpha \in \mathrm{Br}([V/G])$ corresponds to a class in $\mathrm{B}_0(G)$. Then $\alpha_A = 0$ for $A \in \mathcal{B}_G$. So $\alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A)$, thus $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K)$.

Projectively linear actions. Now we consider an action of G on a projective space $V = \mathbb{P}(V^\circ)$. This arises from a representation V° of a cyclic extension $\widetilde{G}$ of G. As for linear actions we have $\mathrm{H}^1(G, \mathrm{Pic}(V)) = 0$. From Example 2.2 we have $\gamma \in \mathrm{H}^2(G)$, with $\mathrm{Am}(V, G) = \langle\gamma\rangle$. We have

$$\mathrm{Br}([V/G]) \cong \mathrm{H}^2(G)/\langle\gamma\rangle.$$

Theorem 5.1. *For a faithful action of a finite group G on a projective space V, corresponding to a faithful linear representation of a central cyclic extension $\widetilde{G}$ of G with associated class $\gamma \in \mathrm{H}^2(G)$, we have*

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \ker\Big(\mathrm{H}^2(G)/\langle\gamma\rangle \to \bigoplus_{A \in \mathcal{B}_G} \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(\gamma)\rangle\Big).$$

Proof. For the forwards containment, let $A \in \mathcal{B}_G$. We form the extension $\widetilde{A}$ of A by restricting the extension $\widetilde{G}$ of G and obtain $\mathrm{B}_0(\widetilde{A}) = 0$ from Lemma 3.3. Bogomolov's result yields

$$\mathrm{Br}_{\mathrm{nr}}(k(V^\circ)^{\widetilde{A}}) = 0,$$

and this gives us what we need, since (with $\ell = |Z|$ in the extension (2.2))

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^A) \cong \mathrm{Br}_{\mathrm{nr}}(k(\mathcal{O}_V(-\ell))^A) \cong \mathrm{Br}_{\mathrm{nr}}(k(\mathcal{O}_V(-1))^{\widetilde{A}}) \cong \mathrm{Br}_{\mathrm{nr}}(k(V^\circ)^{\widetilde{A}})$$

by the stable birational invariance of the unramified Brauer group and the No-Name Lemma (see Section 2). The reverse containment is proved as for linear actions. □

Toric actions. Finally, we consider the G-action on the torus $T = \mathbb{G}_m^d$ given by an injective homomorphism

$$G \hookrightarrow \mathrm{GL}_d(\mathbb{Z}) = \mathrm{GL}(M),$$

where $M = \mathfrak{X}^*(T)$ is the character lattice, and $K = k(T)^G$.

As equivariant compactification we take V to be a smooth projective toric variety, given by the combinatorial data of a G-invariant smooth projective fan of cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$, where $N = \mathfrak{X}_*(T)$ is the cocharacter lattice. (This exists in general; see [14].)

We use a variant of (4.2), involving $\mathrm{Br}([T/G])$:

$$\mathrm{Br}_{\mathrm{nr}}(K) \subset \ker\big(\mathrm{Br}([T/G]) \to \mathrm{Br}(T)\big) \subset \mathrm{H}^2(G, k(T)^\times).$$

The middle group is accessible by the basic exact sequence of Section 2, applied to T. Using the splitting given by the fixed point 1_T and the vanishing of $\mathrm{Pic}(T)$, we obtain

$$\ker\big(\mathrm{Br}([T/G]) \to \mathrm{Br}(T)\big) \cong \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z} \oplus M).$$

According to Saltman [26, Thm. 12], the unramified Brauer group is

$$\mathrm{Br}_{\mathrm{nr}}(K) \cong \ker\Big(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z} \oplus M) \to \bigoplus_{A \in \mathcal{B}_G} \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z} \oplus M)\Big).$$

As in the other cases, the forwards containment is implied by the vanishing of $\mathrm{Br}_{\mathrm{nr}}(k(T)^A)$ for $A \in \mathcal{B}_G$, and the reverse containment holds by (1.1). So Saltman's result follows from the vanishing of $\mathrm{Br}_{\mathrm{nr}}(k(T)^A)$ for $A \in \mathcal{B}_G$, which we explain now, following Barge [2].

There is a G-module M' with $M \oplus M'$ of finite index in a permutation module P (e.g., span of boundary divisors of V in the exact sequence $0 \to M \to P \to \mathrm{Pic}(V) \to 0$, with $M' \subset P$ giving an isomorphism $M' \otimes \mathbb{Q} \to \mathrm{Pic}(V) \otimes \mathbb{Q}$). With associated tori $T_P = \mathrm{Spec}(k[P])$, etc., we have $T_M = T$ and epimorphism $T_P \to T \times T_{M'}$ with finite kernel $F \subset T_P$. On T_P the translation action of F and permutation action of G are linear and, together, yield a semidirect product $F \rtimes G$. For $A \in \mathcal{B}_G$ we have

$$\mathrm{Br}_{\mathrm{nr}}(k(T \times T_M)^A) \cong \mathrm{Br}_{\mathrm{nr}}(k(T_P)^{F \rtimes A}) = 0 \tag{5.2}$$

by Lemma 3.2 and Bogomolov's result. The projection $T \times T_{M'} \to T$ has equivariant section $T \times \{1_{T_{M'}}\}$. Thus the induced map

$$\mathrm{Br}([T/A]) \to \mathrm{Br}([T \times T_{M'}/A])$$

is injective, and we obtain the desired vanishing from (5.2).

6. Grassmannians

We fix notation
$$V = \mathrm{Gr}(r,n) = \mathrm{Gr}(r,U^\circ)$$
for the Grassmannian variety of r-dimensional subspaces of a given n-dimensional k-vector space U°. Here, $1 \le r \le n-1$. Since $\mathrm{Pic}(V) \cong \mathbb{Z}$ any action yields $\mathrm{H}^1(G,\mathrm{Pic}(V)) = 0$, and
$$\mathrm{Br}([V/G]) \cong \mathrm{H}^2(G)/\mathrm{Am}(V,G).$$

Automorphisms. When $r=1$, we have projective space $U = \mathbb{P}(U^\circ)$, with automorphism group $\mathrm{PGL}(U^\circ)$. Suppose $r \ge 2$. It is known classically [12] that when $n \ne 2r$ the automorphism group of V is the same as that of U, i.e., $\mathrm{Aut}(V) = \mathrm{PGL}(U^\circ)$, while for $n = 2r$ there is the identity component $\mathrm{PGL}(U^\circ)$ of $\mathrm{Aut}(V)$ and a second component of automorphisms, given by isomorphisms $U^\circ \to U^{\circ\vee}$.

Amitsur invariant. We recall the Amitsur invariant of a projectively linear action (Section 2). Let $G \to \mathrm{PGL}(U^{\circ\vee})$ define a right action of G on U, with extension (2.2) and compatible
$$\widetilde{G} \to \mathrm{GL}(U^{\circ\vee}).$$
We obtain $\gamma \in \mathrm{H}^2(G)$, with $\mathrm{Am}(U,G) = \langle\gamma\rangle$.

Lemma 6.1. *Let a homomorphism $G \to \mathrm{PGL}(U^{\circ\vee})$ determine G-actions on U and on V. If the action on U gives rise to $\gamma \in \mathrm{H}^2(G)$, with $\mathrm{Am}(U,G) = \langle\gamma\rangle$, then for the action on V we have $\mathrm{Am}(V,G) = \langle r\gamma\rangle$.*

Proof. We consider an extension (2.2) with sufficiently divisible $\ell = |Z|$. Applying the rth exterior power yields the extension
$$1 \to Z/\mu_r \to \widetilde{G}/\mu_r \to G \to 1,$$
thus $\mathrm{Am}(\mathbb{P}(\bigwedge^r U^\circ), G) = \langle r\gamma\rangle$. We conclude by applying Lemma 2.1 to the Plücker embedding $V \to \mathbb{P}(\bigwedge^r U^\circ)$. □

Lemma 6.2. *Let the notation be as in Lemma 6.1. Then*
$$\mathrm{Br}_{\mathrm{nr}}(k(U)^G) \cong \mathrm{Br}_{\mathrm{nr}}(k(U\times V)^G).$$

Proof. By Example 2.2 we have a canonical G-linearization of the vector bundle $\underline{U}^\circ \otimes \mathcal{O}_U(1)$ on U, hence also of the sum of r copies $\underline{U}^{\circ\oplus r} \otimes \mathcal{O}_U(1)$. A similar argument supplies a canonical linearization of the tautological bundle S on V, pulled back by the projection $\mathrm{pr}_2\colon U\times V \to V$ and tensored with $\mathrm{pr}_1^*\mathcal{O}_U(1)$, hence as well of $\mathrm{pr}_2^*S^{\oplus r} \otimes \mathrm{pr}_1^*\mathcal{O}_U(1)$. We have a G-equivariant birational equivalence
$$\underline{U}^{\circ\oplus r} \otimes \mathcal{O}_U(1) \sim_G \mathrm{pr}_2^*S^{\oplus r} \otimes \mathrm{pr}_1^*\mathcal{O}_U(1)$$

and conclude by the stable birational invariance of the unramified Brauer group and the No-Name Lemma. □

Lemma 6.3. *Let the notation be as in Lemma 6.1 and A an abelian subgroup of G of index d. We suppose that d divides r, the order of γ is d, and $\gamma \in \ker(\mathrm{res}^2_A)$. Then $V^G \neq \emptyset$.*

Proof. We prove the result by induction on r. For the base case $r = d$, since $\mathrm{res}^2_A(\gamma) = 0$ there is a lift $A \to \mathrm{GL}(U^{\circ\vee})$ of the restriction to A of the homomorphism $G \to \mathrm{PGL}(U^{\circ\vee})$. Therefore $U^A \neq \emptyset$. We take $z \in U^A$. Then the linear span $\Sigma \subset U^\circ$ of the G-orbit of z is G-invariant. Lemma 6.1 implies $\dim(\Sigma) = d$, so $[\Sigma] \in V^G$.

If $r > d$, then we take $\Sigma \subset U^\circ$ as above, $\dim(\Sigma) = d$, and let the condition to contain Σ define a Schubert variety in V, isomorphic to $\mathrm{Gr}(r - d, n - d)$. The induction hypothesis is applicable and yields a fixed point. □

Case of projectively linear automorphisms. Let G act on V via a homomorphism $G \to \mathrm{PGL}(U^{\circ\vee})$. By Lemma 6.1, we have

$$\mathrm{Br}([V/G]) \cong \mathrm{H}^2(G)/\langle r\gamma\rangle.$$

Theorem 6.4. *Let a faithful linear action of a finite group G on a projective space $U = \mathbb{P}(U^\circ)$ be given, with associated class $\gamma \in \mathrm{H}^2(G)$. Then, for the induced action of G on the Grassmannian $V = \mathrm{Gr}(r, U^\circ)$, we have*

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \ker\Big(\mathrm{H}^2(G)/\langle r\gamma\rangle \to \bigoplus_{A\in\mathcal{B}_G} \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(r\gamma)\rangle\Big).$$

Proof. As in other cases, we divide the assertion into a forwards containment and a reverse containment. The forwards containment follows from the claim, that for $A \in \mathcal{B}_G$ we have $\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$. The reverse containment holds by (1.1).

We establish the claim. Let $A \in \mathcal{B}_G$ and $\alpha \in \mathrm{Br}([V/A])$. If α lies in $\mathrm{Br}_{\mathrm{nr}}(k(V)^A)$, then the image of α in $\mathrm{Br}([U\times V/A])$ lies in $\mathrm{Br}_{\mathrm{nr}}(k(U\times V)^A)$, which by Lemma 6.2 is isomorphic to $\mathrm{Br}_{\mathrm{nr}}(k(U)^A)$. So by Theorem 5.1,

$$\alpha \in \langle \mathrm{res}^2_A(\gamma)\rangle/\langle \mathrm{res}^2_A(r\gamma)\rangle. \tag{6.1}$$

We write $A \cong \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}/f\mathbb{Z}$ with $e \mid f$ and let d denote the order of the quotient group in (6.1). So, $d = \gcd(r, s)$, where s is the order of $\mathrm{res}^2_A(\gamma)$ in $\mathrm{H}^2(A) \cong \mathbb{Z}/e\mathbb{Z}$. We consider the subgroups $A'' \subseteq A' \subseteq A$, corresponding to

$$\mathbb{Z}/\tfrac{e}{s}\mathbb{Z} \oplus \mathbb{Z}/f\mathbb{Z} \subseteq \mathbb{Z}/\tfrac{de}{s}\mathbb{Z} \oplus \mathbb{Z}/f\mathbb{Z} \subseteq \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}/f\mathbb{Z}.$$

We have $r\gamma \in \ker(\mathrm{res}^2_{A'})$, with the quotient group in (6.1) mapping isomorphically to $\langle \mathrm{res}^2_{A'}(\gamma)\rangle$. As well, $\gamma \in \ker(\mathrm{res}^2_{A''})$. Lemma 6.3 is applicable and gives $V^{A'} \neq \emptyset$. We apply Lemma 4.1 to conclude $\alpha = 0$. □

General case. Theorem 6.4 gives a complete treatment of faithful actions on Grassmannians, except when $r \geq 2$ and $n = 2r$, which we suppose from now on. With the classical terminology [12], $\mathrm{Aut}(V)$ consists of *collineations*, given by projective linear automorphisms of U°, and *correlations*, given by projective isomorphisms $U^\circ \to U^{\circ\vee}$. In formulas, for $\psi \in \mathrm{GL}(U^\circ)$ the collineation $L_{[\psi]}$ of $[\psi] \in \mathrm{PGL}(U^\circ)$ is

$$L_{[\psi]}([\Sigma]) = [\psi(\Sigma)],$$

while the correlation $C_{[\varphi]}$, for an isomorphism $\varphi\colon U^\circ \to U^{\circ\vee}$, is

$$C_{[\varphi]}([\Sigma]) = [\Sigma'] \qquad \text{with} \qquad \varphi(\sigma)(\sigma') = 0 \quad \forall\, \sigma \in \Sigma,\, \sigma' \in \Sigma'.$$

We have

$$C_{[\varphi]} \circ C_{[\varphi]} = L_{[\varphi^{-1\vee}\circ\varphi]}. \tag{6.2}$$

As well, $C_{[\varphi]}$ and $L_{[\psi]}$ commute if and only if

$$[\psi^\vee \circ \varphi \circ \psi] = [\varphi]. \tag{6.3}$$

Theorem 6.5. *Let a faithful action of a finite group G on a Grassmannian $V = \mathrm{Gr}(r,n) = \mathrm{Gr}(r,U^\circ)$ be given, $\dim(U^\circ) = n$, and let $\beta \in \mathrm{H}^2(G)$ be the class associated with the projective linear action on Plücker coordinates $G \to \mathrm{PGL}(\bigwedge^r U^{\circ\vee})$. Then we have*

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \ker\Big(\mathrm{H}^2(G)/\langle\beta\rangle \to \bigoplus_{A\in\mathcal{B}_G} \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(\beta)\rangle\Big).$$

Proof. We have $\mathrm{Am}(V,G) = \langle\beta\rangle$ by Lemma 2.1, applied to the Plücker embedding. The statement is thus just Theorem 6.4, unless $r \geq 2$ and $n = 2r$, and the action of G involves correlations; we suppose this from now on. We need to show that for $A \in \mathcal{B}_G$ we have $\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$. This is already known (proof of Theorem 6.4) unless the action of A involves correlations; we suppose this as well. For the index 2 subgroup A' of A, where the action is by collineations, we have $\mathrm{Br}_{\mathrm{nr}}(k(V)^{A'}) = 0$.

Let $\alpha \in \mathrm{Br}([V/A]) \cong \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(\beta)\rangle$. If $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A)$, then α lies in the kernel of $\mathrm{Br}([V/A]) \to \mathrm{Br}([V/A'])$. The nontriviality of this kernel forces the cyclic group $\mathrm{H}^2(A)$ to be of even order and the order of β_A to be odd. Then we conclude by Lemma 4.1, using the following lemma for the existence of a fixed point. □

Lemma 6.6. *Let A be a bicyclic group, acting on $V = \mathrm{Gr}(r,n)$, $n = 2r$. We suppose that if $r \geq 2$ then the action involves correlations. We let*

$\beta \in \mathrm{H}^2(A)$ be the class, associated with the projective linear action on Plücker coordinates. Then β is 2-torsion, and we have

$$\beta = 0 \qquad \textit{if and only if} \qquad V^A \neq \emptyset.$$

Proof. If $r = 1$ then the assertions are clear, so we suppose $r \geq 2$. We may write

$$A \cong \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}/f\mathbb{Z},$$

where the respective generators are a correlation $C_{[\varphi]}$ and a collineation $L_{[\psi]}$. They commute. In fact, the corresponding equation (6.3) may be strengthened to

$$\psi^\vee \circ \varphi \circ \psi = \varphi \tag{6.4}$$

by suitably rescaling ψ. From (6.4) and its equivalent form

$$\psi^\vee \circ \varphi^\vee \circ \psi = \varphi^\vee \tag{6.5}$$

we obtain

$$\psi \circ \varphi^{-1\vee} \circ \varphi = \varphi^{-1\vee} \circ \varphi \circ \psi. \tag{6.6}$$

By (6.2) and (6.6), the action of A' (by collineations) lifts to a linear action. So β lies in the kernel of $\mathrm{H}^2(A) \to \mathrm{H}^2(A')$ and thus is 2-torsion.

Existence of a fixed point clearly implies that β vanishes. It remains to show that the vanishing of β implies the existence of a fixed point. We do this by induction on r, where the base case $r = 1$ is already clear.

We consider

$$\varphi_+ = \frac{1}{2}(\varphi + \varphi^\vee) \qquad \text{and} \qquad \varphi_- = \frac{1}{2}(\varphi - \varphi^\vee),$$

which determine a symmetric, respectively skew-symmetric bilinear form on U°. By (6.4)–(6.5) the analogous identities for φ_+ and φ_- also hold. In particular, ψ induces an automorphism of $\ker(\varphi_+)$.

If φ_+ is degenerate, i.e., $\ker(\varphi_+) \neq 0$, then we may take $v \in \ker(\varphi_+)$ to be an eigenvector of ψ. There is a Schubert variety in V, of r-dimensional spaces containing and orthogonal to v (with respect to φ_-). We apply the induction hypothesis and obtain a fixed point.

It remains to treat the case that φ_+ is nondegenerate. Choosing an orthonormal basis of U° for the associated symmetric bilinear form, with dual basis of $U^{\circ\vee}$, we get a representing matrix

$$B = I + B_-$$

for φ, where I denotes the identity matrix, and the matrix B_- represents φ_- and is skew-symmetric. The representing matrix for $\varphi^{-1\vee} \circ \varphi$ is

$$C = (B^{-1})^t B.$$

We let D denote the representing matrix for ψ; then

$$D^t B D = B \qquad \text{and} \qquad DC = CD.$$

Suppose $B_- \neq 0$. An orthogonal change of basis can be made to bring the matrix B_- into a normal form [18, §XI.4]. In the simplest case this is a block diagonal matrix with 2×2-blocks

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \qquad \lambda \in k^\times, \tag{6.7}$$

and possibly an additional zero block. Generally there can be larger blocks, skew-symmetric analogues of the larger Jordan blocks. But these, if present, would obstruct the diagonalizability of C. Since some power of C is identity, C is diagonalizable, and the normal form of B_- has all nonzero blocks of the form (6.7). The fact that D commutes with C implies that D preserves the eigenspaces of C. (Always $\lambda^2 \neq -1$, since B is invertible, and eigenvalues $1\pm\lambda\sqrt{-1}$ of B correspond to eigenvalues $(1\pm\lambda\sqrt{-1})/(1\mp\lambda\sqrt{-1})$ of C.) We conclude by choosing an eigenvector and appealing to the induction hypothesis, as in the previous case.

We are left with the case $B_- = 0$. Then $B = I$, and the matrix D is orthogonal. The fixed locus

$$V^{C_{[\varphi]}} = \{[\Sigma] \in V \mid \Sigma^\perp = \Sigma\}$$

is a disjoint union of two copies of the maximal orthogonal Grassmannian $\mathrm{SO}_n/\mathrm{P}_r$ (parabolic subgroup P_r corresponding to an end root of the Dynkin diagram D_r), acted upon transitively by the orthogonal group. We fix $[\Sigma] \in V^{C_{[\varphi]}}$ and a lift $\rho \in \mathrm{GL}(\bigwedge^r U^\circ)$ of $C_{[\varphi]}$. A nontrivial homomorphism from the orthogonal group to $\{\pm 1\}$ is defined by $\omega \mapsto \lambda'/\lambda$, where λ and λ' are the respective eigenvalues of $\bigwedge^r \Sigma$ and $\bigwedge^r \omega(\Sigma)$:

$$\rho(v) = \lambda v, \qquad \rho(\textstyle\bigwedge^r\omega(v)) = \lambda'\textstyle\bigwedge^r\omega(v) \qquad \text{for } v \in \textstyle\bigwedge^r \Sigma.$$

This has to be the determinant. So, $\beta = 0$ implies $\det(D) = 1$. Then $L_{[\psi]}$ maps each component of $V^{C_{[\varphi]}}$ to itself, and $V^A \neq \emptyset$. □

7. Flag varieties

We fix a k-vector space U° of dimension n, a positive integer m, and positive integers $r_1, \ldots, r_m$ with

$$1 \le r_1 < \cdots < r_m \le n-1.$$

In this section we extend our treatment to the partial flag variety

$$V = \mathrm{F}\ell(r_1, \ldots, r_m; n) = \mathrm{F}\ell(r_1, \ldots, r_m; U^\circ)$$

of nested subspaces of dimensions $r_1, \ldots, r_m$ of U°. When $m = 1$ this is just a Grassmannian variety (Section 6), so we assume $m \ge 2$.

Automorphisms. We obtain a complete description of $\mathrm{Aut}(V)$ from [16]. There is an identity component $\mathrm{PGL}(U^\circ)$, which is the full automorphism group except when the integers $r_1, \dots, r_m$ satisfy the symmetry condition

$$r_i + r_{m+1-i} = n, \qquad \forall\, i.$$

In that case, as in Section 6, $\mathrm{Aut}(V)$ has a second component, consisting of correlations. The action on

$$\mathrm{Pic}(V) \cong \mathbb{Z}^m$$

is trivial (when $\mathrm{Aut}(V) = \mathrm{PGL}(U^\circ)$) or by an involutive permutation (when the symmetry condition holds). So,

$$\mathrm{H}^1(G, \mathrm{Pic}(V)) = 0,$$

and

$$\mathrm{Br}([V/G]) \cong \mathrm{H}^2(G)/\,\mathrm{Am}(V,G).$$

Projectively linear action. Suppose that G acts on V via a homomorphism $G \to \mathrm{PGL}(U^{\circ\vee})$. Let $\gamma \in \mathrm{H}^2(G)$ be the associated class (Example 2.2). Applying Lemma 2.1 to the natural morphism from V to the product of the Grassmannians $\mathrm{Gr}(r_i, U^\circ)$, we obtain

$$\mathrm{Am}(V,G) = \langle r_1\gamma, \dots, r_m\gamma\rangle = \langle q\gamma\rangle, \qquad q = \gcd(r_1, \dots, r_m).$$

Theorem 7.1. *Let a faithful linear action of a finite group G on a projective space $U = \mathbb{P}(U^\circ)$ be given, with associated class $\gamma \in \mathrm{H}^2(G)$. Then, for the induced action of G on the flag variety $V = \mathrm{F}\ell(r_1, \dots, r_m; U^\circ)$ we have*

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \ker\Big(\mathrm{H}^2(G)/\langle q\gamma\rangle \to \bigoplus_{A\in\mathcal{B}_G} \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(q\gamma)\rangle\Big),$$

where $q = \gcd(r_1, \dots, r_m)$.

The proof is similar to the case of Grassmannians (Theorem 6.4). We collect the analogous preliminary results.

Lemma 7.2. *Let the notation be as in Theorem 7.1. Then*

$$\mathrm{Br}_{\mathrm{nr}}(k(U)^G) \cong \mathrm{Br}_{\mathrm{nr}}(k(U\times V)^G).$$

Proof. The argument is similar to the case of a Grassmannian (Lemma 6.2), but on V we have m nested tautological bundles

$$S_1 \subset \dots \subset S_m$$

of ranks $r_1 < \dots < r_m$. We have an equivariant birational equivalence

$$\underline{U}^{\circ\oplus r_m} \otimes \mathcal{O}_U(1) \sim_G \mathrm{pr}_2^*(S_1^{\oplus r_1} \oplus S_2^{\oplus r_2 - r_1} \oplus \dots \oplus S_m^{\oplus r_m - r_{m-1}}) \otimes \mathrm{pr}_1^*\mathcal{O}_U(1)$$

of G-linearized bundles and conclude as before. ∎

Lemma 7.3. *Let the notation be as in Theorem 7.1 and A an abelian subgroup of G of index d. We suppose that d divides q, the order of γ is d, and $\gamma \in \ker(\mathrm{res}^2_A)$. Then $V^G \neq \emptyset$.*

Proof. We prove the result by induction on r_m. By Lemma 6.3 there exists $[\Sigma] \in \mathrm{Gr}(r_1, U^\circ)^G$. We conclude by applying the induction hypothesis to the Schubert variety of $\Sigma_1 \subset \cdots \subset \Sigma_m$ with $\Sigma_1 = \Sigma$. □

Proof of Theorem 7.1. The argument is just as in the proof of Theorem 6.4. To establish the claim, that $\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$ for $A \in \mathcal{B}_G$, we consider $\mathrm{res}^2_A(\gamma)$, whose order we denote by s, so the quotient group $\langle \mathrm{res}^2_A(\gamma)\rangle / \langle \mathrm{res}^2_A(q\gamma)\rangle$ has order $d = \gcd(q, s)$; we only need to consider elements of this quotient group, by Lemma 7.2. Subgroups $A'' \subseteq A' \subseteq A$ are defined just as before, and we conclude with Lemmas 7.3 and 4.1. □

Remark 7.4. Here, and also in the case of Grassmannians (Section 6), in case of a projectively linear action with $\gamma = 0$, i.e., coming from a linear action, the action of G on V is stably linearizable. We apply the construction of the proof of Lemma 7.2, respectively Lemma 6.2, just without the factor U and twist by $\mathcal{O}_U(1)$.

Action involving correlations. Suppose $r_1, \ldots, r_m$ satisfy the symmetry condition and the action of G on V involves correlations. An index 2 subgroup G' acts by collineations with an associated class $\gamma \in \mathrm{H}^2(G')$.

Let $q = \gcd(r_1, \ldots, r_{[m/2]})$. If m is odd, then $n = 2r_{(m+1)/2}$, and as in Section 6 we have $\beta \in \mathrm{H}^2(G)$, associated with the projective linear action on Plücker coordinates $G \to \mathrm{PGL}(\bigwedge^{r_{(m+1)/2}} U^{\circ\vee})$. We have

$$\mathrm{Am}(V, G) = \begin{cases} \langle \mathrm{cores}^2_{G'}(q\gamma)\rangle, & \text{if } m \text{ is even,} \\ \langle \beta, \mathrm{cores}^2_{G'}(q\gamma)\rangle, & \text{if } m \text{ is odd,} \end{cases}$$

where $\mathrm{cores}^2_{G'} \colon \mathrm{H}^2(G') \to \mathrm{H}^2(G)$ is the corestriction map. This comes by applying Lemma 2.1 to the product of Grassmannians $\mathrm{Gr}(r_i, U^\circ)$. For $i = 1, \ldots, [m/2]$ the projective representation associated with the G-action on $\mathrm{Gr}(r_i, U^\circ) \times \mathrm{Gr}(r_{m+1-i}, U^\circ)$ is obtained from $G' \to \mathrm{PGL}(U^{\circ\vee})$ by two operations. The first, $\bigwedge^{r_i}$, multiplies the associated class by r_i. The second, leading to the corestriction, is tensor induction [3, §2B].

Theorem 7.5. *Let a faithful action of a finite group G on a flag variety $V = \mathrm{F}\ell(r_1, \ldots, r_m; U^\circ)$ be given, with $m \geq 2$. Suppose that the action of G involves correlations, with index 2 subgroup G' acting by collineations leading to $\gamma \in \mathrm{H}^2(G')$. Let β be the class associated with the projective linear action on Plücker coordinates $G \to \mathrm{PGL}(\bigwedge^{r_{(m+1)/2}} U^{\circ\vee})$ when m is* *odd,* 0 *when* m *is even. Set* $q = \gcd(r_1, \dots, r_{[m/2]})$*. Then*

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \ker \Big(\mathrm{H}^2(G)/\langle \beta, \mathrm{cores}^2_{G'}(q\gamma)\rangle \\ \to \bigoplus_{A\in\mathcal{B}_G} \mathrm{H}^2(A)/\langle \mathrm{res}^2_A(\beta), \mathrm{res}^2_A(\mathrm{cores}^2_{G'}(q\gamma))\rangle \Big).$$

Proof. We argue as in the proof of Theorem 6.5. For $A \in \mathcal{B}_G$, we show $\mathrm{Br}_{\mathrm{nr}}(k(V)^A) = 0$. This is known (proof of Theorem 7.1) when $A \subset G'$, so we suppose this is not the case. Following the proof of Lemma 6.6, we have the index 2 subgroup $A' = A \cap G'$, whose action lifts to a linear action. We are done, provided we can show $\mathrm{res}^2_A(\beta) = 0$ implies $V^A \neq \emptyset$.

We suppose $\mathrm{res}^2_A(\beta) = 0$. Since A' acts linearly, it suffices to show that $\mathrm{Gr}(r_{(m+1)/2}, U^\circ)^A \neq \emptyset$ when m is odd, respectively $\mathrm{Gr}(r_{m/2}, U^\circ)^{A'}$ contains a point $[\Sigma]$ sent by the correlations in A to a point

$$[\Sigma'] \in \mathrm{Gr}(r_{\frac{m}{2}+1}, U^\circ) \qquad \text{with} \qquad \Sigma \subset \Sigma'$$

when m is even. The argument is as in the proof of Lemma 6.6, exactly so when m is odd, differing slightly in the treatment of the last case when m is even. When $B_- = 0$ (notation of the proof of Lemma 6.6), the fixed locus of $\mathrm{Gr}(r_{m/2}, U^\circ)$ (m even) for the correlation is a single copy of an orthogonal Grassmannian, thus has a fixed point. □

8. General approach via destackification

Let $\mathcal{X} = [V/G]$ be given, where V is a smooth projective rational variety and G acts generically freely. We suppose that $\mathrm{Br}(\mathcal{X})$ has been determined, as outlined in Section 4, in particular, an element of $\mathrm{Br}(\mathcal{X})$ is given by an element of $\mathrm{H}^2(G, k(V)^\times)$. Here we describe a procedure to decide whether a given element of $\mathrm{Br}(\mathcal{X})$ lies in $\mathrm{Br}_{\mathrm{nr}}(k(V)^G)$.

Root stacks. Let $\mathcal{X}$ be a smooth DM stack and $\mathcal{D}$ a divisor on $\mathcal{X}$. For a positive integer r there is the *root stack*

$$\sqrt[r]{(\mathcal{X}, \mathcal{D})}$$

of [9, §2], [1, App. B], which is again smooth, provided $\mathcal{D}$ is smooth. The root stack has the same set of k-points and the same coarse moduli space as $\mathcal{X}$, but has stabilizer groups extended by μ_r along $\mathcal{D}$.

The *iterated root stack* along a simple normal crossing divisor $\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_\ell$ on $\mathcal{X}$ [9, Defn. 2.2.4] is determined by an ℓ-tuple of positive integers $\mathbf{r} = (r_1, \dots, r_\ell)$. This stack $\sqrt[\mathbf{r}]{(\mathcal{X}, \mathcal{D})}$ is obtained by iteratively performing the r_ith root stack construction along each divisor $\mathcal{D}_i$.

An in-depth treatment of the birational geometry of DM stacks, including background on topics such as root stacks, is given in [24].

Set-up. To start, we replace $\mathcal{X} = [V/G]$ by a smooth DM stack $\mathcal{X}'$ with smooth coarse moduli space and proper birational morphism to $\mathcal{X}$.

This is achieved via functorial destackification [4], [5]. The outcome is a sequence of stacky blow-ups whose composite $\mathcal{X}' \to \mathcal{X}$ is as desired. Here, a stacky blow-up is either a usual blow-up along a smooth center or a root stack operation along a smooth divisor. The coarse moduli space X' of $\mathcal{X}'$ is a smooth projective variety with a simple normal crossing divisor $D = D_1 \cup \dots \cup D_\ell$ on X', such that $\mathcal{X}' \cong \sqrt[\mathbf{r}]{(X', D)}$ is an iterated root stack of D.

The morphism $\mathcal{X}' \to \mathcal{X}$ is not necessarily representable. Indeed, a (nontrivial) root stack operation adds stabilizers along a divisor. The corresponding *relative* coarse moduli space is a stack X' with representable morphism to $\mathcal{X}$. Since $\mathcal{X}$ has a representable morphism to BG, so does X', i.e., $\mathsf{X}' \cong [V'/G]$ for some projective variety V'. The variety V' is normal, but not necessarily smooth. We have the diagram

$$\begin{array}{ccccc} \mathcal{X}' & \longrightarrow & [V'/G] & \longrightarrow & X' \\ & \searrow & \downarrow & & \\ & & \mathcal{X} & & \end{array}$$

with 2-commutative triangle. The vertical morphism is representable, induced by a G-equivariant birational proper morphism $V' \to V$.

Let $M = k(V)$. Suppose we are given $\beta \in \mathrm{H}^2(G, M^\times)$, representing $\alpha \in \mathrm{Br}([V/G])$. We explain how to check whether α has vanishing residue along a divisor of X'. It is only necessary to check this for the finitely many divisors of X', where $\mathcal{X}'$ has nontrivial generic stabilizer. We have $\alpha \in \mathrm{Br}_{\mathrm{nr}}(M^G)$ if and only if these residues vanish.

Let $D' \subset X'$ be such a divisor, and let D be a divisor in V', mapping to D' in X'. We let Z denote the stabilizer and I the inertia of D, so I is cyclic and central in Z. The induced action of $\overline{Z} = Z/I$ on D is faithful, and we have $k(D)^{\overline{Z}} \cong k(D')$. Let $n = |I|$.

By the standard behavior of residue under extensions [27, Thm. 10.4], the residue of α along D' in X' is equal to the residue of the restriction of α to $\mathrm{Br}(M^Z)$ along D/Z in V'/Z.

We introduce notation for DVRs, fraction fields, and residue fields:

- V'/Z: The local ring of V'/Z at the generic point of D/Z will be denoted by R; fraction field $K = M^Z$, residue field $\kappa = k(D)^{\overline{Z}}$.
- V'/I: The local ring of V'/I at the generic point of D will be denoted by S; fraction field $L = M^I$, residue field $\lambda = k(D)$.
- V': The local ring of V' at the generic point of D will be denoted by T; fraction field M, residue field λ.

The respective maximal ideals will be denoted by $\mathfrak{m}_R$, etc.

Residue I. Certainly, a necessary condition for the vanishing of the residue of α along D' is the vanishing of the residue of the restriction of α to $\mathrm{Br}(L)$ along D. We explain the computation of this residue. The restriction of α is represented by

$$\beta|_I \in \mathrm{H}^2(I, M^\times) \cong L^\times/\mathrm{N}_{M/L}(M^\times) = S^\times/\mathrm{N}_{M/L}(T^\times).$$

Let $v \in S^\times$ be a representative of $\beta|_I$. Then the residue of the restriction of α to $\mathrm{Br}(L)$ along D is

$$[\bar{v}] \in \lambda^\times/\lambda^{\times n}.$$

If $[\bar{v}] \neq 0$, then we have detected a nontrivial residue of α, and we stop the computation.

Reduction to cocycle for $\overline{Z}$. Continuing with the above notation, we suppose $[\bar{v}] = 0$. By making a suitable choice of representative v we may suppose that

$$v \in 1 + \mathfrak{m}_S.$$

We let $E \subset 1 + \mathfrak{m}_S$ denote the subgroup generated by $(1 + \mathfrak{m}_B)^n$ and the Galois orbit of v. We define $L' = L(E^{1/n})$ and $M' = L'M$; these are Kummer extensions of L. We now show that, there is a Kummer extension K'/K with $K'L = L'$ and $[K' : K] = [L' : L]$.

A choice of maximal ideal of the integral closure of S in L' determines, by localization, a DVR S' with residue field λ. The Kummer pairing of $\mathrm{Gal}(L'/L)$ with E extends to a pairing

$$\mathrm{Gal}(L'/K) \times E \to \mu_n.$$

The induced homomorphism $\mathrm{Gal}(L'/K) \to \mathrm{Hom}(E, \mu_n) \cong \mathrm{Gal}(L'/L)$ determines a direct product decomposition

$$\mathrm{Gal}(L'/K) \cong \mathrm{Gal}(L'/L) \times \overline{Z}$$

and thus a Kummer extension

$$K' = L'^{\overline{Z}}$$

of K with $K'L = L'$. The corresponding DVR R' has residue field κ.

If we replace the tower of fields $M/L/K$ by $M'/L'/K'$ and pass from $\beta|_Z \in \mathrm{H}^2(Z, M^\times)$ to $\beta' \in \mathrm{H}^2(Z, M'^\times)$, the residue does not change, and we have $v \in (L'^\times)^n$. So

$$\beta' \in \ker\big(\mathrm{H}^2(Z, M'^\times) \to \mathrm{H}^2(I, M'^\times)\big).$$

Residue II. We keep the above notation but revert to the notation $M/L/K$ for the tower of fields. So we have reduced to the case

$$\beta|_Z \in \ker\big(\mathrm{H}^2(Z, M^\times) \to \mathrm{H}^2(I, M^\times)\big).$$

Then, by the Hochschild-Serre spectral sequence and Hilbert's Theorem 90, $\beta|_Z$ is the image, under the inflation map, of some

$$\gamma \in \mathrm{H}^2(\overline{Z}, L^\times).$$

Since the $\overline{Z}$-Galois extension L/K is associated with a unramified extension of DVRs, the residue is determined by the procedure described in [19, §III.2]. We apply the valuation

$$\mathrm{val}\colon L^\times \to \mathbb{Z}$$

to obtain $\mathrm{val}(\gamma) \in \mathrm{H}^2(\overline{Z}, \mathbb{Z})$. Now the residue is the class associated with $\mathrm{val}(\gamma)$ under the isomorphism

$$\mathrm{Hom}(\overline{Z}, \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{H}^2(\overline{Z}, \mathbb{Z}).$$

Example 8.1. For the quotient stack $[\mathbb{P}^3/\mathfrak{K}_4]$ of Example 4.2, with Brauer group of order 2 generated by α, destackification is achieved by

- blowing up the fixed points to produce exceptional divisors E_i ($i \in \{0, \dots, 3\}$),
- blowing up the proper transforms of the intersections of pairs of coordinate hyperplanes to yield exceptional divisors E_{ij} (i, $j \in \{0, \dots, 3\}$, $i < j$), and
- blowing up the intersections of the proper transforms of the exceptional divisors from the first blow-up with the proper transforms of the coordinate hyperplanes, leading to exceptional divisors E'_{cd} (c, $d \in \{0, \dots, 3\}$, $c \neq d$).

As indicated in [25, Rem. 3.3], since only $\mathbb{Z}/2\mathbb{Z}$ and $\mathfrak{K}_4$ occur as stabilizer groups, destackification is achieved with just ordinary blow-ups (no nontrivial root stack operations). So $\mathcal{X}' = [V'/\mathfrak{K}_4]$. Along the divisors E_{ij} and E'_{cd} the generic stabilizer has order 2. Let $D \subset V'$, over $D' \subset X'$, be one of the divisors with nontrivial generic stabilizer. In local coordinates x, y, z, we have D given by $x = 0$, where $\mathfrak{K}_4$ acts by distinct nontrivial characters on x and y and acts trivially on z. We have $|I| = 2$ and $\beta \in \mathrm{H}^2(\mathfrak{K}_4, k(x, y, z)^\times)$, given by a μ_2-valued cocycle and image under the inflation map of $[x^2] \in \mathrm{H}^2(\mathfrak{K}_4/I, k(x^2, y, z)^\times)$ (with the conventions of Section 2 for cyclic group cohomology). The residue is given by the nontrivial homomorphism $\mathfrak{K}_4/I \to \mathbb{Q}/\mathbb{Z}$.

Example 8.2. Consider the action of

$$G = \mathfrak{A}_4 \cong \langle (135)(246), (12)(34), (12)(56) \rangle \subset \mathfrak{S}_6$$

on $V = \overline{\mathcal{M}}_{0,6}$. This is a *nonstandard* $\mathfrak{A}_4$ in $\mathfrak{S}_6$, *not* fixing a plane in the Segre cubic model. Actions fixing a plane, such as the Klein 4-group $\mathfrak{K}_4 \subset G$, are birational to actions on toric varieties, see [10, Section 6]. Restriction to the Klein 4-group induces an isomorphism

$$\mathrm{H}^2(G) \cong \mathrm{H}^2(\mathfrak{K}_4) \cong \mathbb{Z}/2\mathbb{Z}.$$

As well, V^G is nonempty, with

$$\mathrm{H}^1(G, \mathrm{Pic}(V)) \cong \mathrm{H}^1(\mathfrak{K}_4, \mathrm{Pic}(V)) \cong \mathbb{Z}/2\mathbb{Z}.$$

So

$$\mathrm{Br}([V/G]) \cong \mathrm{Br}([V/\mathfrak{K}_4]) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

It is known that $\mathrm{Br}_{\mathrm{nr}}(k(V)^{\mathfrak{K}_4}) = 0$ (since the $\mathfrak{K}_4$-action is birational to a toric action, and the rationality of such a quotient is a special case of [20, Thm. 1.2 and 1.3]); consequently,

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) = 0.$$

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