INTERMEDIATE JACOBIANS AND LINEARIZABILITY

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Abstract. We develop an equivariant version of the formalism of intermediate Jacobian torsor obstructions, and apply it to conic bundles over rational surfaces, quadric surface bundles over $\mathbb{P}^1$, and Fano threefolds.

1. Introduction

Let X be a rational variety, over $\mathbb{C}$, equipped with a generically free, regular action of a finite group G. A fundamental problem in higher-dimensional birational geometry is to identify *linearizable* and *projectively linearizable* actions, i.e., actions that are birational to G-actions on $\mathbb{P}(V)$, where V is a linear representation of G, respectively, a linear representation of a central extension of G (see, e.g., [HT22, Section 2] for definitions of basic terms). Even the classification of birational types of involutions in dimension 3 is an open problem, see [Pro13].

We develop an equivariant version of the formalism of intermediate Jacobian torsor obstructions from [HT21a] and [BW23], and apply it to conic bundles over $\mathbb{P}^2$, quadric surface bundles over $\mathbb{P}^1$, and Fano threefolds. We pursue the analogy with arithmetic considerations in [HT21b], [HT21a], [BW23], [HT21c], [KP23], [FJS$^+$23] and [JJ23], focused on rationality of geometrically rational threefolds over nonclosed fields.

There are certain similarities between birational geometry over nonclosed fields and equivariant birational geometry; rationality should be viewed as analogous to linearizability, and birationality to a Brauer-Severi variety as analogous to projective linearizability. In the study of rationality, the absolute Galois group of the ground field acts on geometric invariants, such as the Picard group, and all geometric constructions have to take into account Galois symmetries. In the study of (projective) linearizability, the action of the automorphism group limits the choices for birational transformations.

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On the other hand, there are also substantial differences:

- Existence of fixed points is not an equivariant birational invariant for actions of nonabelian groups, and is not required for (projective) linearizability.
- The action of cyclic groups on projective space is always linearizable, while a Galois twist of $\mathbb{P}^1$ by a cyclic group is not necessarily rational, as is the case for $\mathfrak{S}_2 \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on a conic over $\mathbb{R}$ without $\mathbb{R}$-points.
- Rationality of smooth quadric hypersurfaces $X_2 \subset \mathbb{P}^n$, $n \geq 2$, is completely settled, and is equivalent to the existence of rational points, while linearizability is still an open problem for $n \geq 4$, see [TYZ23, Section 9].
- Some varieties are rational over any field but fail to be linearizable for automorphisms, e.g., a quintic del Pezzo surface or a quintic del Pezzo threefold, which are birationally rigid, and not linearizable, for $G = \mathfrak{A}_5$, see [HT23, Section 8.2] and [CS16].

Main results and constructions. In Sections 2 and 3 we develop equivariant versions of the theory of intermediate Jacobians and related torsor obstructions. Regular actions of finite groups on X yield, by functoriality, actions on a smooth group scheme $\mathbf{CH}^2_{X/\mathbb{C}}$ whose $\mathbb{C}$-valued points are naturally identified with the Chow group $\mathrm{CH}^2(X)$ of codimension-two cycles. This gives rise to obstructions to projective linearizability analogous to those in [HT21a, Theorem 6.3] and [BW23, Theorem C]:

Theorem 1.1 (Theorem 3.3)**.** *Let X be a smooth projective rational threefold over $\mathbb{C}$ with a regular, projectively linearizable, action of a finite group G. Then there exists a smooth projective (possibly disconnected) curve C with a regular G-action such that for any G-invariant connected component M of $\mathbf{CH}^2_{X/\mathbb{C}}$ there exist a G-invariant connected component $N \subset \mathrm{Pic}(C)$ and an equivariant isomorphism of G-varieties*

$$M \cong N.$$

In such situations, we say that the equivariant intermediate Jacobian torsor (IJT) obstruction vanishes. The nonvanishing of this obstruction allows to prove failure of (projective) linearizability in many new cases, e.g., for conic bundles (see Section 5, Examples 5.10 and 5.12), and quadric surface bundles over $\mathbb{P}^1$ (see Section 6, Examples 6.9 and 6.10).

Furthermore, this gives *criteria* for (projective) linearizability, e.g., for actions of cyclic groups on conic bundles $X \to \mathbb{P}^2$ with quartic discriminant, in Theorem 5.4, or quadric surface bundles $\mathcal{X} \to \mathbb{P}^1$ in Theorem 6.4. Among rational Fano threefolds, projective linearizability

of actions on smooth complete intersections of two quadrics $X_{2,2} \subset \mathbb{P}^5$ was settled in [HT22], via a reduction to rationality considerations over nonclosed fields; here we give an alternative proof. We establish a similar criterion for actions on X_{16}: they are projectively linearizable if and only if there exists a G-invariant rational cubic curve on the variety. We elucidate issues arising for Fano threefolds of type X_{18}, see Theorem 7.1 and the discussion in Section 7.

Moreover, the intermediate Jacobian torsor formalism yields new general results concerning birationality of nonlinearizable actions:

Theorem 1.2 (Corollary 3.5)**.** *Let X_i, for $i = 1, 2$, be smooth projective rationally connected threefolds with a regular action of a finite group G. Assume that their intermediate Jacobians are the Jacobians of smooth projective curves C_i, with transitive action of G on their connected components. Assume that*

- *the connected components of C_i have genus ≥ 2, for $i = 1, 2$,*
- *the equivariant IJT obstruction does not vanish for X_1,*
- *C_1 is not G-equivariantly isomorphic to C_2.*

Then X_1 is not G-equivariantly birational to X_2.

In particular, this applies to *involutions* acting on rational conic bundles over rational surfaces or quadric surface bundles over $\mathbb{P}^1$. Recall that the classification of conjugacy classes of involutions in the Cremona group $\mathrm{Cr}_2(\mathbb{C})$, the group of birational automorphisms of $\mathbb{P}^2$, is based on the study of *non-uniruled divisors* in the fixed locus of the involution, see, e.g., [Bla07]. In higher dimensions, birational involutions are much more difficult to control: there is too much flexibility. The paper [Pro13] gives a rough classifications of involutions in $\mathrm{Cr}_3(\mathbb{C})$, in presence of *non-uniruled* divisorial components in the fixed locus. Even then, some of the cases listed in [Pro13, Theorem 1.2] may overlap. In Examples 5.10, 5.12, 6.9 and 6.10, no such divisors exist; to the best of our knowledge, these are the first instances when one is able to distinguish conjugacy classes of such involutions.

An arithmetic version of Theorem 1.2 gives many nonrational, geometrically rational, threefolds which are not birational to each other over the ground field; e.g., intersections of two quadrics $X_{2,2}$ considered in [HT21a] and [BW23], or conic bundles in [FJS$^+$23].

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2. Equivariant intermediate Jacobians

Throughout, we work over the complex numbers $\mathbb{C}$. Let X be a smooth projective rationally connected threefold, $\mathrm{NS}(X)$ its Neron-Severi group, and

$$\mathrm{IJ}(X) := \mathrm{H}^3(X,\mathbb{C})/(\mathrm{H}^1(X,\Omega^2_X)\oplus \mathrm{H}^3(X,\mathbb{Z}))$$

its intermediate Jacobian. This complex torus carries a principal polarization θ_X induced by the cup product

$$\wedge^2 \mathrm{H}^3(X,\mathbb{Z}) \to \mathrm{H}^6(X,\mathbb{Z}) \cong \mathbb{Z},$$

so that $(\mathrm{IJ}(X),\theta_X)$ is a principally polarized abelian variety, and we can consider $\mathrm{IJ}(X)$ as a smooth projective variety over $\mathbb{C}$.

Let $\mathrm{CH}^2(X)$ be the group of codimension-two cycles on X, over $\mathbb{C}$, up to rational equivalence, and $\mathrm{CH}^2(X)_{\mathrm{alg}} \subset \mathrm{CH}^2(X)$ the subgroup of cycles algebraically equivalent to 0. Put

$$\mathrm{NS}^2(X) := \mathrm{CH}^2(X)/\mathrm{CH}^2(X)_{\mathrm{alg}},$$

it is a finitely generated abelian group.

Let A be an abelian variety over $\mathbb{C}$ and

$$\phi : \mathrm{CH}^2(X)_{\mathrm{alg}} \to A(\mathbb{C})$$

a group homomorphism. It is called *regular* if for any smooth connected variety T over $\mathbb{C}$, $t_0 \in T(\mathbb{C})$, and codimension-two cycle $Z \in Z^2(T\times X)$, the map

$$\begin{array}{rcl} T(\mathbb{C}) & \to & A(\mathbb{C}), \\ t & \mapsto & \phi(Z_t - Z_{t_0}), \end{array}$$

is induced by a morphism $T \to A$, defined over $\mathbb{C}$. Such a morphism is unique if it exists. An example of a regular homomorphism is the Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{CH}^2(X)_{\mathrm{alg}} \to \mathrm{IJ}(X)(\mathbb{C}).$$

This is bijective when X is smooth rationally connected by [BS83, Theorem I(i)], and *universal*, i.e., initial object in the category of regular

homomorphisms from $\mathrm{CH}^2(X)_{\mathrm{alg}}$, see [Mur85], [Kah21]. This identification endows $\mathrm{CH}^2(X)_{\mathrm{alg}}$ with the structure of a principally polarized abelian variety, and we denote this scheme by $(\mathbf{CH}^2_{X/\mathbb{C}})^0$.

Let $\mathrm{CH}^2(X)^\gamma$ be the preimage of $\gamma \in \mathrm{NS}^2(X)$ under the map

$$\mathrm{CH}^2(X) \to \mathrm{NS}^2(X).$$

It is in bijection with $\mathrm{CH}^2(X)_{\mathrm{alg}}$ (via translation by an element of $\mathrm{CH}^2(X)^\gamma$), which is unique up to a unique translation by elements in $\mathrm{CH}^2(X)_{\mathrm{alg}}$. This defines a scheme structure on $\mathrm{CH}^2(X)^\gamma$, and a group scheme structure on $\mathrm{CH}^2(X)$, denoted by $(\mathbf{CH}^2_{X/\mathbb{C}})^\gamma$ and $\mathbf{CH}^2_{X/\mathbb{C}}$, respectively. When X is rational, $\mathbf{CH}^2_{X/\mathbb{C}}$ is the group scheme that represents the functor $\mathrm{CH}^2_{X/\mathbb{C},\mathrm{fppf}}$ of Chow groups of codimension-two cycles on X, constructed in [BW23, Theorem 3.1].

A regular action of a finite group G on X induces a G-action on algebraic cycles, and on $\mathrm{CH}^2(X)$. It also yields an action on cohomology and a regular action on the intermediate Jacobian $\mathrm{IJ}(X)$. Compatibility with the cup product implies that the polarization $\theta_X \in \mathrm{NS}(\mathrm{IJ}(X))$ is G-invariant.

Lemma 2.1. *Let X be a smooth projective rationally connected threefold with a regular G-action. Then the Abel-Jacobi map*

$$\mathrm{AJ} : \mathrm{CH}^2(X)_{\mathrm{alg}} \to \mathrm{IJ}(X)(\mathbb{C})$$

is G-equivariant.

Proof. We recall the construction: let γ be a codimension-two cycle which is algebraically equivalent to 0. Algebraic equivalence and homological equivalence coincide, provided the Chow group of zero cycles on X is supported on a surface [BS83, Theorem I(ii)], which is the case, by the assumption. Thus γ is homologically equivalent to 0. Let α be a topological 3-cycle with boundary $\partial\alpha = \gamma$. Then the linear functional

$$\int_\alpha \cdots \in (\mathrm{H}^1(X, \Omega^2_X)^\vee / \mathrm{H}_3(X, \mathbb{Z})) \cong \mathrm{IJ(X)},$$

is well-defined and is the image of α under the Abel-Jacobi map. For any $g \in G$, we have $g\gamma = g\partial\alpha = \partial g\alpha$ on the level of topological cycles. □

In particular, the G-action on $\mathrm{CH}^2(X)_{\mathrm{alg}}$ is induced by a regular G-action on $(\mathbf{CH}^2_{X/\mathbb{C}})^0$.

Lemma 2.2. *The G-action on $\mathrm{CH}^2(X)$ is induced by a regular G-action on $\mathbf{CH}^2_{X/\mathbb{C}}$.*

Proof. We need to verify that for $g \in G$ and $\gamma \in \mathrm{NS}^2(X)$, the action

$$g : (\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma}(\mathbb{C}) \to (\mathbf{CH}^2_{X/\mathbb{C}})^{g\gamma}(\mathbb{C}), \quad [Z] \mapsto [gZ],$$

is algebraic. Let Z_0 be an effective 1-cycle representing γ. The above action is induced by a composition of the morphisms:

$$(\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma} \to (\mathbf{CH}^2_{X/\mathbb{C}})^{0} \to (\mathbf{CH}^2_{X/\mathbb{C}})^{0} \to (\mathbf{CH}^2_{X/\mathbb{C}})^{g\gamma},$$

mapping

$$[Z] \mapsto [Z] - [Z_0] \mapsto [gZ] - [gZ_0] \mapsto [gZ],$$

where the middle map is the G-action on $(\mathbf{CH}^2_{X/\mathbb{C}})^0$ and the other two are translations. □

A G-action on X induces a natural G-action on $\mathrm{Chow}^2(X)$, the Chow variety of codimension-two cycles, via $Z \mapsto gZ$, for $g \in G$.

Theorem 2.3. *Let X be a smooth projective rationally connected threefold with a regular action by a finite group G. The Abel-Jacobi map*

$$\mathrm{AJ} : \mathrm{Chow}^2(X) \to \mathbf{CH}^2_{X/\mathbb{C}}$$

is a G-equivariant morphism.

Proof. As in, e.g., [HT21a, Proposition 2.5], one verifies that the Abel-Jacobi map is a well-defined morphism. The G-equivariance of

$$\mathrm{AJ} : \mathrm{Chow}^2(X)(\mathbb{C}) \to \mathrm{CH}^2(X), \quad Z \mapsto [Z],$$

is clear from the construction. □

3. Equivariant intermediate Jacobian torsors and obstructions

Abelian varieties with group actions. We continue to work over $\mathbb{C}$. A G-abelian variety is an abelian variety with an action of a finite group G, preserving the origin. Let A be a G-abelian variety. A principal polarization $\theta_A \in \mathrm{NS}(A)$ is called G-equivariant if it is G-invariant with respect to the G-action on $\mathrm{NS}(A)$; in this case, (A, θ_A) is called a G-equivariant principally polarized abelian variety. A G-equivariant homomorphism

$$\iota : (A, \theta_A) \to (B, \theta_B)$$

of G-equivariant principally polarized abelian varieties is a homomorphism $\iota : A \to B$ of abelian varieties such that $\iota^*\theta_B = \theta_A$. We record:

Lemma 3.1. *Let*

$$\iota : (A, \theta_A) \to (B, \theta_B)$$

be a G-equivariant homomorphism of G-equivariant principally polarized abelian varieties, which is a closed embedding. Then there exists a G-equivariant principally polarized abelian variety $(A', \theta_{A'})$ such that

$$(B, \theta_B) \simeq (A \times A', p_1^*\theta_A + p_2^*\theta_{A'}),$$

as G-equivariant principally polarized abelian varieties, where p_i's are natural projections.

Proof. Consider the induced primitive, G-equivariant, embedding

$$\iota_* : \mathrm{H}_1(A, \mathbb{Z}) \to \mathrm{H}_1(B, \mathbb{Z}).$$

The principal polarization $\theta_B \in \mathrm{H}^2(B, \mathbb{Z}) = \wedge^2\mathrm{H}^1(B, \mathbb{Z})$ defines an alternating form on $\mathrm{H}_1(B, \mathbb{Z})$, and its restriction to $\mathrm{H}_1(A, \mathbb{Z})$ is the alternating form induced by θ_A. This alternating form is compatible with the G-action on these lattices. Let Λ be the orthogonal complement of $\mathrm{H}_1(A, \mathbb{Z})$ in $\mathrm{H}_1(B, \mathbb{Z})$. Then

$$\mathrm{H}_1(A, \mathbb{Z}) \oplus \Lambda = \mathrm{H}_1(B, \mathbb{Z}),$$

since θ_B and θ_A are principal polarizations. Since the alternating form is compatible with G-action, Λ is G-invariant and θ_B defines a non-degenerate alternating form on Λ which is selfdual.

The groups $\mathrm{H}_1(A, \mathbb{Z})$ and $\mathrm{H}_1(B, \mathbb{Z})$ admit integral Hodge structures of weight -1, which are compatible with ι_* and preserved by the G-action. Since θ_B is a Hodge class, this induces a principally polarized integral Hodge structure of weight -1 on Λ which is compatible with G-action on it, and Λ can be identified with $\mathrm{H}_1(A', \mathbb{Z})$, where A' is a G-equivariant principally polarized abelian variety. □

Corollary 3.2. *A G-equivariant principally polarized abelian variety admits a decomposition as a product of indecomposable G-equivariant principally polarized abelian varieties, which is unique, up to permutation of the factors.*

An indecomposable G-equivariant principally polarized abelian variety is also called *irreducible*.

Intermediate Jacobian torsor obstructions. The following theorem gives an obstruction to equivariant projective linearizability; it is analogous to [HT21a, Theorem 6.3] and [BW23, Theorem C]:

Theorem 3.3. *Let X be a smooth projective rational threefold with a regular, projectively linearizable, G-action. Then there exists a smooth projective (possibly disconnected) curve C with a regular G-action such*

that for any $\gamma \in \mathrm{NS}^2(X)^G$, one has a G-equivariant isomorphism of G-varieties

$$(\mathbf{CH}^2_{X/\mathbb{C}})^\gamma \simeq \mathrm{Pic}^m(C),$$

for some G-invariant class m. Moreover, we have

$$(\mathbf{CH}^2_{X/\mathbb{C}})^0 \simeq \mathrm{Pic}^0(C),$$

as G-equivariant principally polarized abelian varieties.

Proof. We have a G-equivariant birational map

$$\phi : X \dashrightarrow \mathbb{P}^3,$$

where $\mathbb{P}^3$ is equipped with a regular action of G. Resolving the indeterminacy of ϕ equivariantly, we have equivariant birational morphisms

$$X \xleftarrow{g} W \xrightarrow{f} \mathbb{P}^3,$$

where W is a smooth projective threefold with a regular G-action. By functorial weak factorization (see [AT19]), f and g are compositions of equivariant blowups and downs with G-irreducible smooth centers. Here G-irreducible means that the action on connected components of the smooth center is transitive. The assertion is reduced to a single G-equivariant blowup

$$\psi : X_1 \to X_2,$$

where one of the X_i's satisfies the statement of the assertion, and we need to prove the statement for the other X_j.

Suppose that X_2 satisfies the statement, i.e., there exists a smooth projective G-curve C such that for any $\gamma \in \mathrm{NS}^2(X_2)^G$,

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^\gamma \cong \mathrm{Pic}^m(C),$$

for some m, as G-equivariant varieties, and when $\gamma = 0$,

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^0 \cong \mathrm{Pic}^0(C),$$

as G-equivariant principally polarized abelian varieties. Suppose that the center of ψ is a G-invariant and G-irreducible curve Γ. The blowup formula

$$\mathrm{CH}^2(X_1) = \mathrm{CH}^2(X_2) \oplus \mathrm{CH}^1(\Gamma), \tag{3.1}$$

is G-equivariant, and ψ induces a G-equivariant isomorphism

$$\mathrm{H}^3(X_1, \mathbb{Z}) \cong \mathrm{H}^3(X_2, \mathbb{Z}) \oplus \mathrm{H}^1(\Gamma, \mathbb{Z}(-1)),$$

compatible with the cup product as well as the Hodge structures. This induces an isomorphism

$$\mathrm{IJ}(X_1) \cong \mathrm{IJ}(X_2) \times \mathrm{Pic}^0(\Gamma),$$

as principally polarized abelian varieties, and an isomorphism

$$(\mathbf{CH}^2_{X_1/\mathbb{C}})^0 \cong \mathrm{Pic}^0(C \sqcup \Gamma), \tag{3.2}$$

as G-equivariant principally polarized abelian varieties.

For any $\gamma \in \mathrm{NS}^2(X_1)^G$, (3.1) induces a G-equivariant isomorphism of varieties

$$(\mathbf{CH}^2_{X_1/\mathbb{C}})^\gamma \simeq \mathrm{Pic}^m(C \sqcup \Gamma),$$

for some m, which is a translation of (3.2), as claimed.

When the center of ψ is 0-dimensional, there is a similar blowup formula for CH^2, after replacing $\mathrm{CH}^1(\Gamma)$ by a permutation module of $\mathbb{Z}$. In particular, connected components of $\mathbf{CH}^2_{X_2/\mathbb{C}}$ are identified with connected components of $\mathbf{CH}^2_{X_1/\mathbb{C}}$.

In the other direction, when X_1 satisfies the assertion, the proof is reversed. Assume that ψ is a blowup of a G-invariant and G-irreducible curve Γ. Then we have a G-equivariant blowup formula:

$$\mathrm{CH}^2(X_1) = \mathrm{CH}^2(X_2) \oplus \mathrm{CH}^1(\Gamma).$$

As before, $\mathrm{Pic}^0(C) \simeq (\mathbf{CH}^2_{X_2/\mathbb{C}})^0 \times \mathrm{Pic}^0(\Gamma)$ as G-equivariant principally polarized abelian varieties. When every connected component of Γ has genus 0, the assertion is clear. Thus, we may assume that every connected component of Γ has genus $g \geq 1$. Using Corollary 3.2 and the Torelli theorem for curves, we can write $C = C' \sqcup \Gamma$, where C' is a smooth projective G-curve, and we have

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^0 \cong \mathrm{Pic}^0(C').$$

Then, for any $\gamma \in \mathrm{NS}^2(X_2)^G$,

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^\gamma \oplus \mathrm{Pic}^0(\Gamma) \cong \mathrm{Pic}^m(C),$$

for some m, as G-varieties. Since an isomorphism

$$(\mathrm{CH}^2(X_2))^\gamma \oplus \mathrm{Pic}^0(\Gamma) \cong \mathrm{Pic}^m(C) = \mathrm{Pic}^{m'}(C') \oplus \mathrm{Pic}^0(\Gamma)$$

is a translation of an isomorphism

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^0 \oplus \mathrm{Pic}^0(\Gamma) \cong \mathrm{Pic}^0(C) = \mathrm{Pic}^0(C') \oplus \mathrm{Pic}^0(\Gamma),$$

and the translation preserves the direct sum structure, we see that

$$(\mathbf{CH}^2_{X_2/\mathbb{C}})^\gamma \cong \mathrm{Pic}^{m'}(C'),$$

as G-varieties. When the center of ψ is 0-dimensional, the proof is similar, and we omit it. □

The intermediate Jacobian torsor formalism yields new obstructions to equivariant birationality of nonlinearizable actions on rationally connected threefolds, and is a new tool to distinguish finite subgroups of the Cremona group $\mathrm{Cr}_3(\mathbb{C})$, up to conjugation in this group. As an application of Theorem 3.3, we obtain:

Theorem 3.4. *Let X_i, with $i = 1, 2$, be smooth projective rationally connected G-varieties of dimension 3 such that their intermediate Jacobians are irreducible G-equivariant principally polarized abelian varieties. Assume that*

- *the equivariant IJT-obstruction does not vanish for X_1,*
- *$\mathrm{IJ}(X_1)$ is not isomorphic to $\mathrm{IJ}(X_2)$ as G-equivariant principally polarized abelian varieties.*

Then X_1 is not G-equivariantly birational to X_2.

Proof. Suppose that X_1 is G-equivariantly birational to X_2. By assumption, there is a G-invariant class γ of codimension-two cycles on X_1 such that

$$(\mathbf{CH}^2(X_1))^\gamma \ncong \mathrm{Pic}^e(C),$$

for any smooth projective G-curve C and any e. As in Theorem 3.3, this shows that $\mathrm{IJ}(X_1)$ must arise in the decomposition of $\mathrm{IJ}(X_2)$. Since they are irreducible, as G-principally polarized abelian varieties, this implies that $\mathrm{IJ}(X_1) \cong \mathrm{IJ}(X_2)$. This contradicts the assumption. □

Corollary 3.5. *Let X_i, with $i = 1, 2$, be smooth projective rationally connected G-varieties of dimension 3 such that their equivariant intermediate Jacobians are the Jacobians of G-irreducible smooth projective curves C_i whose connected components have genus $g_i \geq 2$. Assume that*

- *the equivariant IJT-obstruction does not vanish for X_1,*
- *C_1 is not G-equivariantly isomorphic to C_2.*

Then X_1 is not G-equivariantly birational to X_2.

Proof. By Torelli theorem, $\mathrm{Pic}^0(C_1)$ and $\mathrm{Pic}^0(C_2)$ are not isomorphic to each other as G-equivariant principally polarized abelian varieties. The assertion follows from Theorem 3.4. □

4. Equivariant Prym varieties

We consider G-equivariant Prym varieties, following the constructions in [FJS+23, Section 4] in the arithmetic context.

Equivariant Prym varieties. Fix a G-equivariant étale finite morphism $\varpi : \widetilde{\Delta} \to \Delta$ between connected smooth projective G-curves, of degree 2. The norm map

$$\mathrm{Nm} : \mathrm{Pic}(\widetilde{\Delta}) \to \mathrm{Pic}(\Delta),$$

is a G-equivariant homomorphism of group schemes. A G-equivariant embedding $r : \Delta \hookrightarrow W$ into a smooth projective rational G-surface induces a G-equivariant morphism of group schemes

$$r^* : \mathrm{Pic}(W) \to \mathrm{Pic}(\Delta).$$

The $\mathrm{Pic}(W)$-polarized Prym scheme is defined as

$$\mathrm{PPrym}^{\mathrm{Pic}(W)}(\widetilde{\Delta}/\Delta) := \mathrm{Pic}(\widetilde{\Delta}) \times_{\mathrm{Pic}(\Delta)} \mathrm{Pic}(W),$$

it is a group scheme with diagonal G-action. The identity component of $\mathrm{PPrym}^{\mathrm{Pic}(W)}(\widetilde{\Delta}/\Delta)$ is the identity component of the kernel of the norm map Nm which is the Prym variety $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$ from [Bea77]. This is a principally polarized abelian variety, and G acts on it. The polarization associated with a theta divisor on $\mathrm{Pic}^0(\widetilde{\Delta})$ is a G-principal polarization. Its restriction to $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$ is twice a principal polarization on $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$. In particular, $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$ admits a G-invariant principal polarization.

For $D \in \mathrm{Pic}(W)$, denote the fiber of

$$\mathrm{PPrym}^{\mathrm{Pic}(W)}(\widetilde{\Delta}/\Delta) \to \mathrm{Pic}(W)$$

above D by V_D; it is a G-equivariant torsor of V_0 when D is G-invariant. Since $\varpi_*\varpi^* D = 2D$, we have

$$V_{2D+D'} = \varpi^* D + V_{D'}, \quad \forall D, D' \in \mathrm{Pic}(W),$$

compatibly with G-actions. The group scheme V_0 consists of two connected components,

$$P = \mathrm{Prym}(\widetilde{\Delta}/\Delta), \quad \widetilde{P} = V_0 \setminus P.$$

Both carry the G-action, and $\widetilde{P}$ is an equivariant 2-torsion torsor of P.

For $W = \mathbb{P}^2$, $\Delta \subset \mathbb{P}^2$ is a G-invariant smooth plane quartic, and H the hyperplane class of $\mathbb{P}^2$ we have:

Proposition 4.1 ([Mum74, Section 6, Equation (6.1)])**.** *The parity of h^0 is constant on each of the two connected components of V_H, and is different on these components.*

Definition 4.2 ([FJS+23, Definition 4.4])**.** Let $P^{(1)}$ be the connected component of V_H on which h^0 is even, and let $\widetilde{P}^{(1)}$ be the connected

component of V_H on which h^0 is odd. For each $m \in \mathbb{Z}_{\geq 0}$ and $e = 0, 1$, we define

$$P^{(2m+e)} = P^{(e)} + mH, \quad \widetilde{P}^{(2m+e)} = \widetilde{P}^{(e)} + mH,$$

where $P^{(0)} = P$ and $\widetilde{P}^{(0)} = \widetilde{P}$. These are G-equivariant torsors of $\mathrm{Prym}(\widetilde{\Delta}/\Delta)$.

Intermediate Jacobians and Prym varieties. We follow [Bea77] and [FJS+23]. Let X be a smooth projective rational threefold over $\mathbb{C}$ with a regular action of a finite group G, and admitting a G-equivariant standard conic bundle structure $\pi : X \to W$, where

- W is a smooth projective rational G-surface,
- the discriminant curve $\Delta \subset W$ is smooth,
- the (equivariant) étale double cover $\varpi : \widetilde{\Delta} \to \Delta$ is irreducible.

Lemma 4.3 ([FJS+23, Proposition 5.3])**.** *Let γ be the class of a line in a singular fiber of π. Then γ is G-invariant and we have a G-equivariant exact sequence:*

$$0 \to \mathbb{Z}\gamma \to \mathrm{NS}^2(X) \xrightarrow{\pi_*} \mathrm{NS}(W) \to 0.$$

Proof. Since π is G-equivariant, π_* is as well. Moreover, G maps any line in the singular fibers of π to a line in a singular fiber. Since $\widetilde{\Delta}$ is irreducible, γ is G-invariant. The exactness of the sequence follows from [FJS+23, Proposition 5.3]. □

The following is an analog of a key result in [FJS+23], comparing torsors of intermediate Jacobians and Prym varieties:

Theorem 4.4 ([FJS+23, Theorems 5.1 and 5.8])**.** *For $\pi : X \to W$ as above, there is a G-equivariant surjective morphism of group schemes*

$$\mathbf{CH}^2_{X/\mathbb{C}} \to \mathrm{PPrym}^{\mathrm{Pic}(W)}(\widetilde{\Delta}/\Delta)$$

that is an isomorphism when restricted to each connected component. Moreover, this induces an isomorphism

$$(\mathbf{CH}^2_{X/\mathbb{C}})^0 \to \mathrm{Prym}(\widetilde{\Delta}/\Delta)$$

of G-equivariant principally polarized abelian varieties.

Proof. A morphism as stated is established in [FJS+23, Theorem 5.1], so we only need to show that it is G-equivariant. We recall the constructions in [FJS+23, Section 5]:

We have a G-equivariant embedding $\iota : \Delta \hookrightarrow X$ with G-invariant image. Let $\epsilon : X' \to X$ be the G-equivariant blowup along $\iota(\Delta)$. Let $X_\Delta := \pi^{-1}(\Delta)$ and S be its proper transform in X', embedded via

$j : S \hookrightarrow X'$. Note that X_Δ is G-invariant, and so is S. Then S admits a G-equivariant $\mathbb{P}^1$-bundle $p : S \to \widetilde{\Delta}$. We have a homomorphism

$$p_* j^* \epsilon^* : \mathrm{CH}^2(X) \to \mathrm{CH}^1(\widetilde{\Delta}) = \mathrm{Pic}(\widetilde{\Delta}),$$

where $\epsilon^* : \mathrm{CH}^2(X) \to \mathrm{CH}^2(X')$ and $j^* : \mathrm{CH}^2(X') \to \mathrm{CH}^2(S)$ are the refined Gysin homomorphisms from [Ful98, Section 6.6]. By [Ful98, Theorem 6.2], these are G-equivariant, so the above homomorphism is also G-equivariant. By [FJS+23, Proposition 5.6], the homomorphism

$$(p_* j^* \epsilon^*, \pi_*) : \mathrm{CH}^2(X) \to \mathrm{Pic}(\widetilde{\Delta}) \times \mathrm{Pic}(W)$$

is G-equivariant, with image contained in $\mathrm{PPrym}^{\mathrm{Pic}(W)}(\widetilde{\Delta}/\Delta)$. This is the homomorphism used in [FJS+23, Theorem 5.1]. The last statement follows from [FJS+23, Lemma 5.5 and Theorem 5.8]. □

5. Conic bundles

We turn to the linearizability of actions on smooth rational Fano threefolds X over $\mathbb{C}$ admitting an equivariant conic bundle structure

$$\pi : X \to \mathbb{P}^2,$$

such that the relative Picard rank of π is 1 and the *discriminant curve* $\Delta \subset \mathbb{P}^2$ is a smooth plane quartic. We develop a version of intermediate Jacobian torsor obstructions in this setting, following [FJS+23], and establish a criterion for linearizability of actions of cyclic groups.

Classification. The classification of Fano threefolds implies that X is one of the following, see [MM81], [MM83], [MM03], and [Mat95]:

(1) a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched along a smooth divisor of type $(2, 2)$,
(2) the blowup of a quadric hypersurface in $\mathbb{P}^4$ along a genus 2, degree 6 curve,
(3) the blowup of a smooth intersection of two quadrics in $\mathbb{P}^5$ along a plane conic.

In all cases, X is rational, see, e.g., [Pro18, Corollary 5.6.1], and the automorphism group $\mathrm{Aut}(X)$ is finite.

A G-action on X induces a G-action on the Picard group $\mathrm{Pic}(X)$, which has to preserve the extremal rays of the nef cone. In Cases (2)-(3), this implies that the G-action necessarily preserves the blowup presentation. Thus, the linearization problem is reduced to the case of G-actions on quadrics (considered in [TYZ23]) or the case of intersections of two quadrics (considered in [HT23]). Linearizability is still largely open in Case (2), see [TYZ23, Section 9]. In Case (3), projective linearizability is equivalent to the existence of a G-stable line.

The main example. Thus, we focus on Case (1), the double cover

$$\delta : X \to \mathbb{P}^1 \times \mathbb{P}^2, \tag{5.1}$$

ramified in a smooth divisor of bi-degree $(2,2)$, with projections

$$\begin{array}{ccc} X & \xrightarrow{\pi_2} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \pi_1} & & \\ \mathbb{P}^1 & & \end{array}$$

where π_2 is a standard conic bundle, with discriminant a smooth quartic $\Delta \subset \mathbb{P}^2$ and discriminant double cover $\varpi : \widetilde{\Delta} \to \Delta$.

A regular G-action on X induces actions of G on $\mathbb{P}^1, \mathbb{P}^2$, Δ, and $\widetilde{\Delta}$, so that π_1, π_2 and $\varpi : \widetilde{\Delta} \to \Delta$ are G-equivariant. Let $\mathcal{F}_1(X/\mathbb{P}^1)$ be the Fano variety of lines in the fibers of π_1; it carries the G-action as well, and we have an equivariant Stein factorization

$$\mathcal{F}_1(X/\mathbb{P}^1) \to C \to \mathbb{P}^1,$$

where C is a smooth projective curve of genus 2 equipped with a regular G-action.

Standard linearization construction.

Proposition 5.1 ([Pro18])**.** *Assume that G is cyclic and $\widetilde{\Delta}^G \neq \emptyset$. Then the G-action on X is linearizable.*

Proof. Let H_1, H_2 be the pullbacks of hyperplane classes from $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively. Let $\ell \cup \ell'$ be a singular fiber of π_2 corresponding to the G-fixed point on $\tilde{\Delta}$, i.e., both ℓ and ℓ' are G-stable. Let $p : \widehat{X} \to X$ the blowup of X along ℓ, with exceptional divisor E. Then $\widehat{X}$ is a smooth weak Fano threefold with a regular G-action induced by the G-action on X.

Let $L = p^*H_1 + p^*H_2 - E$; we have $L^3 = 2$. The linear system $|L|$ is 4-dimensional and base point free, the associated morphism

$$\Phi_{|L|} : \widehat{X} \to \mathbb{P}^4$$

defines a G-equivariant birational morphism $\widehat{X} \to Q \subset \mathbb{P}^4$ to a quadric threefold which is a quadric cone of rank 4; with linear G-action on the ambient $\mathbb{P}^4$. Now ℓ and ℓ' meet at a G-fixed point. The fiber S of $\pi_1 : X \to \mathbb{P}^1$ containing this fixed point is G-stable. Let $\widehat{S}$ be its strict transform on $\widehat{X}$. Then $L^2.\widehat{S} = 1$, i.e., Q contains a G-stable plane. Since G is cyclic, Q contains a G-fixed smooth point. Projection from this point yields an equivariant birational map to $\mathbb{P}^3$. $\square$

Intermediate Jacobians and torsors.

Theorem 5.2. [Bru08], [FJS+23, Theorem 4.5] *We have equivariant isomorphisms*

$$P^{(0)} = \mathrm{Prym}(\widetilde{\Delta}/\Delta) \cong \mathrm{Pic}^0(C), \quad P^{(1)} \cong \mathrm{Pic}^1(C).$$

Moreover, the first isomorphism is an isomorphism of G-equivariant principally polarized abelian varieties.

Proof. Choose a general $x \in C$ with image $t \in \mathbb{P}^1$. The fiber S_t of p is a smooth quadric surface and x corresponds to a ruling of lines on S_t. Let ℓ be a general line in this ruling. The pushfoward of ℓ to $\mathbb{P}^2$ via π_2 is a line so that $\pi_2^*\Delta.\ell = 4$. Since x and ℓ are general, ℓ meets $\pi_2^*\Delta$ at smooth points of $\pi_2^*\Delta$. In particular, $\pi_2^*\Delta \cap \ell$ and its pushforward to $\widetilde{\Delta}$ define a degree 4 effective 0-cycle α on $\widetilde{\Delta}$. Since ℓ is parametrized by $\mathbb{P}^1$, the linear equivalence class of α only depends on x, and we have a natural G-equivariant morphism

$$C \to \mathrm{Pic}(\widetilde{\Delta}).$$

Since the pushfoward of α to Δ is the intersection of Δ and a line on $\mathbb{P}^2$, the above morphism actually defines

$$C \to V_{K_\Delta}.$$

Since α is parametrized by $\mathbb{P}^1$, we have $h^0(\widetilde{\Delta}, \mathcal{O}(\alpha)) \geq 2$. By Clifford's theorem, $h^0(\widetilde{\Delta}, \mathcal{O}(\alpha)) = 2$. Hence we have a G-equivariant morphism

$$C \to P^{(1)},$$

inducing a G-equivariant morphism

$$\mathrm{Pic}^1(C) \to P^{(1)},$$

which is an isomorphism by [Bru08] and [FJS+23, Theorem 4.5]. The above construction of this morphism is slightly different from the one in [Bru08] and [FJS+23, Theorem 4.5], however they are compatible and they define the same morphism. (See [FJS+23, Proposition 6.3(iii)] for more details.) Finally, the above G-equivariant isomorphism induces a G-equivariant isomorphism

$$\mathrm{Pic}^0(C) \to P^{(0)}.$$

For the last claim see, e.g., [Bru08, Section 5, Case 4]. □

Let γ_1 be the class of lines in the fibers of $\pi_1 : X \to \mathbb{P}^1$ and γ_2 the class of lines in the fibers of $\pi_2 : X \to \mathbb{P}^2$. Then $\mathrm{NS}^2(X)$ is generated by γ_1 and γ_2 and both classes are G-invariant.

Theorem 5.3 ([FJS+23, Theorem 6.4])**.** *In the above setting, we have equivariant isomorphisms:*

$$(\mathbf{CH}^2_{X/\mathbb{C}})^{m\gamma_1+n\gamma_2} \cong \begin{cases} P & m,n \text{ are even;} \\ \widetilde{P} & m \text{ is even and } n \text{ is odd;} \\ P^{(1)} & m \text{ is odd and } n \text{ is even;} \\ \widetilde{P}^{(1)} & m,n \text{ are odd.} \end{cases}$$

Proof. The analogous isomorphisms in [FJS+23, Theorem 6.4] are induced by the morphism in Theorem 4.4, which is equivariant. □

Equivariant IJT obstructions. The following is the equivariant analog of [FJS+23, Theorem 1.5]:

Theorem 5.4. *Let G be a cyclic group and $\delta : X \to \mathbb{P}^1 \times \mathbb{P}^2$ a smooth G-double cover ramified along a smooth divisor of bi-degree $(2,2)$. Then X is G-linearizable if and only if $\widetilde{P}$ or $\widetilde{P}^{(1)}$ is a trivial torsor.*

Remark 5.5. The triviality of $\mathrm{Br}(k)$ in the assumptions in [FJS+23, Theorem 1.5] translates into the condition $\mathrm{H}^2(G, k^\times) = 0$.

We start with auxiliary constructions. Consider

$$\overline{\mathcal{M}}_{0,0}(X,\gamma), \quad \gamma \in \mathrm{NS}^2(X),$$

the coarse moduli space of stable maps of genus 0 and class γ and put

$$\mathcal{F}_{1,1}(X) := \overline{\mathcal{M}}_{0,0}(X, \gamma_1+\gamma_2).$$

Proposition 5.6. *The moduli space $\mathcal{F}_{1,1}(X)$ is smooth and the Abel-Jacobi map*

$$\mathrm{AJ} : \mathcal{F}_{1,1}(X) \to (\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}$$

is a $\mathbb{P}^1$-fibration.

Proof. There are two types of stable maps parameterized by $\mathcal{F}_{1,1}(X)$:

(1) an embedding $f : \mathbb{P}^1 \to X$ of class $\gamma_1+\gamma_2$;
(2) the domain consists of two $\mathbb{P}^1$'s, one is isomorphically mapping to a line R_1 in a fiber of π_1 and the other to a line R_2 in a fiber of π_2.

These have trivial automorphisms. Thus, for the first claim, we only need to show the smoothness the moduli stack

$$\overline{\mathcal{M}}_{0,0}(X, \gamma_1+\gamma_2).$$

Consider (1), an embedding $f : R \cong \mathbb{P}^1 \to X$, and let $\ell := \pi_2(f(R))$ be its image in $\mathbb{P}^2$, a line. Its preimage $S_\ell := \pi_2^{-1}(\ell)$ is a normal

surface with only canonical singularities. Consider the exact sequences of tangent and normal bundles

$$0 \to T_R \to T_{S_\ell}|_R \to Q \to 0,$$

and

$$0 \to Q \to N_{R/X} \to N_{S_\ell/X}|_R \to 0.$$

Since $N_{R/X}$ is locally free, Q is torsion-free so that it is locally free. Thus, $T_{S_\ell}|_R$ is also torsion-free. We have

$$-K_{S_\ell}.R = (-K_X - S_\ell).R = 3 - 1 = 2$$

This indicates that $c_1(Q).R = 0$ and $Q \cong \mathcal{O}$. We also have $S_\ell.R = 1$, so that $N_{S_\ell/X}|_R \cong \mathcal{O}(1)$ and $N_{R/X} \cong \mathcal{O} \oplus \mathcal{O}(1)$; we have $\mathrm{H}^1(R, N_{R/X}) = 0$. Thus $[f]$ is a smooth point of $\mathcal{F}_{1,1}(X)$.

Consider an $f : R = R_1 \cup R_2 \to X$, from Case (2). As above,

$$N_{R_1/X} \cong \mathcal{O} \oplus \mathcal{O} \quad \text{and} \quad N_{R_2/X} \cong \mathcal{O} \oplus \mathcal{O}(-1),$$

so that $\mathrm{H}^1(R_i, f^*T_X|_{R_i}) = 0$ for $i = 1, 2$. Since

$$T_X|_{R_1} \cong \mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O},$$

it follows from [BLRT23, Proposition 2.3] that

$$\mathrm{H}^1(R, f^*T_X) = 0.$$

By the discussion of [Tes09, Section 1.1], $[f] \in \mathcal{F}_{1,1}(X)$ is a smooth point. Hence $\mathcal{F}_{1,1}(X)$ is a smooth projective variety of dimension 3; and the Abel-Jacobi map

$$\mathrm{AJ} : \mathcal{F}_{1,1}(X) \to (\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}$$

is well-defined. We proceed to show show that it is a $\mathbb{P}^1$-fibration.

A stable map $[f : R \to X] \in \mathcal{F}_{1,1}(X)$ defines a line $\ell := \pi_2(f(R))$ in $\mathbb{P}^2$ and thus a morphism to the dual

$$\mathcal{F}_{1,1}(X) \to (\mathbb{P}^2)^\vee.$$

We denote its Stein factorization by

$$\mathcal{F}_{1,1}(X) \to B \to (\mathbb{P}^2)^\vee.$$

The pullback $S_\ell := \pi_2^{-1}(\ell)$ is a normal projective surface with only canonical singularities, and $f(R)$ is a Weil divisor on S_ℓ. A fiber of $\mathcal{F}_{1,1}(X) \to B$ is the linear system $|f(R)|$ of the divisorial sheaf $\mathcal{O}_{S_\ell}(f(R))$ which is isomorphic to $\mathbb{P}^1$, thus $\mathcal{F}_{1,1}(X) \to B$ is a $\mathbb{P}^1$-fibration, and the Abel-Jacobi map factors as

$$\mathcal{F}_{1,1}(X) \to B \to (\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}.$$

The assertion follows if we show that AJ is dominant with connected fibers. Recall that the Abel-Jacobi map is induced by

$$\mathcal{F}_{1,1}(X) \to \mathrm{Pic}^4(\widetilde{\Delta}),$$

given in the proof of Theorem 4.4, which factors as

$$\mathcal{F}_{1,1}(X) \to B \to \mathrm{Pic}^4(\widetilde{\Delta}).$$

To show that $B \to \mathrm{Pic}^4(\widetilde{\Delta})$ is injective, fix $[f] \in \mathcal{F}_{1,1}(X)$ and let D be its image in $\mathrm{Pic}^4(\widetilde{\Delta})$. We claim that $h^0(\widetilde{\Delta}, \mathcal{O}(D)) = 1$. Otherwise, by Clifford's theorem, $h^0(\widetilde{\Delta}, \mathcal{O}(D)) = 2$. However, by Theorem 5.3, the image of the Abel-Jacobi map is in $\widetilde{P}^{(1)}$. This contradicts the fact that $h^0(\widetilde{\Delta}, \mathcal{O}(D))$ is even. Thus we have $h^0(\widetilde{\Delta}, \mathcal{O}(D)) = 1$. This implies that $B \to \mathrm{Pic}^4(\widetilde{\Delta})$ is injective. □

The following is analogous to [FJS+23, Proposition 6.5]:

Proposition 5.7. *Let G be a cyclic group. Assume that $\widetilde{P}^{(1)}$ is a trivial P-torsor. Then X is G-linearizable.*

Proof. Assume that $\widetilde{P}^{(1)}$ has a G-fixed point. By Theorem 5.3,

$$(\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2},$$

has a fixed point. Since G is cyclic, Proposition 5.6 implies that

$$\mathcal{F}_{1,1}(X)$$

has at least two fixed points. In particular, $\pi_1 : X \to \mathbb{P}^1$ admits two G-invariant sections. Projection from one of the sections gives a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, with a G-fixed section. Such a $\mathbb{P}^2$-bundle is G-linearizable. □

Consider the coarse moduli space

$$\mathcal{F}_{2,1}(X) = \overline{\mathcal{M}}_{0,0}(X, 2\gamma_1 + \gamma_2).$$

Dimension count shows that it generically parameterizes a stable map from $\mathbb{P}^1$ which is an embedding into X.

Consider $\mathrm{Sym}^8(\widetilde{\Delta}) \to \mathrm{Pic}^8(\widetilde{\Delta})$ and denote the preimage of $\widetilde{P}^{(2)} \subset \mathrm{Pic}^8(\widetilde{\Delta})$ by $\widetilde{S}^{(2)}$. Then

$$\widetilde{S}^{(2)} \to \widetilde{P}^{(2)}$$

is a $\mathbb{P}^3$-bundle over $\widetilde{P}^{(2)}$. For a general smooth irreducible curve C of type $2\gamma_1+\gamma_2$, $C \cap \pi_2^{-1}(\Delta)$ induces a degree-8 divisor on $\widetilde{\Delta}$ which defines a class on $\widetilde{P}^{(2)}$. After taking a G-equivariant resolution $\widetilde{\mathcal{F}}_{2,1}(X)$, we have a G-equivariant birational morphism

$$\widetilde{\mathcal{F}}_{2,1}(X) \to \widetilde{S}^{(2)}.$$

The following is analogous to [FJS+23, Theorem 6.7]:

Proposition 5.8. *Assume that G is cyclic and $\widetilde{P}$ is a trivial P-torsor. Then X is G-linearizable.*

Proof. If $\widetilde{P}$ is trivial then so is $\widetilde{P}^{(2)}$, and $\widetilde{S}^{(2)}$ has a fixed point. Since the existence of fixed points is an equivariant birational invariant, for actions of abelian groups on smooth projective varieties, both $\widetilde{\mathcal{F}}_{2,1}(X)$ and $\mathcal{F}_{2,1}(X)$ have a fixed point. Thus, there is a G-invariant section of $\pi_1 : X \to \mathbb{P}^1$. Projecting from it gives a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. The generic fiber X_η is isomorphic to $\mathbb{P}^2_\eta$ with a G-invariant line. Such bundles are linearizable, for cyclic actions. □

Proof of Theorem 5.4. If X is G-linearizable, then Theorems 3.3, 5.2, and 5.3 show that $\widetilde{P}$ and $\widetilde{P}^{(1)}$ are isomorphic to P or $P^{(1)}$. Since

$$[\widetilde{P}^{(1)}] = [P^{(1)}] + [\widetilde{P}] \in \mathrm{H}^1(G, P),$$

$\widetilde{P}$ or $\widetilde{P}^{(1)}$ is a trivial torsor.

Conversely, when $\widetilde{P}^{(1)}$, respectively $\widetilde{P}$, is trivial, we apply Proposition 5.7, respectively 5.8.

□

Examples of (non)linearizable actions. As before, $\delta : X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a double cover ramified along a smooth divisor of bidegree $(2, 2)$.

Proposition 5.9. *Let $\tau \in \mathrm{Aut}(X)$ be an involution such that*

- *the induced action on $\mathbb{P}^1$ via $\pi_1 : X \to \mathbb{P}^1$ is trivial,*
- *the induced action on $\mathbb{P}^2$ via $\pi_2 : X \to \mathbb{P}^2$ is non-trivial, and*
- *$\widetilde{\Delta}^\tau = \emptyset$.*

Then X is not $\langle\tau\rangle$-projectively linearizable.

Proof. Since cyclic actions on projective space are linearizable, we may assume that the induced action on $\mathbb{P}^2$ is given by

$$\bar{\tau} : [x : y : z] \mapsto [x : y : -z].$$

Then the discriminant curve has the form

$$z^4 + z^2 Q_2(x, y) + Q_4(x, y),$$

where $Q_2, Q_4 \in \mathbb{C}[x, y]$ are homogeneous of degree 2 and 4, respectively. Since we assume that Δ is smooth, $Q_4 = 0$ has 4 distinct roots.

Suppose that X is $\langle\tau\rangle$-projectively linearizable. By Theorem 3.3, there exists a smooth connected projective curve C of genus 2 with $\mathfrak{S}_2$-action such that $(\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}$ is equivariantly isomorphic to $\mathrm{Pic}^m(C)$, for some m. The quotient of C by a nontrivial involution is either

an elliptic curve, or $\mathbb{P}^1$. In either case, C has a fixed point, and thus $\mathrm{Pic}^m(C)$ is equivariantly isomorphic to $\mathrm{Pic}^0(C)$, with a fixed point.

We conclude that $(\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}$ has a fixed point. Since

$$\mathcal{F}_{1,1}(X) \to (\mathbf{CH}^2_{X/\mathbb{C}})^{\gamma_1+\gamma_2}$$

is a $\mathbb{P}^1$-fibration, $\mathcal{F}_{1,1}(X)$ also has a fixed point. Let $f : R \to X$ be a stable map which is $\langle\tau\rangle$-stable. If R is reducible, then $f(R)$ consists of two lines in fibers of π_1 and π_2 respectively. However, τ never preserves a line in fibers of π_2 because $\widetilde{\Delta}^\tau = \emptyset$. Thus R is irreducible and $R_1 = f(R)$ is $\langle\tau\rangle$-stable. If R_1 is not fixed, then R_1 must be a bisection of π_1 because the induced action on $\mathbb{P}^1$ is trivial. This contradicts the fact that the class of R_1 is $\gamma_1 + \gamma_2$. We conclude that R_1 is fixed. Let ℓ be the image of R_1 via $\pi_2 : X \to \mathbb{P}^2$. Then ℓ is a line, fixed by the group action, i.e., ℓ is defined by $z = 0$. Let S be the pullback of ℓ via $\pi_2 : X \to \mathbb{P}^2$. Then S is a smooth del Pezzo surface of degree 4 because $Q_4 = 0$ has distinct roots.

Consider the double cover $S \to \mathbb{P}^1 \times \ell$ and let R' be the image of R_1 via this map. Then $R_1 \to R'$ is birational. This means that the pullback of R' via $S \to \mathbb{P}^1 \times \ell$ consists of two curves R_1 and R_2 and the involution is swapping those curves. This is a contradiction. □

Example 5.10. Consider the double cover:

$$X : w^2 = t_0^2 q_0 + 2t_0 t_1 q_{01} + t_1^2 q_1,$$

with

$$\begin{aligned} q_0 &:= z^2 + a_{0,0}x^2 + a_{0,1}xy + a_{0,2}y^2, \\ q_{01} &:= a_{01,0}x^2 + a_{01,1}xy + a_{01,2}y^2, \\ q_1 &:= z^2 + a_{1,0}x^2 + a_{1,1}xy + a_{1,2}y^2, \end{aligned}$$

where $a_{i,j} \in \mathbb{C}$. The discriminant curve Δ is given by

$$z^4 + z^2 Q_2(x, y) + Q_4(x, y) = 0,$$

where $Q_2, Q_4 \in \mathbb{C}[x, y]$ are homogeneous polynomials of degree 2 and 4, respectively. We assume that Δ is smooth and consider the involutions

$$\begin{aligned} \mu &: ([t_0 : t_1], [x : y : z], w) \mapsto ([t_0 : t_1], [x : y : -z], w), \\ \sigma &: ([t_0 : t_1], [x : y : z], w) \mapsto ([t_0 : t_1], [x : y : z], -w). \end{aligned}$$

The involution μ is linearizable, since $\widetilde{\Delta}$ has a $\langle\mu\rangle$-fixed point. The action of $\tau := \sigma\mu$ is not projectively linearizable, by Proposition 5.9.

Remark 5.11. The Burnside formalism of [KT22] applies in Example 5.10 to $G \simeq \mathfrak{S}_2^2 = \langle\sigma, \tau\rangle$: the σ-action fixes a del Pezzo surface of

degree 2, and the residual action fixes a curve of genus 3. The formalism of incompressible symbols of, e.g., [TYZ23, Section 3.6] shows that the G-action on X is not projectively linearizable.

Example 5.12. Consider the double cover:

$$X : w^2 = t_0^2 q_0 + 2t_0 t_1 q_{01} + t_1^2 q_1,$$

with

$$\begin{aligned} q_0 &:= a_{0,0}x^2 + a_{0,1}y^2 + a_{0,2}z^2, \\ q_{01} &:= a_{01,0}x^2 + a_{01,1}y^2 + a_{01,2}z^2, \\ q_1 &:= a_{1,0}x^2 + a_{1,1}y^2 + a_{1,2}z^2, \end{aligned}$$

where $a_{i,j} \in \mathbb{C}$. In this case, the discriminant $\Delta \subset \mathbb{P}^2$ is defined by an equation of the form

$$Q(x^2, y^2, z^2) = 0.$$

We assume that Δ is a smooth quartic and consider the involutions

$$\begin{aligned} \tau_1 &: ([t_0 : t_1], [x : y : z], w) \mapsto ([t_0 : t_1], [-x : y : z], w) \\ \tau_2 &: ([t_0 : t_1], [x : y : z], w) \mapsto ([t_0 : t_1], [x : -y : z], w) \\ \tau &: ([t_0 : t_1], [x : y : z], w) \mapsto ([t_0 : t_1], [x : y : -z], -w). \end{aligned}$$

Then $\tau_1\tau_2 = \tau$ and $G := \langle \tau_1, \tau_2 \rangle \simeq \mathfrak{S}_2^2$ does not contain the involution σ from Example 5.10. For $i = 1, 2$, we have $\widetilde{\Delta}^{\tau_i} \neq \emptyset$, so that X is $\langle \tau_i \rangle$-projectively linearizable. However, by Proposition 5.9, X is not $\langle \tau \rangle$-projectively linearizable, and thus not G-projectively linearizable.

Remark 5.13. In Examples 5.10 and 5.12, (projective) linearizability is equivalent to the vanishing of equivariant IJT obstructions by Theorem 5.4. It follows from Corollary 3.5 that equivariant birational classes of conic bundles X, X' with involutions τ, respectively, τ', are different as soon as $\widetilde{\Delta}, \widetilde{\Delta}'$ are not equivariantly isomorphic. In these examples, there are no non-uniruled divisors in the fixed locus of the involutions; previously known obstructions do not apply to these conic bundles, see [Pro13].

The following example shows that $\widetilde{\Delta}^G \neq \emptyset$ is not a necessary condition for linearizability.

Example 5.14. Here we prove that X is always $\langle \sigma \rangle$-linearizable, where σ is the unique nontrivial involution acting trivially on $\mathbb{P}^1 \times \mathbb{P}^2$. This follows from the triviality of $\widetilde{P}^{(1)}$, as a P-torsor.

Let ℓ be a bitangent line to Δ such that $\ell \cap \Delta$ consists of two points. Then S_ℓ contains two A_1-singularities, and there is a curve

R of class $\gamma_1+\gamma_2$ mapping to ℓ and passing through these two singularities. Indeed, let $\widetilde{S}_\ell \to S_\ell$ be the minimal resolution. A singular fiber of $\pi_1|_{S_\ell} : S_\ell \to \mathbb{P}^1$ at $t_0 = 0$ contains a line ℓ' of class γ_1, and such a line passes through one singular point but not the other. We also have $-K_{S_\ell}.\ell' = 1$. The fibration $\widetilde{S}_\ell \to \ell$ comes with two singular fibers, both consisting of two (-1)-curves connected by one (-2)-curve. If we contract all (-1)-curves of these singular fibers, we get a weak del Pezzo surface of degree 8. The strict transform of ℓ' is a conic of self-intersection 0. This implies that this weak del Pezzo surface is a smooth quadric surface. The existence of R follows. Then the existence of R induces a fixed point on $\widetilde{P}^{(1)}$.

When $\ell \cap \Delta$ consists of one point, S_ℓ contains a unique singularity of type A_3. The minimal resolution $\widehat{S}_\ell$ of S_ℓ contains a chain of three (-2)-curves, with two (-1)-curves attached to the edges. Then there is a curve of class $\gamma_1+\gamma_2$ mapping to ℓ whose strict transform on $\widehat{S}_\ell$ meets the central (-2)-curve of the chain. This induces a fixed point on $\widetilde{P}^{(1)}$.

6. Quadric surface bundles

Let $\mathcal{X}$ be a smooth projective threefold over $\mathbb{C}$ admitting a quadric surface bundle structure

$$\pi : \mathcal{X} \to \mathbb{P}^1.$$

Such threefolds are rational: by Tsen's theorem, there exists a section of π, projecting from it gives birationality to a relative $\mathbb{P}^2$-bundle, thus birationality $X \sim \mathbb{P}^3$. However, this may not work equivariantly.

Here, we explore equivariant birational geometry of such quadric surface bundles, establish a criterion for linearizability of actions of cyclic groups, and provide examples of nonlinearizable actions.

Geometry. We follow [HT12]. The intermediate Jacobian is given by

$$\mathrm{IJ}(\mathcal{X}) = \mathrm{Pic}^0(C),$$

where $C \to \mathbb{P}^1$ is the discriminant double cover, a hyperelliptic curve ramified in the discriminant $\mathfrak{d} \subset \mathbb{P}^1$ of π (reduced, since $\mathcal{X}$ is smooth, by assumption). Let $|\mathfrak{d}|$ be the degree of the discriminant divisor,

$$g(C) = \frac{|\mathfrak{d}|}{2} - 1$$

the genus of C, and ω_π the relative dualizing sheaf of π. The height of a section $s : \mathbb{P}^1 \to \mathcal{X}$ is defined as

$$h_{\omega_\pi^{-1}}(s) := \deg(s^*\omega_\pi^{-1}),$$

it is equal to the degree of the normal bundle N_s. The space of sections of π of height h is denoted by

$$\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, h),$$

and its closure in the coarse moduli space of stable maps of genus 0 by

$$\overline{\mathrm{Sec}}(\mathcal{X}/\mathbb{P}^1, h).$$

The expected dimension of this moduli space is $h+2$. Let $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)$ the relative Fano variety of lines, with the Stein factorization

$$\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C \to \mathbb{P}^1.$$

Lemma 6.1 ([HT12, Proposition 2])**.** *Assume that $C \to \mathbb{P}^1$ is non-split. Fix a class γ of sections of π of sufficiently large height. Then the Abel-Jacobi map*

$$\mathrm{AJ} : \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, \gamma) \to (\mathbf{CH}^2_{X/\mathbb{C}})^\gamma$$

is an open immersion composed with a projective bundle over $(\mathbf{CH}^2_{X/\mathbb{C}})^\gamma$.

Proof. We include a proof, following [HT12]. Since the Brauer group of $\mathbb{C}(C)$ is trivial, $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C$ is the projectivization $\mathbb{P}_C(V)$ of a rank 2 vector bundle V over C. Let $\mathcal{O}_{\mathbb{P}_C(V)}(1)$ be the relative polarization of $\mathbb{P}_C(V)$. There is a bijection between sections of $\mathcal{X}/\mathbb{P}^1$ and sections of $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)/C$, as in [HT12, Section 3], with a natural identification

$$\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, h) = \mathrm{Sec}(\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)/C, d),$$

where $d = \deg(s^*\mathcal{O}_{\mathbb{P}_C(V)}(1))$ and $s : C \to \mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)$ is a section. By [HT12, (3.2)],

$$h = 2d + \deg(V) - \frac{|\mathfrak{d}|}{2}.$$

We have a natural morphism

$$\mathrm{Sec}(\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)/C, d) \to \mathrm{Pic}^d(C), \quad s \mapsto s^*\mathcal{O}_{\mathbb{P}_C(V)}(1).$$

Let $\mathcal{L}$ be the universal line bundle on $C \times \mathrm{Pic}^d(C)$. When d is sufficiently large, $(\rho_2)_*(\mathcal{L} \otimes \rho_1^* V)$ is locally free; here ρ_i are the projections onto the factors in $C \times \mathrm{Pic}^d(C)$. By [HT12, Proposition 2], the moduli space

$$\mathrm{Sec}(\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)/C, d)$$

is a Zariski open in the projectivization $\mathbb{P}((\rho_2)_*(\mathcal{L} \otimes \rho_1^* V))$ over $\mathrm{Pic}^d(C)$.

To show that this morphism coincides with the Abel-Jacobi map, we follow [LT24, Example 10.4]: after rescaling we may assume that $\mathcal{O}_{\mathbb{P}_C(V)}(1)$ admits a section C'; we let B' be a section of π corresponding to C'. We denote the blow up of $\mathcal{X}$ along B' by $\phi : \mathcal{X}' \to \mathcal{X}$. Let $E \subset \mathcal{X}$ be the divisor swept out by lines of π meeting at B', we denote its strict

transform by $E' \subset \mathcal{X}'$. This is a $\mathbb{P}^1$-bundle over C' and we denote this projection by $\eta : E' \to C'$. Consider the homomorphism

$$\mathrm{H}_3(\mathcal{X}, \mathbb{Z}) \xrightarrow{\phi^*} \mathrm{H}_3(\mathcal{X}', \mathbb{Z}) \xrightarrow{\cap E'} \mathrm{H}_1(E', \mathbb{Z}) \xrightarrow{\eta_*} \mathrm{H}_1(C', \mathbb{Z}),$$

where ϕ^* is the homology pullback defined using the Poincáre duality twice. It respects the Hodge structures, giving a homomorphism

$$\mathrm{H}^{2,1}(\mathcal{X})^\vee \to \mathrm{H}^{2,1}(\mathcal{X}')^\vee \to \mathrm{H}^{1,0}(E')^\vee \to \mathrm{H}^{1,0}(C')^\vee.$$

Fix homologous sections B_0, B_1 of π and denote the strict transforms of B_i via β by B_i'. There is a real 3-cycle α such that $\partial\alpha = B_1 - B_0$ and $\partial\phi^*\alpha = B_1' - B_0'$. The Abel-Jacobi map sends B_1 to

$$\int_\alpha \cdot \in \mathrm{H}^{2,1}(\mathcal{X})^\vee / \mathrm{H}_3(\mathcal{X}, \mathbb{Z}).$$

This is mapped by the above homomorphism to $\int_{\phi^*\alpha \cap E'} \cdot$ and $\int_{\eta_*(\phi^*\alpha \cap E')} \cdot$. The boundary of $\eta_*(\phi^*\alpha \cap E)$ is given by

$$\eta(B_1' \cap E') - \eta(B_0' \cap E').$$

This gives rise a morphism

$$\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, \gamma) \to \mathrm{IJ}(\mathcal{X}) \to \mathrm{Jac}(C) \cong \mathrm{Pic}^0(C).$$

This composition is compatible with the construction of [HT12, Proposition 2] described above. On the other hand, since the fibers are open subsets of projective spaces, the Abel-Jacobi map must factor as

$$\mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, \gamma) \to \mathrm{Pic}^0(C) \to \mathrm{IJ}(\mathcal{X}).$$

□

Automorphisms. Let $G \subseteq \mathrm{Aut}(\mathcal{X})$ be a finite group. Assume that π is equivariant for G. Easy examples of projectively nonlinearizable actions on $\mathcal{X} = \mathbb{P}^1 \times Q$, where Q is a smooth quadric surface, are:

- generically free action of $\mathfrak{S}_2^2$ on $\mathbb{P}^1$ and any action on Q – this is not linearizable since there are no fixed points on $\mathcal{X}$;
- $\mathfrak{S}_2$ acting on $\mathbb{P}^1$ and $\mathfrak{D}_6$ acting on Q – via the Burnside formalism of [KT22], see [CTZ23, Theorem 5.1].

The main example. Consider

$$\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

given by the vanishing of a sufficiently general form F of bi-degree $(n, 2)$. Projection to the first factor $\mathcal{X} \to \mathbb{P}^1$ is a quadric surface bundle of relative Picard rank 1, splitting over the discriminant double cover $C \to \mathbb{P}^1$ and ramified over the discriminant divisor $\mathfrak{d}$ of degree $|\mathfrak{d}| = 4n$; so that $g(C) = 2n - 1$.

Intermediate Jacobians and torsors. Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a smooth quadric surface bundle with a regular, compatible, action by a finite group G. This induces a G-action on $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1)$ so that

$$\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to \mathbb{P}^1,$$

and the Stein factorization

$$\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C \to \mathbb{P}^1$$

are G-equivariant. Let γ_ℓ be the class of lines in fibers of π.

Lemma 6.2. *Assume that $C \to \mathbb{P}^1$ is non-split. Then γ_ℓ is G-invariant and we have a G-equivariant isomorphism*

$$(\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma_\ell} \cong \mathrm{Pic}^1(C).$$

Moreover, we have a G-equivariant isomorphism

$$\mathrm{IJ}(\mathcal{X}) \cong (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^0 \cong \mathrm{Pic}^0(C),$$

as G-equivariant principally polarized abelian varieties.

Proof. Note that $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C$ is a $\mathbb{P}^1$-bundle over C. The Abel-Jacobi map

$$\mathrm{AJ} : \mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma_\ell},$$

factors as

$$\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C \to (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma_\ell}$$

and induces a G-equivariant isomorphism

$$\mathrm{Pic}^1(C) \cong (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma_\ell},$$

as in the proof of Lemma 6.1.

The claim concerning principal polarizations follows from the fact that, in the notation of the proof of Lemma 6.1, $\mathcal{X}'$ is the blowup of a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ along a curve C'. Indeed, one can use the blowup formula to verify compatibility of cup products. □

Linearization criterion.

Lemma 6.3. *Let G be a cyclic group and $\pi : \mathcal{X} \to \mathbb{P}^1$ a G-equivariant quadric surface bundle with a G-invariant section. Then the G-action on $\mathcal{X}$ is linearizable.*

Proof. Projecting from a G-invariant section gives an equivariant birationality of $\mathcal{X}$ to a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Since G is cyclic, the generic fiber $\mathbb{P}^2_\eta$ admits a G-invariant line. Such a $\mathbb{P}^2_\eta$ is linearizable, and we obtain a G-equivariant birationality $\mathcal{X} \sim_G \mathbb{P}^1 \times \mathbb{P}^2$, and thus linearizability. □

Theorem 6.4. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a smooth quadric surface bundle of relative Picard rank 1 and $|\mathfrak{d}| \geq 6$. Let $G \subseteq \mathrm{Aut}(\mathcal{X})$ be a cyclic group such that π is G-equivariant. Then the G-action on $\mathcal{X}$ is linearizable if and only if there exists a G-invariant section of π.*

Proof. If π admits a G-invariant section, Lemma 6.3 applies. Conversely, assume that the G-action on $\mathcal{X}$ is linearizable. Fix a class γ of sections. Since $\mathcal{X}$ has Picard rank 2 and G respects π, γ is a G-invariant class. By Theorem 3.3, we have

$$(\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma} \cong \mathrm{Pic}^e(C),$$

for some e. By Lemma 6.2, we have

$$[(\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma+k\gamma_\ell}] = [e+k][\mathrm{Pic}^1(C)].$$

In particular, whenever $e+k \equiv 0 \pmod{2g(C)-2}$, this torsor is trivial.

Let k be sufficiently large so that the height of $\gamma + k\gamma_\ell$ is large and $e + k \equiv 0 \pmod{2g(C)-2}$. By Lemma 6.1, the Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{Sec}(\mathcal{X}/\mathbb{P}^1, \gamma + k\gamma_\ell) \to (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma+k\gamma_\ell}$$

is a composition of an open immersion followed by a projective bundle

$$\mathcal{P} \to (\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma+k\gamma_\ell}.$$

Since $(\mathbf{CH}^2_{\mathcal{X}/\mathbb{C}})^{\gamma+k\gamma_\ell}$ has a fixed point, and G is cyclic, $\mathcal{P}$ has a fixed point as well. This implies that $\overline{\mathrm{Sec}}(\mathcal{X}/\mathbb{P}^1, \gamma + k\gamma_\ell)$ has a fixed point, proving that there is a G-invariant section of π. □

Effective results. Here we establish an effective version of Theorem 6.4. First we recall:

Proposition 6.5 ([HT12, Proposition 14])**.** *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a smooth quadric surface bundle with discriminant degree $|\mathfrak{d}|$ such that*

- *the discriminant cover $C \to \mathbb{P}^1$ is non-trivial, and*
- *for any section $s : \mathbb{P}^1 \to \mathcal{X}$, we have*

$$h_{\omega_\pi^{-1}}(s) \geq -\frac{|\mathfrak{d}|}{2}.$$

Then $\mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C$ is the projectivization of a semi-stable bundle V over C.

After rescaling, we may, and will, assume that $\deg(V) = 0$ or 1.

Proposition 6.6. *With the assumptions of Proposition 6.5, consider*

$$\mathbb{P}_C(V) = \mathcal{F}_1(\mathcal{X}/\mathbb{P}^1) \to C$$

and suppose that $|\mathfrak{d}| \geq 6$ *and*

$$h \geq \frac{3}{2}|\mathfrak{d}| - 4 + 4\deg(V).$$

Then the statement of Lemma 6.1 is valid.

Proof. We use notation from Lemma 6.1. By Proposition 6.5, V is semi-stable. By Riemann-Roch, there exists a section

$$s_0 : C \to \mathbb{P}_C(V)$$

with

$$d_0 = \deg(s^*\xi) \leq g(C) = \frac{|\mathfrak{d}|}{2} - 1,$$

where $\xi = \mathcal{O}_{\mathbb{P}_C(V)}(1)$. The numerical class of the canonical divisor of $\mathbb{P}_C(V)$ is given by

$$K_{\mathbb{P}_C(V)} = -2\xi + (2g(C) - 2 + \deg(V))\lambda,$$

where λ is the class of fibers of $\mathbb{P}_C(V) \to C$. Thus the degree of N_{s_0} is

$$2d_0 - \deg(V).$$

We then glue

$$b \geq 2g(C) - 2d_0 + \deg(V) = |\mathfrak{d}| - 2 - 2d_0 + \deg(V)$$

fibers of $\mathbb{P}_C(V) \to C$ to $s_0(C)$. By assumption,

$$d_0 = \frac{h_0}{2} - \frac{\deg(V)}{2} + \frac{|\mathfrak{d}|}{4} \geq -\frac{\deg(V)}{2}.$$

Using [GHS03, Lemma 2.6] and smoothing arguments, the existence of sections is guaranteed as soon as

$$d \geq |\mathfrak{d}| - 2 + \frac{3}{2}\deg(V),$$

which in turn follows when

$$h \geq \frac{3}{2}|\mathfrak{d}| - 4 + 4\deg(V).$$

Assuming $d \geq 2g(C)$, it follows from Proposition 6.5 and [LRT23, Corollary 2.7] that $V \otimes s^*\mathcal{O}_{\mathbb{P}_C(V)}(1)$ is globally generated and

$$\mathrm{H}^1(C, V \otimes s^*\mathcal{O}_{\mathbb{P}_C(V)}(1)) = 0,$$

where $s : C \to \mathbb{P}_C(V)$ is a section such that $\deg(s^*\xi) = d$. This implies that $(\rho_2)_*(\mathcal{L} \otimes \rho_1^*V)$ is locally free. The above condition translates to

$$h \geq 4g(C) + \deg(V) - \frac{|\mathfrak{d}|}{2} = \frac{3}{2}|\mathfrak{d}| + \deg(V) - 4.$$

□

Here is an effective version of Theorem 6.4:

Theorem 6.7. *Let G be a cyclic group and*

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

a G-equivariant quadric surface bundle of relative Picard rank 1 and $|\mathfrak{d}| \geq 6$. With the assumptions of Proposition 6.6, the G-action on $\mathcal{X}$ is linearizable if and only if there exists a G-invariant section of height

$$\leq \frac{7}{2}|\mathfrak{d}| + 4\deg(V) - 14.$$

Proof. In the proof of Theorem 6.4, the height of $\gamma + k\gamma_\ell$ needs to be $\geq \frac{3}{2}|\mathfrak{d}| - 4 + 4\deg(V)$ and we also must have $e + k \equiv 0 \mod (|\mathfrak{d}| - 4)$. This is possible assuming the height is $\geq \frac{7}{2}|\mathfrak{d}| + 4\deg(V) - 14$. □

Example 6.8. Let $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a smooth divisor of type $(n, 2)$, where $n \geq 2$; with $\pi : \mathcal{X} \to \mathbb{P}^1$ a quadric surface bundle of relative Picard rank 1 with square-free discriminant and $|\mathfrak{d}| = 4n \geq 8$. By [HT12, Section 4, Case 1], $\deg(V) \equiv n \pmod 2$.

Any section $s : \mathbb{P}^1 \to \mathcal{X}$ has height of the form $2 - n + 2a$, with $a = \deg(s^*H) \geq 0$, where H is the pullback of the hyperplane class from $\mathbb{P}^3$. Since

$$2 - n \geq -\frac{|\mathfrak{d}|}{2} = -2n,$$

the assumptions of Proposition 6.5 are verified. By Theorem 6.7, a cyclic action on $\pi : \mathcal{X} \to \mathbb{P}^1$ is projectively linearizable if and only if there exists a G-invariant section of height $\leq 14n + 4\deg(V) - 14$.

Examples of actions. We consider $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^3$, with $n \geq 2$, defined by the vanishing of a form F of bi-degree $(n, 2)$. Let

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

be the associated quadric surface bundle.

Example 6.9. Let

$$F := \sum_{i=0}^{n} t_0^i t_1^{n-i}(f_i(x, y) + g_i(z, w)),$$

where f_i, g_i are general binary quadratic forms, so that $\mathfrak{d}$ is reduced, and thus $\mathcal{X}$ smooth. The involution

$$\sigma : ([t_0 : t_1], [x : y : z : w]) \mapsto ([t_0 : t_1], [-x : -y : z : w])$$

acts trivially on $\mathbb{P}^1$, and any invariant section of π is pointwise fixed. However, the fixed locus of σ consists of two nonrational curves, thus there are no invariant sections. By Theorem 6.4, the $\langle\sigma\rangle$-action on $\mathcal{X}$ is not (projectively) linearizable.

Example 6.10. Let

$$F := t_0^2 f(x, y, z, w) + t_1^2 g(x, y, z, w),$$

where f, g are general quadratic forms in 4 variables. Consider

$$\sigma : ([t_0 : t_1], [x : y : z : w]) \mapsto ([-t_0 : t_1], [x : y : z : w]).$$

Note that σ is the covering involution of $\pi_2 : \mathcal{X} \to \mathbb{P}^3$. If we had a $\langle\sigma\rangle$-invariant section $R \subset \mathcal{X}$, then $\pi_2|_R : R \to \pi_2(R) = R'$ would be of degree 2. Since R is smooth, R' must be smooth as well, and the branch divisor of $R \to R'$ reduced, showing that R cannot be rational, contradiction. Thus, the $\langle\sigma\rangle$-action is not projectively linearizable.

Remark 6.11. Examples 6.9 and 6.10 produce moduli of conjugacy classes of involutions in $\mathrm{Cr}_3(\mathbb{C})$, by varying the underlying curve C in the moduli space and applying Corollary 3.5. In these examples, the non-uniruled divisorial component of the fixed locus of the involution σ is empty, and the previously known obstructions do not distinguish these equivariant birational classes.

7. Fano threefolds

Let X be a smooth projective Fano threefold of Picard rank 1 and $\mathcal{F}_d(X)$ the variety of rational curves of anticanonical degree d. Over nonclosed fields, rationality of geometrically rational varieties of this type has been recently settled in [HT22], [HT21a], [BW23], [HT21c], [KP23]. There are three types of such Fano varieties:

- rational over any field: the quintic del Pezzo threefold $V_5 \subset \mathbb{P}^4$,
- rational if and and only if there are rational points: forms of $\mathbb{P}^3$, the smooth quadric $Q \subset \mathbb{P}^4$, $X_{12} \subset \mathbb{P}^8$ and $X_{22} \subset \mathbb{P}^{13}$,
- rational if and only if there are rational points and rational curves, defined over the ground field, in prescribed degrees:
  - $X_{2,2} \subset \mathbb{P}^5$, intersection of two quadrics – lines,
  - $X_{16} \subset \mathbb{P}^{10}$ – twisted cubics,
  - $X_{18} \subset \mathbb{P}^{13}$ – conics.

In [KP23], the intermediate Jacobian torsor obstructions of [HT21b], [HT21a], and [BW23] are reinterpreted in the framework of derived categories.

The equivariant situation is markedly different. First of all, the quintic del Pezzo threefold V_5 is *not* projectively linearizable for the action of $\mathfrak{A}_5 \subset \mathrm{Aut}(V_5) = \mathsf{PGL}_2(\mathbb{C})$, by the main result of [CS16]. In fact, V_5 is birationally rigid for this action. Projective linearizability of quadrics is an open problem; linearizability is unknown even for some actions of $\mathfrak{S}_3$. In [BvBT23, Section 5] there is an example of a

nonlinearizable action by the Frobenius group $\mathfrak{F}_8$ on X_{12} without any obstructions in the derived category.

The following theorem is a partial equivariant analog:

Theorem 7.1. *Let X be a smooth projective Fano threefold of the following types:*

(1) *an intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$,*
(2) *a prime Fano threefold of genus 9, $X_{16} \subset \mathbb{P}^{10}$,*
(3) *a prime Fano threefold of genus 10, $X_{18} \subset \mathbb{P}^{11}$.*

Let G be a finite group acting generically freely and regularly on X. Assume that the G-action is projectively linearizable. Then

$$\mathcal{F}_d(X)^G \neq \emptyset, \tag{7.1}$$

for $d = 2, 3,$ or 2, respectively.

In Cases (1) and (2), this condition is also sufficient for projective linearizability.

We treat the cases separately.

Intersections of two quadrics. This case was settled in [HT22], via reduction to nonclosed fields. We supply an argument based on the equivariant intermediate Jacobian torsor formalism of Section 2. After the completion of this paper, F. Scavia sent us his unpublished note [Sca21] with a similar proof.

The variety of lines $\mathcal{F}_2(X_{2,2})$ is a torsor under $\mathrm{IJ}(X) = \mathrm{Pic}^0(C)$. When if $\mathcal{F}_2(X_{2,2})^G \neq \emptyset$, there exists a G-stable line, which yields a standard G-equivariant birationality with $\mathbb{P}^3$. To prove the converse, we need:

Lemma 7.2. *Let $X_{2,2}$ be a smooth complete intersection of two quadrics in $\mathbb{P}^5$ with a regular G-action on $X_{2,2}$. Then the Abel-Jacobi map defines a G-equivariant isomorphism*

$$\mathrm{AJ} : \mathcal{F}_2(X_{2,2}) \cong (\mathbf{CH}^2_{X_{2,2}/\mathbb{C}})^2.$$

Moreover, as a G-equivariant torsor, we have

$$2[\mathcal{F}_2(X_{2,2})] = [\mathrm{Pic}^1(C)],$$

where C is a smooth projective curve of genus 2 with a regular G-action.

Proof. The non-trivial part of the first statement is the G-equivariant property, established in Theorem 2.3. For the second statement, we need to show that we have a G-equivariant isomorphism

$$(\mathbf{CH}^2_{X_{2,2}/\mathbb{C}})^4 \cong \mathrm{Pic}^1(C),$$

for some C. We have $X_{2,2} \subset \mathbb{P}^5$, with a regular G-action on the ambient $\mathbb{P}^5$, so that $X_{2,2}$ is the G-invariant base locus of a pencil of quadrics. Let $\beta : Y \to \mathbb{P}^5$ be the G-equivariant blowup along $X_{2,2}$, it admits a G-equivariant quadric fibration $\pi : Y \to \mathbb{P}^1$. Let $\mathcal{W} \to \mathbb{P}^1$ be the relative Fano variety of planes of $Y/\mathbb{P}^1$, with Stein factorization

$$\mathcal{W} \to C \to \mathbb{P}^1,$$

here C is a smooth projective curve of genus 2 with a regular G-action. The proof of [LT19, Proposition 7.4] shows that $\mathcal{W} = \mathcal{F}_2(X_{2,2})$, the space of plane conics on $X_{2,2}$. We have the Abel-Jacobi map

$$\mathrm{AJ} : \mathcal{W} \to (\mathbf{CH}^2_{X_{2,2}/\mathbb{C}})^4.$$

The fibers of $\mathcal{W} \to C$ are orthogonal Grassmannians, and thus rational. This shows that the Abel-Jacobi map factors as

$$\mathcal{W} \to C \to (\mathbf{CH}^2_{X_{2,2}/\mathbb{C}})^4.$$

This induces a G-equivariant isomorphism

$$\mathrm{Pic}^1(C) \cong (\mathbf{CH}^2_{X_{2,2}/\mathbb{C}})^4.$$

□

Now, assume that the G-action on $X_{2,2}$ is projectively linearizable. By Lemma 7.2,

$$2[\mathcal{F}_2(X_{2,2})] = [\mathrm{Pic}^1(C)].$$

On the other hand, by Theorem 3.3, there exists a smooth projective G-curve C' such that

$$[\mathcal{F}_2(X_{2,2})] = [\mathrm{Pic}^e(C')],$$

for some e. By Torelli, C' is G-equivariantly isomorphic to C. Thus,

$$0 = 2e[\mathrm{Pic}^1(C)] = [\mathrm{Pic}^1(C)],$$

because C has genus 2. This implies that $\mathcal{F}_2(X_{2,2})$ is a trivial $\mathrm{IJ}(X_{2,2})$-torsor, as claimed.

Prime Fano threefolds of degree 16. Consider a G-Fano threefold $X_{16} \subset \mathbb{P}^{10}$ with projectively linearizable G-action. Then there exists a smooth projective curve C, of genus 3, with a regular G-action such that

$$[(\mathbf{CH}^2_{X_{16}/\mathbb{C}})^1] = [\mathrm{Pic}^e(C)],$$

for some e. The variety of conics $\mathcal{F}_2(X_{16})$ is a $\mathbb{P}^1$-bundle over C, see, e.g., [KP23, Theorem 9.2]. It follows that

$$2e[\mathrm{Pic}^1(C)] = [(\mathbf{CH}^2_{X_{16}/\mathbb{C}})^2] = [\mathrm{Pic}^1(C)].$$

On the other hand, since D has genus 3, we have

$$4[\mathrm{Pic}^1(C)] = 0.$$

Together, these imply that $[(\mathbf{CH}^2_{X_{16}/\mathbb{C}})^3] = 0$. The Abel-Jacobi map induces a G-equivariant isomorphism

$$\mathrm{AJ} : \mathcal{F}_3(X_{16}) \to (\mathbf{CH}^2_{X_{16}/\mathbb{C}})^3,$$

and the assertion of Theorem 7.1 follows.

Over nonclosed fields, we have the converse: the existence of a twisted cubic curve over the ground field implies birationality to (a form of) V_5, and these are rational, over any ground field.

In the equivariant context, we follow the arguments of [KP23]. Suppose that we have a G-stable twisted cubic R. Assume R does not admit a bisecant line. Then the Sarkisov link with center at R yields an equivariant birationality to V_5 (see [KP23, Theorem 5.10]). However, this does not immediately suffice to conclude projective linearizability. Recall that $\mathrm{Aut}(V_5) = \mathsf{PGL}_2$. We have already mentioned that V_5 is rigid for the action of $\mathfrak{A}_5 \subset \mathsf{PGL}_2$, by [CS16]. However, $\mathfrak{A}_5$ cannot act on X_{16}, since it does not act on the Jacobian of a genus 3 curve. Linearizations of cyclic and dihedral group actions on V_5 are given in Example 4.34, respectively, Example 4.37 of [ACC$^+$23]. The remaining case of a possible $\mathfrak{S}_4$-action on V_5 does not allow an equivariant birationality to X_{16}; indeed this would involve blowing up an $\mathfrak{S}_4$-invariant genus 3 and degree 9 curve $T \subset V_5$, which would intersect the $\mathfrak{S}_4$-invariant hyperplane section of V_5 in 9 points, which is impossible (we are grateful to I. Cheltsov for providing us with the proof and the reference). When R admits a bisecant line, the intersection of the linear span of R with X_{16} is the union of R and a bisecant line ℓ which is G-stable. Then equivariant birationality to $\mathbb{P}^3$ can be realized using the Sarkisov link with center at ℓ (see [KP23, Theorem 5.8]).

In other cases, according to [KP23, Lemma 2.9], either we have a G-stable line or there is a G-stable union of three lines. When there is a G-stable line, one may argue as above. Finally when we have a G-stable union of three lines, they meet at a single point x which is G-fixed. By [KP23, Lemma 5.11 and Lemma 5.16], the double projection from x yields equivariant birationality to a complete intersection of three quadrics $\overline{X}_{16}$ containing a G-stable plane Π. Projection from Π provides an equivariant birational map

$$\overline{X}_{16} \dashrightarrow \mathbb{P}^3,$$

see, e.g., [KP23, Corollary 5.15]. Thus X_{16} is projectively linearizable.

Prime Fano threefolds of degree 18. Let $X_{18} \subset \mathbb{P}^{11}$ be a projectively linearizable G-Fano threefold. Then there is a smooth projective genus 2 curve C with a regular G-action such that

$$[(\mathbf{CH}^2_{X_{18}/\mathbb{C}})^1] = [\mathrm{Pic}^e(C)],$$

for some e, and therefore

$$[(\mathbf{CH}^2_{X_{18}/\mathbb{C}})^2] = e[\mathrm{Pic}^2(C)] = 0.$$

The Abel-Jacobi map gives a G-equivariant isomorphism

$$\mathrm{AJ} : \mathcal{F}_2(X_{18}) \cong (\mathbf{CH}^2_{X_{18}/\mathbb{C}})^2,$$

and thus $\mathcal{F}_2(X_{18})^G \neq \emptyset$.

Again, over a nonclosed field K, we have a converse statement, which requires, in addition to a conic $R \subset X_{18}$ over K, a (sufficiently general) rational point $x \in X_{18}(K)$. These yield birationality of X_{18} to a sextic del Pezzo fibration $\varphi : \mathcal{X} \to \mathbb{P}^1$, together with a bisection, coming from R, and a trisection, from x, over K. Sextic del Pezzo surfaces are rational over a nonclosed field, provided they have points over degree 2 and degree 3 extensions; thus, $\mathcal{X}$ is rational over $K(\mathbb{P}^1)$, and X_{18} is rational over K. A similar construction in the equivariant context would require, in addition to a G-invariant conic, a G-fixed point. However, even a section of a sextic del Pezzo fibration φ does not guarantee linearizability, see [CTZ23, Section 5, Case (F)]!

It would be interesting to settle the (projective) linearizability of actions on X_{18}.

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