EQUIVARIANT GEOMETRY OF THE SEGRE CUBIC AND THE BURKHARDT QUARTIC

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Abstract. We study linearizability and stable linearizability of actions of finite groups on the Segre cubic and Burkhardt quartic, using techniques from group cohomology, birational rigidity, and the Burnside formalism.

1. Introduction

Let G be a finite group. We study generically free G-actions on rational Fano threefolds, over an algebraically closed field of characteristic zero, up to equivariant birationality. This is part of a long-standing program to identify finite subgroups of the Cremona group Cr_3 (see, e.g., [51] for background and references concerning this problem). One of the main tools in this area of research is the Equivariant Minimal Model Program (EMMP), and in particular the study of *birational rigidity* (BR). Among the principal achievements is the classification of finite simple groups that can act on rationally connected threefolds [47]. There is a wealth of results towards distinguishing conjugacy classes of embeddings of simple groups into the Cremona group, e.g., $\mathfrak{A}_5$ (see [19]). There are also many interesting problems: even the classification of involutions in Cr_3 is still open [49].

There are two particularly intriguing examples of rational threefolds with large automorphisms: the *Segre cubic* $X_3 \subset \mathbb{P}^4 \subset \mathbb{P}^5$, given by

$$\sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} x_i = 0, \tag{1.1}$$

with $\mathrm{Aut}(X_3) = \mathfrak{S}_6$, acting via permutation of variables, and the *Burkhardt quartic* X_4 that can be defined in $\mathbb{P}^4 \subset \mathbb{P}^5$ by the vanishing

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of elementary symmetric polynomials in 6 variables of degree 1 and 4

$$\sum_{1\leq i<j<k<l\leq 6} x_i x_j x_k x_l = \sum_{i=1}^{6} x_i = 0, \tag{1.2}$$

and as such carries the action of $\mathfrak{S}_6$. However, the full automorphism group of the Burkhardt quartic is $\mathsf{PSp}_4(\mathbb{F}_3)$, of order 25920. Another standard form of the Burkhardt quartic is

$$\big\{y_1(y_1^3+y_2^3+y_3^3+y_4^3+y_5^3)+3y_2y_3y_4y_5=0\big\}\subset\mathbb{P}^4, \tag{1.3}$$

which we will often use in this paper.

Our goal is to identify subgroups in $\mathfrak{S}_6$ and $\mathsf{PSp}_4(\mathbb{F}_3)$ whose actions on X_3 and X_4 are (projectively) *nonlinearizable*, i.e., not equivariantly birational to linear (or projectively linear) actions on $\mathbb{P}^3$. To do this, we explore the range of applicability of group cohomology, birational rigidity, and the Burnside group formalism [41].

Our main results are:

Theorem 1 (Theorem 6.3)**.** *Let $G\subseteq\mathfrak{S}_6=\mathrm{Aut}(X_3)$. The G-action on X_3 is linearizable if and only if one of the following conditions holds:*

- *G fixes a singular point on X_3,*
- *G is contained in the nonstandard $\mathfrak{S}_5'$, up to conjugation,*
- *$G=C_2^2$, X_3 contains three G-invariant planes, and $\mathrm{Sing}(X)$ splits as a union of five C_2^2-orbits of length 2.*

Moreover, when the G-action is not linearizable, it is not stably linearizable.

There are 55 conjugacy classes of nontrivial subgroups $G\subseteq\mathfrak{S}_6$, and 19 of these give rise to nonlinearizable actions on X_3.

Theorem 2. *Let $G\subseteq\mathsf{PSp}_4(\mathbb{F}_3)=\mathrm{Aut}(X_4)$. The G-action on X_4 is nonlinearizable if at least one of the following conditions holds:*

1. *$\mathrm{rkCl}(X_4)^G=1$,*
2. *G contains an involution that swaps two coordinates in $\mathbb{P}^5$,*
3. *G contains a subgroup G' such that $\mathrm{H}^1(G',\mathrm{Pic}(\widetilde{X}_4))\neq 0$,*

where $\widetilde{X}_4$ is the standard resolution of X_4.

This gives nonlinearizability for 103 out of 115 conjugacy classes of nontrivial subgroups of $\mathsf{PSp}_4(\mathbb{F}_3)$.

Theorem 3. *Among the remaining* 12 *conjugacy classes of subgroups* $G \subset \mathsf{PSp}_4(\mathbb{F}_3)$*, the* G*-action is linearizable if* G *is conjugate to one of the subgroups*

$$C_2, C_3, C_3', C_2^2, C_4, C_5, C_6, C_9,$$

explicitly described in Section 7.

See [21] for additional information concerning the actions of the groups in Theorems 1, 2, 3.

Thus, we settle completely the (stable) linearizability problem for the Segre cubic threefold X_3. After excluding known cases when either there is a G-fixed singular point in X_3, and the action is linearizable via projection from this point, or G is conjugated to a subgroup of the nonstandard $\mathfrak{S}_5$ in $\mathfrak{S}_6$, and such actions have been treated in [3, 34], the remaining analysis hinges on the existence of G-stable planes. A key observation in the remaining cases is that if X_3 does *not* contain a G-stable plane, then there is a cohomological obstruction to stable linearizability, and if it does contain such a plane, then X_3 is birational to a singular, toric, intersection of two quadrics in $\mathbb{P}^5$, which can be analyzed via toric geometry.

As an auxiliary tool, we settle the (stable) linearizability problem for translation-free actions with fixed points on algebraic tori in dimension 3, in Theorem 5.1.

For the Burkhardt quartic X_4, we settle the linearizability problem for all finite subgroups of $\mathsf{PSp}_4(\mathbb{F}_3)$ except for 4 (conjugacy classes of) subgroups isomorphic to

$$\mathfrak{S}_3,\ \mathfrak{D}_5,\ \mathfrak{D}_6,\ C_3 \rtimes C_4,$$

which are described explicitly in Section 7. We do not know whether or not the actions of these 4 groups are linearizable. For the subgroups $\mathfrak{S}_3$ and $\mathfrak{D}_6$ we present an equivariant birational map from X_4 to a smooth quadric threefold, see Section 7.

The proofs of Theorems 1, 2, and 3 imply the following corollary.

Corollary 4. *Let* G *be a group acting faithfully on* X_3 *and* X_4 *and such that both actions are not linearizable. Then there is no* G*-equivariant birational map* $X_3 \dashrightarrow X_4$*, with the possible exception when* $G \simeq C_2^2$*, conjugate to*

$$\big\langle (1\ 2)(3\ 4), (1\ 2)(5\ 6) \big\rangle \subset \mathfrak{S}_6 = \mathrm{Aut}(X_3),$$

and the corresponding subgroup in $\mathrm{Aut}(X_4) \subset \mathsf{PGL}_5$ *generated by*

$$\begin{pmatrix} -1 & 2 & 2 & 2 & 2 \\ 1 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

It would be interesting to clarify what happens in this exceptional case.

Here is the roadmap of the paper: in Section 2 we recall basic notions and constructions in equivariant birational geometry, and tabulate results of computations for the Segre cubic and the Burkhardt quartic. In Section 3 we recall the main tools from equivariant birational rigidity. Section 4 presents a simplified version of the Burnside formalism from [41], based on incompressible symbols, that allows to prove new cases of nonlinearizability. In Section 5 we adopt Kunyavski's rationality analysis of 3-dimensional tori over nonclosed fields [43] to the equivariant context. In Section 6 we turn to the Segre cubic threefold; we address the linearizability of the Burkhardt quartic in Section 7. In Section 8, we prove that the Burkhardt quartic X_4 is G-birationally rigid if $G \subseteq \mathsf{PSp}_4(\mathbb{F}_3)$ satisfies $\mathrm{rk}\,\mathrm{Cl}(X_4)^G = 1$.

Throughout, we consider group actions from the right, by `Magma` conventions; C_n denotes cyclic groups of order n; $\mathfrak{D}_n$ denotes dihedral groups of order $2n$; $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote alternating groups and symmetric groups of degree n, respectively.

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2. Background

Linearizability. Let G be a group and X a G-variety, i.e., an algebraic variety with a generically free action of G. We are interested in the following properties of G-actions:

- *Linearizability:* X is G-equivariantly birational to $\mathbb{P}(V)$, the projectivization of a linear representation V of G;
- *Stable linearizability:* $X \times \mathbb{P}^m$, with trivial G-action on the second factor, is linearizable.

One may also consider the related notions of (stable) *projective* linearizability, where the G-action on X is compared to the G-action on $\mathbb{P}(V)$, the projectivization of a representation V of a *central extension* of G. Our focus in this paper is on linearizability, since projectively linear actions on the Segre cubic and the Burkhardt quartic are linear.

Equivariant resolution of singularities (over fields of characteristic zero) allows to reduce to the case when X is smooth.

Obstructions. Let X be a smooth projective G-variety, over an algebraically closed field of characteristic zero. The geometric action induces an action on invariants of X, such as the Picard group $\mathrm{Pic}(X)$. Among necessary conditions for stable linearizability of the G-action on X is

(SP) $\mathrm{Pic}(X)$ is a stably permutation G-module.

This condition is not easy to verify, in practice. On the other hand, it implies the more tractable condition

(H1) $\mathrm{H}^1(G', \mathrm{Pic}(X)) = \mathrm{H}^1(G', \mathrm{Pic}(X)^\vee) = 0, \quad \forall G' \subseteq G.$

This can be checked in `Magma`, when the G-action on $\mathrm{Pic}(X)$ is known explicitly. Note that these conditions are equivalent for $G = C_2$.

It should be pointed out that not all failures of birationality are explained by invariants of the G-action on Picard groups, see Section 4 and Remark 7.3. For singular G-varieties it is also useful to consider the induced G-action on the class group $\mathrm{Cl}(X)$; this is particularly relevant to the study of G-birational rigidity, see Section 3 and 8.

Remark 2.1. If X is a singular G-variety, then $\mathrm{H}^1(G, \mathrm{Cl}(X))$ is not a G-birational invariant. For instance, let $X = X_3$ be the Segre cubic in $\mathbb{P}^5$ and $G = C_2$, acting on X via swapping of two coordinates. Then $\mathrm{Cl}(X) = \mathbb{Z}^6$ and $\mathrm{H}^1(G, \mathrm{Cl}(X)) = \mathbb{Z}/2$. But G fixes a singular point on X and is thus linearizable.

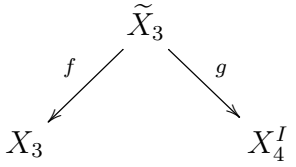
Rational surfaces. Actions of finite groups on Del Pezzo surfaces have been extensively studied in [28, 50]. The G-action on Picard groups of Del Pezzo surfaces of degrees $4, 3, 2, 1$ factors through subgroups of Weyl groups

$$W(\mathsf{D}_5), \quad W(\mathsf{E}_6), \quad W(\mathsf{E}_7), \quad W(\mathsf{E}_8),$$

respectively. Subgroups satisfying **(H1)** have been enumerated in [58]; the paper [50] contains examples of such subgroups of $W(\mathsf{D}_n)$, acting on Picard groups of conic bundles over $\mathbb{P}^1$.

Applications of the Burnside formalism from [41] to threefolds require a detailed understanding of birationality of G-actions on surfaces.

Segre cubic. Let X_3 be the Segre cubic in $\mathbb{P}^4$. We have the following $\mathfrak{S}_6$-equivariant diagram



where f is the blowup of the 10 singular points of the cubic, g is the anticanonical morphism, and X_4^I is the Igusa quartic threefold in $\mathbb{P}^4$. We say that f is the standard resolution of singularities of X_3. Recall that $\widetilde{X}_3$ is $\mathfrak{S}_6$-equivariantly isomorphic to $\bar{\mathcal{M}}_{0,6}$, the moduli space of 6 points on $\mathbb{P}^1$, which has a natural $\mathfrak{S}_6$-action permuting the 6 points.

The group $\mathfrak{S}_6$ has an outer automorphism, and thus two conjugacy classes of subgroups $\mathfrak{S}_5$; one of them we call *standard*, it acts trivially on one of the indices, and the other *nonstandard*; we shall denote it by $\mathfrak{S}_5'$. By [3, Proposition 4.1], the action of the standard $\mathfrak{S}_5$ on the Segre cubic, via permutation of 5 variables in (1.1), is birationally rigid; the action of the nonstandard $\mathfrak{S}_5'$ is linearizable.

Recall that $\mathrm{Pic}(\widetilde{X}_3)^{\mathfrak{S}_6}$ is generated by 2 classes, corresponding to the birational contractions to the Segre cubic and the Igusa quartic. The following table provides additional information about ranks of the invariant Picard group and class group, as one changes the action:

Group	$\mathfrak{S}_6$	$\mathfrak{A}_6$	$\mathfrak{S}_5$	$\mathfrak{S}_5'$	$\mathfrak{A}_5$	$\mathfrak{A}_5'$	$\mathfrak{S}_3 \wr C_2$	$C_2 \times \mathfrak{S}_4$	$C_2 \times \mathfrak{S}_4''$
$\mathrm{rk}\,\mathrm{Pic}(\widetilde{X}_3)^G$	2	2	2	3	2	3	3	3	4
$\mathrm{rk}\,\mathrm{Cl}(X_3)^G$	1	1	1	2	1	2	1	1	2

Burkhardt quartic. We also record ranks of invariants in the Picard group of $\widetilde{X}_4$, the standard resolution of singularities of the Burkhardt quartic X_4 obtained by blowing up all its singular points, for various subgroups $G \subseteq \mathsf{PSp}_4(\mathbb{F}_3)$:

Group	$\mathsf{PSp}_4(\mathbb{F}_3)$	$C_2^4.\mathfrak{A}_5$	$\mathfrak{S}_6$	$C_2.\mathfrak{A}_4 \wr C_2$	$\mathrm{SL}_2(\mathbb{F}_3) : \mathfrak{A}_4$	$\mathrm{SL}_2(\mathbb{F}_3)$
$\mathrm{rk}\,\mathrm{Pic}(\widetilde{X}_4)^G$	2	3	3	4	5	7,9,11
$\mathrm{rk}\,\mathrm{Cl}(X_4)^G$	1	1	1	1	2	3,3,5

where the last entry reflects the different conjugacy classes. The full table, obtained with `Magma`, is available at [21]. Furthermore, we have:

Proposition 2.2. *Let X_4 be the Burkhardt quartic and $G \subseteq \mathsf{PSp}_4(\mathbb{F}_3)$ a subgroup of its automorphism group. Then* $\operatorname{rk} \mathrm{Cl}(X_4)^G = 1$ *if and only if G contains a subgroup conjugate to one of the following subgroups*

$$\mathfrak{F}_5,\ C_2^4,\ C_2^2 \rtimes C_4,\ C_2 \times \mathfrak{D}_4,\ \mathfrak{S}_4,\ \mathfrak{S}_4',\ C_3 \rtimes \mathfrak{D}_4,\ \mathfrak{S}_3^2,\ C_3^2 \rtimes C_4,\ \mathfrak{A}_5,$$

explicitly specified in [21].

This extends [15, Corollary 2.10] and [13, Corollary 5.4], which listed the corresponding subgroups of $\mathfrak{S}_6$.

3. Birational rigidity

Let X be a Fano threefold with at most terminal singularities and $G \subseteq \mathrm{Aut}(X)$ a finite subgroup. Suppose that

$$\operatorname{rk} \mathrm{Pic}^G(X) = 1.$$

If X is smooth, then X is a G-Mori fiber space (over a point), and X lies in 25 deformation families described in [48, Theorem 1.2]. If X is singular, then it may fail to be $G\mathbb{Q}$-factorial, i.e., we may have

$$\operatorname{rk} \mathrm{Cl}^G(X) > \operatorname{rk} \mathrm{Pic}^G(X),$$

so that X is not necessarily a G-Mori fiber space. However, we can always take a $G\mathbb{Q}$-factorialization of X, and then apply EMMP to obtain a G-equivariant birational map from X to some G-Mori fiber space.

On the other hand, if $\operatorname{rk} \mathrm{Cl}^G(X) = 1$, then X is a G-Mori fiber space. In this case, one can try to describe all G-birational maps from X to other G-Mori fiber spaces. Every such map can be decomposed into a sequence of G-Sarkisov links [23, 31], which have a more restricted structure. If there are no G-Sarkisov links that start at X, then X is the only G-Mori fiber space that is G-birational to X and

$$\mathrm{Bir}^G(X) = \mathrm{Aut}^G(X),$$

i.e., X is G-birationally super-rigid. We say that X is G-birationally rigid if every G-Sarkisov link that starts at X also ends at X, which means that X is not G-birational to other G-Mori fiber spaces, but X may admit non-biregular G-birational selfmaps.

Remark 3.1. If $X \not\simeq \mathbb{P}^3$, $\operatorname{rk} \mathrm{Cl}^G(X) = 1$, and X is G-birationally rigid, then the G-action on X is not (projectively) linearizable.

If X is not G-birational to any G-Mori fiber space with a positive dimensional base (a conic bundle or a Del Pezzo fibration), we say that X is G-solid. G-birationally rigid and G-solid Fano threefolds are studied in [14, 15, 18, 17, 19, 20, 12, 16, 11], with a special focus on rational threefolds. These studies are based on the following technical result, which is the engine of the G-equivariant Sarkisov program:

Theorem 3.2 ([19, Theorem 3.3.1])**.** *Suppose that* $\operatorname{rk}\operatorname{Cl}^G(X) = 1$*, and let* $\chi\colon X \dashrightarrow V$ *be a* G*-birational non-biregular map such that*

- V *has terminal singularities,*
- $\operatorname{rk}\operatorname{Cl}^G(V) = \operatorname{rk}\operatorname{Pic}^G(V)$*, and*
- *there exists a* G*-equivariant Mori fiber space* $\pi\colon V \to Z$*.*

Set

$$\mathcal{M} = \chi_*^{-1}\big(|-pK_V + \pi^*(H)|\big),$$

for $p \gg 0$*, and a sufficiently general very ample divisor* $H \in \operatorname{Pic}(Z)$ *such that* $[\pi^*(H)]$ *is* G*-invariant. Then* $\mathcal{M}$ *is a* G*-invariant non-empty mobile linear system, and the singularities of the log pair* $(X, \lambda\mathcal{M})$ *are not canonical for* $\lambda \in \mathbb{Q}_{>0}$ *such that* $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_X$*.*

This is a G-equivariant version of the classical Noether–Fano inequality. EMMP and Theorem 3.2 give a simple criterion for G-birational super-rigidity:

Corollary 3.3. *Suppose that* $\operatorname{rk}\operatorname{Cl}^G(X) = 1$*. Then* X *is* G*-birationally super-rigid if and only if for every* G*-invariant non-empty mobile linear system* $\mathcal{M}$ *on* X*, the log pair* $(X, \lambda\mathcal{M})$ *has canonical singularities for* $\lambda \in \mathbb{Q}_{>0}$ *such that* $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_X$*.*

There is a similar (albeit more technical) criterion for G-birational rigidity, see [19, Chapter 3]. If X is toric, and G contains the maximal torus in $\operatorname{Aut}(X)$, a criterion for G-solidity is given in [12].

Usually, Corollary 3.3 is applicable when $(-K_X)^3$ is "sufficiently small" or when the group G is "sufficiently large". For instance, for $(-K_X)^3 = 2$, arguing as in the proof of [14, Theorem A], we obtain:

Theorem 3.4. *Let* $X \subset \mathbb{P}(1,1,1,1,3)$ *be a hypersurface of degree* 6 *with at most isolated ordinary double points (nodes) and* $G \subseteq \operatorname{Aut}(X)$ *a finite subgroup such that* $\operatorname{rk}\operatorname{Cl}(X)^G = 1$*. Then* X *is* G*-birationally super-rigid.*

This is also expected for nodal quartics, where $(-K_X)^3 = 4$, see [44]:

Conjecture 3.5 ([13, Conjecture 5.2])**.** Let $X \subset \mathbb{P}^4$ be a nodal quartic threefold and $G \subseteq \mathrm{Aut}(X)$ a finite subgroup such that $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$. Then X is G-birationally rigid.

In Section 7, we prove this conjecture for the Burkhardt quartic. Unfortunately, we do not have such precise (conjectural) characterizations of G-birational rigidity for most of the other (singular) Fano threefolds, apart from sporadic results in this direction. For instance, for the Segre cubic X_3, where $(-K_{X_3})^3 = 24$, Avilov found all possibilities for $G \subset \mathrm{Aut}(X_3) \simeq \mathfrak{S}_6$ such that X_3 is G-birationally rigid:

Theorem 3.6 ([4])**.** *Let $X_3 \subset \mathbb{P}^4$ be the Segre cubic and $G \subseteq \mathrm{Aut}(X_3)$ a subgroup such that $\mathrm{rk}\,\mathrm{Cl}(X_3)^G = 1$. Then the following are equivalent:*

(1) *X_3 is G-birationally rigid,*
(2) *X_3 is G-birationally super-rigid,*
(3) *G contains a group isomorphic to $\mathfrak{A}_5$ that leaves invariant a hyperplane section of X_3.*

Returning to general threefolds, if $\mathrm{rk}\,\mathrm{Cl}(X)^G = 2$, then X admits exactly two $G\mathbb{Q}$-factorializations, and we have the following G-equivariant commutative diagram:

(3.1)
$$\begin{array}{ccccccc} & & V & \overset{\varsigma}{\dashrightarrow} & V' & & \\ & \overset{\varphi}{\swarrow} & & \overset{\varpi}{\searrow}\quad \overset{\varpi'}{\swarrow} & & \overset{\varphi'}{\searrow} & \\ Z & & & X & & & Z' \end{array}$$

where ϖ and ϖ' are G-equivariant small resolutions such that

$$\mathrm{rk}\,\mathrm{Pic}(V)^G = \mathrm{rk}\,\mathrm{Cl}(V)^G = 2 = \mathrm{rk}\,\mathrm{Cl}(V')^G = \mathrm{rk}\,\mathrm{Pic}(V')^G,$$

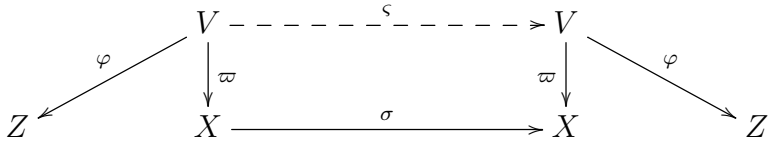
the map ς is a pseudo-automorphism that flops ϖ-contracted curves, both φ and φ' are G-equivariant extremal contractions that can be of the following three types:

- a birational contraction,
- a fibration into Del Pezzo surfaces over $\mathbb{P}^1$,
- a conic bundle over a rational surface.

The diagram (3.1) is an example of a G-Sarkisov link (with X being its *center*). When both V and V' are smooth, such links have been studied in [55, 37, 38, 5, 26, 1, 25, 56]. Note that (3.1) is uniquely determined up to swapping its left and right sides.

If the morphism φ in (3.1) is birational, then Z is a Fano variety with at most terminal singularities such that $\mathrm{rk}\,\mathrm{Cl}(Z)^G = 1$, so we are back to the case when $\mathrm{rk}\,\mathrm{Cl}^G(X) = 1$ with X replaced by Z. Further, if

the normalizer of G in $\mathrm{Aut}(X)$ contains an automorphism σ such that $\mathrm{rk}\,\mathrm{Cl}^{\langle\sigma,G\rangle}(X)=1$, the diagram (3.1) simplifies as

(3.2)


In this case, we say that the G-Sarkisov link is symmetric. For instance, if $X=X_3$ is the Segre cubic, this holds in many (but not all) cases.

4. Burnside formalism

Here we explain a simplified version of the Burnside group formalism introduced in [41], which yields equivariant birational invariants of G-actions on algebraic varieties. We continue to work over an algebraically closed field k of characteristic zero.

Applying equivariant blowups we may assume that the G-action is realized as a regular action on a *standard model* (X,D) of the function field $K=k(X)$, i.e.,

- X is smooth projective, D a normal crossings divisor,
- G acts freely on $U:=X\setminus D$,
- for every $g\in G$ and every irreducible component D, either $g(D)=D$ or $g(D)\cap D=\emptyset$,

see [32, Section 7.2] for details. Given such a model, let

$$\{D_\alpha\}_{\alpha\in\mathcal{A}}$$

be the set of irreducible divisors with nontrivial (thus necessarily cyclic) stabilizers $H_\alpha\subseteq G$; we consider these up to conjugation in G. Each such D_α inherits a residual action of a group $Y_\alpha\subseteq Z_G(H_\alpha)/H_\alpha$. Consider the subset $\mathcal{A}^{\mathrm{inc}}\subseteq\mathcal{A}$ corresponding to those divisors, together with the respective Y_α-action, that cannot be obtained via equivariant blowups of *any* standard model of *any* G-variety.

We have an assignment

$$X\circlearrowleft G \quad\Rightarrow [X\circlearrowleft G]^{\mathrm{inc}}:=\sum_{\alpha/\mathrm{conj}}(H_\alpha, Y_\alpha\circlearrowright k(D_\alpha),(b_\alpha)), \tag{4.1}$$

where the sum is over (G-conjugacy classes of) nontrivial cyclic H_α, of symbols encoding

- the stabilizer H_α of the generic point of D_α,
- the residual action of $Y_\alpha\subseteq Z_G(H_\alpha)/H_\alpha$ on D_α,
- the character b_α of H_α in the normal bundle to D_α

Note that G-conjugation extends to symbols in (4.1), conjugating the Y_α-action as well as the character b_α, see [42, Section 2] for more details.

Proposition 4.1. [42, Proposition 3.4] *The class* $[X \circlearrowleft G]^{\mathrm{inc}}$*, taking values in the free abelian group generated by symbols*

$$(H, Y \circlearrowright k(D), (b)), \quad H \neq 1, \tag{4.2}$$

up to G-conjugation as above, is a well-defined G-birational invariant.

This is a rough invariant, obliterating information from nontrivial stabilizers in higher codimensions; but it already allowed to distinguish actions not accessible with previous methods [42]. Given this, it becomes essential to provide a geometric characterization of incompressible divisorial classes. As explained in [59, Section 3.6], this property *a priori* depends on the ambient group G. However, for some Y-actions on D, there is no such dependence, and we will call such symbols *absolutely incompressible*.

For instance, by [42, Proposition 3.6], in dimension 2, a divisorial symbol (4.2) is absolutely incompressible iff:

- D is a curve of genus ≥ 1, or
- D is a curve of genus 0, and the residual Y action on D is not cyclic.

In dimension 3, sufficient conditions for incompressibility include:

- D is not uniruled,
- D is Y-birational to a Y-solid Pezzo surface,
- the Y-action on D has cohomology: $\mathrm{H}^1(Y, \mathrm{Pic}(D)) \neq 0$,
- the Y-action on D is not equivariantly birational to a Y-action on a $\mathbb{P}^1$-bundle over a curve.

If one is interested in comparing a G-action on a rational threefold to a linear action on $\mathbb{P}^3$, one can exclude symbols (4.2) where D admits a surjection onto a curve of genus ≥ 1, as such symbols are not produced by the algorithm from [42] which computes the class of a linear action in the full Burnside group of [41], see [59, Corollary 6.1].

Thus, for applications to linearizability in dimension 3, we need a classification of incompressible divisorial symbols of the form

$$(H, Y \circlearrowright k(\mathbb{P}^2), (b)). \tag{4.3}$$

This can be obtained by combining classification schemes for G-surfaces, with the Burnside formalism of [41]. There are two complementary approaches: via EMMP, as carried out in [28, Section 8], and using cohomology, as in [50]. The first approach allows to completely settle

the linearization problem for rational surfaces [54]. But in practice, the second approach is simpler to apply [6, 40]. For instance, if X is a minimal G-Del Pezzo surface, then the following are equivalent, by [50, Theorem 1.2]:

- vanishing cohomology:

$$\mathrm{H}^1(G', \mathrm{Pic}(X)) = 0, \quad \text{for all } G' \subseteq G. \tag{4.4}$$

- no element of G fixes a curve of genus ≥ 1,
- either the degree of X is at least 5 or $G = C_3 \rtimes C_4$ and X is G-birational to a specific nonlinearizable G-Del Pezzo surface of degree 4.

While not strictly necessary for the analysis of incompressible symbols, there is also a complete description of conic bundles satisfying (4.4), see [50, Theorems 8.3 and 8.6].

Proposition 4.2. *Let $Y \subset \mathsf{PGL}_3(k)$ be a finite nonabelian group, acting linearly on $D = \mathbb{P}^2$. This action gives rise to an absolutely incompressible divisorial symbol in dimension 3, of the form* (4.3)*, if and only if the action is transitive.*

Proof. By [53], if Y acts transitively, then D is Y-birationally rigid except for $Y = \mathfrak{A}_4$ or $\mathfrak{S}_4$. In particular, such actions are not birational to actions on Hirzebruch surfaces. If $Y = \mathfrak{A}_4$, it follows from [**?**] or the proof of [39, Proposition 43] that D is not Y-birational to a Hirzebruch surface. Alternatively, one can notice that the Klein four subgroup of $\mathfrak{A}_4$ fixes a point in D, while every faithful action of $\mathfrak{A}_4$ on a Hirzebruch surface does not enjoy this property. Same holds for $Y = \mathfrak{S}_4$. Hence, if Y acts transitively on D, then the symbol is absolutely incompressible.

Conversely, if Y fixes a point on $\mathbb{P}^2$, then a Y-equivariant blowup exhibits a Hirzebruch surface, and the symbol is compressible. □

Proposition 4.3 ([46])**.** *Let D be a Del Pezzo surface of degree 6 and $Y \subset \mathrm{Aut}(D)$ a finite subgroup acting transitively on (-1)-curves. If $Y \not\simeq C_6$ and $Y \not\simeq \mathfrak{S}_3$ then D is Y-solid, and the Y-action is not (projectively) linearizable.*

In particular, the corresponding divisorial symbols in dimension 3, of the form (4.3) are absolutely incompressible.

5. Linearizability of G-actions on tori

Recall the structure of automorpisms $\mathrm{Aut}(T)$ of an algebraic torus $T = \mathbb{G}_m^n$, over a field k: there is an exact sequence of groups

$$1 \to T(k) \to \mathrm{Aut}(T) \xrightarrow{\phi} \mathsf{GL}_n(\mathbb{Z}) \to 1,$$

and the homomorphism ϕ admits a section. In particular, the torus T admits automorphisms arising from finite subgroups $\Gamma \subset \mathsf{GL}_n(\mathbb{Z})$.

Let X be a smooth projective T-equivariant compactification of T. Its Picard group has a presentation

$$0 \to \mathfrak{X}^*(T) \to \mathrm{PL} \to \mathrm{Pic}(X) \to 0, \tag{5.1}$$

where PL is the free abelian group spanned by irreducible components of the boundary $X \setminus T$, and $\mathfrak{X}^*(T)$ is the character group of T. In presence of G-actions, the sequence (5.1) is a sequence of Γ-modules, where $\Gamma := \phi(G) \subset \mathsf{GL}_n(\mathbb{Z})$; here PL is a permutation module.

Lists of finite groups Γ, for small n, giving rise to actions on $\mathrm{Pic}(X)$ which satisfy **(SP)**, and thus **(H1)**, can be found in [35]. Linearizability properties of actions of finite subgroups of $\mathrm{Aut}(T)$ have been studied via birational rigidity techniques in [12, 16], where many examples of G-birationally rigid toric Fano threefolds were produced, and the groups G considered typically had a large intersection with $T(k)$. The *stable* linearization problem of toric varieties, with G-actions satisfying $G \cap T(k) = \emptyset$, was settled in [34, Proposition 12].

Linearization of actions on 2-dimensional tori is understood [54]. Let us recall the analysis in dimension 3, following [43]:

Step 1. There are 4 maximal finite subgroups $\Gamma \subset \mathsf{GL}_3(\mathbb{Z})$, and in each case we can fix an explicit (possibly singular) projective toric Fano threefold X on which the Γ-action is regular:

- (F) with $\Gamma_F := C_2 \times \mathfrak{D}_6$, acting on $X = \mathbb{P}^1 \times S$, where S is a degree 6 Del Pezzo surface;
- (C) with $\Gamma_C := C_2 \times \mathfrak{S}_4$, acting on $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- (S) with $\Gamma_S := C_2 \times \mathfrak{S}_4$, acting on $X = X_{2,2} \subset \mathbb{P}^5$, a singular intersection of two quadrics;
- (P) with $\Gamma_P := C_2 \times \mathfrak{S}_4$; acting on the singular divisor

$$X = \{x_0y_0z_0t_0 = x_1y_1z_1t_1\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

If X is singular, we let $\widetilde{X} \to X$ be the equivariant blowup of its singular points. If X is smooth, we let $\widetilde{X} = X$.

Step 2. By [43, Proposition 1], in absence of an obstruction of type **(SP)** for the Γ-module $\mathrm{Pic}(X)$, one of the following holds:

(a) the Γ-module $\mathfrak{X}^*(T)$ splits and the action is birational to a product action,
(b) $\mathfrak{X}^*(T)^\Gamma \neq 0$,
(c) the action is via a subgroup of Γ_C,
(d) the action is via C_4, $\mathfrak{S}_3$, or is linear, via a subgroup of $\mathfrak{S}_4$.

Step 3. Fix the following subgroups in $\mathsf{GL}_3(\mathbb{Z})$:

$$U_1 := \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle,$$

$$W_1 := \left\langle \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

$$W_2 := \left\langle \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

Then $U_1 \simeq C_2 \times C_2$, $W_1 \simeq C_2 \times C_4$, $W_2 \simeq C_2^3$. It follows from [43] that

$$\mathrm{H}^1(U_1, \mathrm{Pic}(\widetilde{X})) \neq 0.$$

Hence, if Γ contains a subgroup conjugate to U_1, then Γ and G do not satisfy **(H1)**. There are exactly 12 conjugacy classes of such subgroups $\Gamma \subset \mathsf{GL}_3(\mathbb{Z})$.

Furthermore, if Γ contains a subgroup conjugate to W_1 or W_2, then it also follows from [43] that the Γ and G-action on $\mathrm{Pic}(\widetilde{X})$ do not satisfy **(SP)** (but W_1 and W_2 do satisfy **(H1)**).

Here we adapt this to the study of linearizability of the actions of these groups on tori, via a case by case study as in *Step 1.* We identify subgroups of $\mathsf{GL}_3(\mathbb{Z})$ with subgroups of $\mathrm{Aut}(X)$ via the standard lifts to $\mathrm{Aut}(T)$ with fixed point $(1,1,1) \in T$.

Case (F): Assume that $\Gamma \subseteq \mathfrak{D}_6$, with trivial action on $\mathbb{P}^1$. The action of $\mathfrak{D}_6$ on S is not linearizable by Proposition 4.3 or [36]. However, by [7, Proposition 12], $\mathbb{P}^1 \times S$ is *linearizable* with the trivial action on $\mathbb{P}^1$. The action of every proper subgroup of $\mathfrak{D}_6$ on S is linearizable.

Conversely, if $\Gamma = \Gamma_F$, then the action on the toric threefold contributes an absolutely incompressible, by Proposition 4.3, symbol

$$(C_2, \mathfrak{D}_6 \circlearrowright k(S), (1)),$$

from the origin in the torus. On the other hand, such symbols do not arise from projectively linear actions, as S is not $\mathfrak{D}_6$-linearizable, nor birational to a product of (projectively) linear actions [60, Example 9.2].

Case (C): The action of Γ_C on $\mathbb{G}_m^3$ is generated by

$$(x_1, x_2, x_3) \mapsto (x_3, x_2, \frac{1}{x_1}), \quad (\frac{1}{x_1}, \frac{1}{x_3}, \frac{1}{x_2}), \quad (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}).$$

A birational change of coordinates $y_i := \frac{1-x_i}{1+x_i}$ yields the action

$$(y_1, y_2, y_3) \mapsto (y_3, y_2, -y_1), \quad (-y_1, -y_3, -y_2), \quad (-y_1, -y_2, -y_3),$$

which is clearly linearizable.

Case (S): By results in Section 6 (Proposition 6.1 and Theorem 6.3), it suffices to establish the linearizability of the action of $\Gamma = C_2^2$ on $\mathbb{G}_m^3$, with coordinates x_1, x_2, x_3, via

$$\sigma : x_1 \leftrightarrow x_3,\ x_2 \mapsto 1/x_1x_2x_3, \quad \tau : x_j \mapsto 1/x_j, \quad j = 1, 2, 3.$$

This action is conjugate to a subgroup of Γ_C in Case (C) and is linearizable.

Case (P): For $\Gamma \subseteq \Gamma_P$, only five groups G do not appear in Case (F), (C) or (S), up to conjugation:

$$\Gamma = \mathfrak{A}_4, C_2 \times \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{S}_4',\ \text{or}\ \Gamma_P.$$

Each of them contains a subgroup conjugate to U_1 and thus is not linearizable. In summary, a subgroup of Γ_P is not linearizable if and only if it contains one of U_1, W_1, or W_2.

We summarize the above discussion:

Theorem 5.1. *Let $T = \mathbb{G}_m^3$ and $G \subset \mathrm{Aut}(T)$ be such that $\Gamma := \phi(G)$ contains U_1, W_1, or W_2. Then the G-action on T is not stably linearizable. Assume that $T^G \neq \emptyset$, i.e., G fixes a point in T. Then*

- *if $\Gamma = C_2 \times \mathfrak{D}_6$, then the action is not linearizable but stably linearizable,*

- *if* $\Gamma \neq C_2 \times \mathfrak{D}_6$ *and does not contain* U_1, W_1, *or* W_2, *then the action is linearizable.*

Proof. If $\phi(G) = U_1$, then G has an (**H1**)-obstruction to stable linearizability. If $\phi(G) = W_1$ or W_2, then G has an (**SP**)-obstruction to stable linearizability. When $T^G \neq \emptyset$, we can assume $(1,1,1) \in T^G$ up to translation. Then we are in one of the cases (F), (C), (S) or (P) discussed above.

Stable linearizability of the $C_2 \times \mathfrak{D}_6$-action is established as in [34], using the equivariant version of the torsor formalism. □

Remark 5.2. The second part of Theorem 5.1 does not hold without the assumption $T^G \neq \emptyset$. For example, consider $\Gamma = C_2^2 \subset \mathsf{GL}_3(\mathbb{Z})$ from Case (S) above. Up to conjugation, we find two translation-free lifts $G \subseteq \mathrm{Aut}(T)$ of Γ, i.e., $\phi(G) = \Gamma$: the standard lift generated by

$$\sigma : y_1 \mapsto \frac{1}{y_1},\ y_2 \leftrightarrow y_3, \quad \tau : y_j \mapsto \frac{1}{y_j}, \quad j = 1,2,3,$$

and a twist of it generated by

$$\sigma' : y_1 \mapsto -\frac{1}{y_1},\ y_2 \leftrightarrow y_3, \quad \tau : y_j \mapsto \frac{1}{y_j}, \quad j = 1,2,3.$$

The standard lift is linearizable. The twisted one is not linearizable as the G-action on the projective model $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ does not have a fixed point. In particular, these two lifts are not equivariantly birational.

6. Geometry of the Segre cubic

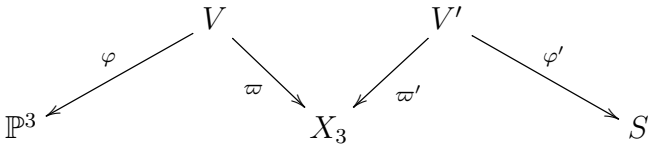
Rationality of forms of the Segre cubic threefold over nonclosed fields has been considered in [30]; there exist nonrational forms over nonclosed fields. All forms over the reals are rational [4, Corollary 2.5].

There are 55 nontrivial conjugacy classes of subgroups of $\mathfrak{S}_6$. By [3, 18], everything is known in the *minimal case*, when

$$G = \mathfrak{A}_5, \quad \mathfrak{S}_5, \quad \mathfrak{A}_6, \quad \mathfrak{S}_6.$$

Namely, there are two $\mathfrak{A}_5$ and $\mathfrak{S}_5$ classes, corresponding to the *standard*, respectively, *nonstandard* embedding of these groups into $\mathfrak{S}_6$. If G is a standard subgroup $\mathfrak{A}_5$, then $\mathrm{rk}\,\mathrm{Cl}(X_3)^G = 1$ and X_3 is G-birationally super-rigid [3]. This also implies that X_3 is G-birationally super-rigid if G is a standard $\mathfrak{S}_5$, $\mathfrak{A}_6$ or the whole group $\mathfrak{S}_6$, and the actions of these groups are not linearizable. Vice versa, if G is a nonstandard subgroup

$\mathfrak{S}_5$, then $\operatorname{rk}\operatorname{Cl}(X_3)^G = 2$, and we have the following nonsymmetric G-Sarkisov link:

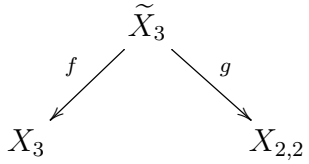


where φ is a blow up of a G-orbit of length 5, both ϖ and ϖ' are flopping contractions, S is a smooth Del Pezzo surface of degree 5, and φ' is a $\mathbb{P}^1$-bundle. In particular, the actions of all subgroups of the nonstandard subgroup $\mathfrak{S}_5$ are linearizable, e.g., the nonstandard subgroup $\mathfrak{A}_5$, the unique subgroup $C_5, \mathfrak{D}_5, \mathfrak{F}_5$.

Next we exclude G-actions with G-fixed singular points, since then X_3 is G-birational to $\mathbb{P}^3$. There are 25 such conjugacy classes; all such G are contained in the unique class of $\mathfrak{S}_3^2 \rtimes C_2$.

Therefore, all subgroups of the nonstandard $\mathfrak{S}_5'$ or $\mathfrak{S}_3^2 \rtimes C_2$ yield linearizable actions. The remaining groups are contained in one of two nonconjugate $C_2 \times \mathfrak{S}_4$; one of them preserve a plane $\Pi \subset X_3$ (this is a *nonstandard* subgroup of $\mathfrak{S}_6$), and another one does not preserve any plane in X_3 (this is a *standard* subgroup, conjugate to the group generated by the involution (12) and permutations of the remaining indices).

Proposition 6.1. *Suppose that X_3 contains a G-invariant plane Π. Then there exists the following G-equivariant diagram:*



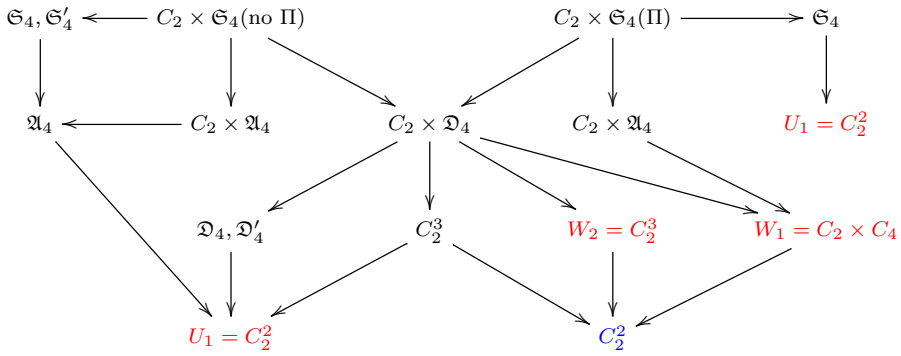
where

- *$X_{2,2} \subset \mathbb{P}^5$ is the unique singular toric complete intersections of two quadrics with six nodes,*
- *f is a small birational morphism, and*
- *g is a blowdown of the proper transform of the plane Π to a smooth point of $X_{2,2}$.*

Moreover, the G-action on $X_{2,2}$ preserves the torus in $X_{2,2}$, and G fixes a point in the torus.

Proof. Unprojecting from the G-invariant plane, we obtain the required commutative diagram. □

The following diagram summarizes the relations between G-actions on X_3, when G is contained in one of two nonconjugate subgroups $C_2 \times \mathfrak{S}_4$ and is not contained in the nonstandard $\mathfrak{S}_5'$ or $\mathfrak{S}_3^2 \rtimes C_2$:



By Proposition 6.1, we can identify subgroups of $\mathrm{Aut}(X_3)$ leaving a plane invariant with subgroups of $\mathrm{Aut}(X_{2,2})$ fixing the origin of the torus, which also can be identified with finite subgroups of $\mathsf{GL}_3(\mathbb{Z})$ as in Section 5. In this way, we identified the subgroups C_2^2, C_2^3 and $C_2 \times C_4$ in the diagram with subgroups of $\mathsf{GL}_3(\mathbb{Z})$, using notations from Section 5:

- $U_1 = C_2^2 \subset \mathsf{GL}_3(\mathbb{Z})$ is the group with **(H1)** obstruction to stable linearizability,
- $W_1 = C_2 \times C_4, W_2 = C_2^3$ are groups with **(SP)** obstruction to stable linearizability.

Moreover, the other subgroup C_2^2 in the diagram can be uniquely characterized by the following geometric conditions:

(1) $\mathrm{Sing}(X)$ splits as a union of five C_2^2-orbits of length 2, and
(2) it is not contained in the nonstandard $\mathfrak{S}_5'$, and
(3) it leaves exactly three planes in X_3 invariant.

Corollary 6.2. *Suppose that the following conditions are satisfied:*

(1) *G does not fix a singular point of X_3, and*
(2) *G is not contained in the nonstandard $\mathfrak{S}_5'$, and*
(3) *X_3 does not contain G-invariant planes.*

Then G does not satisfy **(H1)**, *and the G-action is not stably linearizable.*

Proof. By the diagram above, G contains the subgroup C_2^2 with the **(H1)** obstruction. This can also be checked directly, via `Magma`. □

If G leaves invariant a plane $\Pi \subset X_3$, the linearization problem is reduced to G-actions on a three-dimensional torus with G-fixed points, which was studied in [33, Section 9], and in detail in Section 5. Combining Proposition 6.1, Corollary 6.2 and results in Section 5, we obtain:

Theorem 6.3. *The G-action on X_3 is linearizable if and only if either*

- *G fixes a singular point on X_3, or*
- *G is contained in the nonstandard $\mathfrak{S}_5'$, or*
- *$G = C_2^2$, X_3 contains three G-invariant planes, and $\mathrm{Sing}(X)$ splits as a union of five C_2^2-orbits of length 2.*

Moreover, when the G-action is not linearizable, it is not stably linearizable.

Proof. If one of the first two conditions is satisfied, then G is linearizable, as explained above. If X_3 does not contain G-invariant planes, then G is not stably linearizable by Corollary 6.2. Hence we may assume G does not fix a singular point, G is not contained in the nonstandard $\mathfrak{S}_5'$ and X_3 contains a G-invariant plane Π. By Proposition 6.1, there then exists a G-equivariant birational map from X_3 to the toric intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$.

Going through the group diagram above, we see G is not stably linearizable when G is not conjugate to the C_2^2 characterized in the third condition; this C_2^2 can be identified with the subgroup in $\mathsf{GL}_3(\mathbb{Z})$ generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is linearizable, as explained in Section 5. □

Remark 6.4. The Burnside formalism of [41] *does not* allow to decide the linearizability of the actions of both C_2^2 at the bottom of the lattice diagram above. On the other hand, the formalism of *incompressible* symbols as in Section 4, proves nonlinearizability in several cases; note that these cases are obstructed by **(SP)**, as they contain W_1.

Let $G = C_2 \times \mathfrak{A}_4$, generated by

$$\begin{aligned} \iota &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_3, x_5, x_1, x_6, x_2, x_3), \\ \tau &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_3, x_5, x_1, x_4, x_2, x_6), \\ \sigma &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_5, x_6, x_2, x_1, x_4, x_3). \end{aligned}$$

The fixed locus for the involution ι is a plane $\Pi \subset X_3$ given by

$$x_1 + x_3 = x_2 + x_5 = x_4 + x_6 = 0$$

with a residue $\mathfrak{A}_4$-action on it. This produces an absolutely incompressible divisorial symbol (see Proposition 4.2)

$$\mathfrak{s} := (C_2, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)).$$

The model $X_3 \circlearrowleft G$ is not in standard form. However, G does not leave invariant any irreducible subvariety of X_3 with nontrivial stabilizer except Π. Therefore, no equivariant blow-up of X can possibly contribute the symbol $\mathfrak{s}$ to the class $[X_3 \circlearrowleft G]^{\mathrm{inc}}$. The class then contains the incompressible symbol $\mathfrak{s}$ with multiplicity 1. On the other hand, the algorithm in [41], implemented in [59], shows that $[\mathbb{P}^3 \circlearrowleft G]^{\mathrm{inc}}$ contains $\mathfrak{s}$ with multiplicity 2 for any (projectively) linear action $\mathbb{P}^3 \circlearrowleft G$. We conclude that this G-action on X_3, and thus also the action of $C_2 \times \mathfrak{S}_4$ containing G, are not (projectively) linearizable.

7. Geometry of the Burkhardt quartic

The Burkhardt quartic X_4 can be defined in $\mathbb{P}^4 \subset \mathbb{P}^5$ by (1.2), or in $\mathbb{P}^4$ by (1.3). Up to projectivity, X_4 is the unique quartic threefold with 45 nodes [27], and $\mathrm{Aut}(X_4) = \mathsf{PSp}_4(\mathbb{F}_3)$ acts on $\mathbb{P}^4$ via an irreducible 5-dimensional representation of its central extension $\mathrm{Sp}(\mathbb{F}_3)$. Our goal is to identify subgroups $G \subseteq \mathsf{PSp}_4(\mathbb{F}_3)$ giving rise to (projectively) linearizable actions on X_4.

Arithmetic aspects of the Burkhardt quartic, in particular, its rationality over nonclosed ground fields k, have been explored in [8, 9, 10]. For example, the form (1.3) is rational over $\mathbb{Q}$. For all forms X of X_4 over nonclosed fields of characteristic zero there exist a dominant, degree 6, map $M \to X$, where M is a Brauer-Severi variety of dimension 3 [8, Theorem 1.1]; forms arising from moduli spaces of abelian surfaces are unirational, in particular, their rational points are Zariski dense (see [10] and references therein). It is an open problem to determine which forms X are rational over $\mathbb{Q}$ [9, Question 2.9]; there certainly are k-forms that are not k-rational [10, 8].

Note that [10, Section 3] lists all subgroups $G \subseteq \mathsf{PSp}_4(\mathbb{F}_3)$ with nontrivial cohomology

$$\mathrm{H}^1(G, \mathrm{Pic}(\widetilde{X}_4)),$$

where $\widetilde{X}_4$ is the standard resolution of singularities of the Burkhardt quartic, denoted by $\mathcal{A}_2^*(3)$ in [10]. Recall that this is an obstruction to stable (projective) linearizability of the G-action. In particular, of the

115 conjugacy classes of nontrivial subgroups of $\mathsf{PSp}_4(\mathbb{F}_3)$, only 26 do not have the cohomological obstruction to **(H1)**.

Remark 7.1. Note that $\mathsf{PSp}_4(\mathbb{F}_3) \subset W(\mathsf{E}_6)$, as an index-two subgroup. If we consider the $W(\mathsf{E}_6)$-action on the Picard lattice of a smooth cubic surface, then, by [58], *every* subgroup $G \subseteq W(\mathsf{E}_6)$ arising from a *minimal* action on a cubic surface has nontrivial cohomology.

Proposition 7.2. *Assume that G contains an involution exchanging two coordinates in $\mathbb{P}^5$. Then the G-action is not projectively linearizable and not equivariantly birational to the action on the Segre cubic.*

Proof. We apply the formalism of Section 4. The involution action leads to classes

$$(C_2, Y \circlearrowright k(D), (1)),$$

where D is a quartic K3 surface with 12 nodes. This is an absolutely incompressible symbol, since D is not uniruled. Furthermore, it does not arise from projectively linear actions.

On the other hand, every C_2-action on the Segre cubic fixes a singular point, and the action is linearizable. □

Remark 7.3. There are 12 (conjugacy classes of) subgroups of $\mathsf{PSp}_4(\mathbb{F}_3)$ containing this involution and satisfying **(H1)**:

$$C_2, C_4, C_2^2, C_6, C_6', Q_8, \mathfrak{D}_4, C_2 \times C_6, C_{12}, \mathsf{SL}_2(\mathbb{F}_3), C_3 \times Q_8, C_3 \rtimes \mathfrak{D}_4,$$

in particular, cohomology does not allow to distinguish linearizability from nonlinearizability in these cases; the corresponding actions are specified in [21]. The Burnside obstruction to linearizability in Proposition 7.2 vanishes for $X_4 \times \mathbb{P}^1$, with trivial action on the second factor, see, e.g., [60, Section 3.5]. The **(SP)**-obstruction is also trivial, at least for $G = C_2$. Thus we are led to speculate that the C_2-action on the threefold $X_4 \times \mathbb{P}^1$ *is* linearizable.

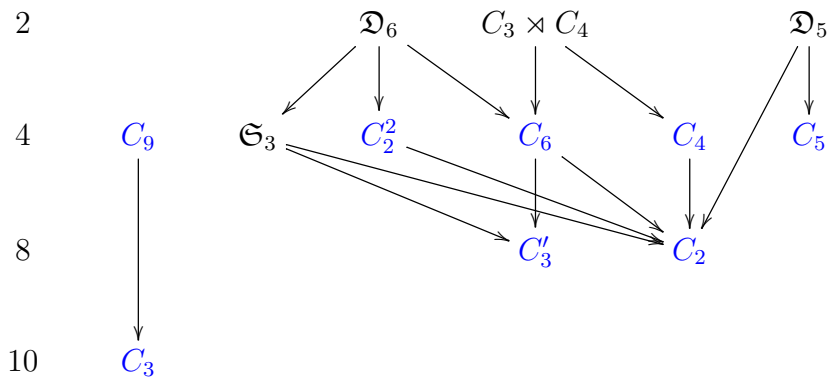
Using Proposition 2.2, we see that $\operatorname{rk} \mathrm{Cl}^G(X_4) = 1$ for 34 out of 115 conjugacy classes of nontrivial subgroups in $\mathsf{PSp}_4(\mathbb{F}_3)$. All these groups are not linearizable by the following result.

Proposition 7.4. *Let $G \subseteq \mathrm{Aut}(X_4)$ be such that $\operatorname{rk} \mathrm{Cl}(X_4)^G = 1$. Then X_4 is G-birationally super-rigid.*

Proof. See Section 8. □

Now, excluding groups with nontrivial cohomological obstructions to linearizability, those containing an involution fixing a K3 surface in X_4,

and those with $\operatorname{rk}\operatorname{Cl}^G(X_4) = 1$, we are left with the following tree of 12 groups, where the left column lists $\operatorname{rk}\operatorname{Cl}(X_4)^G$:



Concretely, put

$$\sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & q^2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_9 = \begin{pmatrix} 2q+1 & 2q+4 & 0 & 0 & 0 \\ -2q-1 & q+2 & 0 & 0 & 0 \\ 0 & 0 & 2q+1 & -q+1 & 2q+1 \\ 0 & 0 & -q+1 & 2q+1 & 2q+1 \\ 0 & 0 & -q-2 & -q-2 & 2q+1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^2 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_6 = \begin{pmatrix} -1 & 2q & 2q^2 & 2 & 2 \\ 1 & q & q^2 & 1 & 2 \\ q^2 & 2 & q & q^2 & q^2 \\ q^2 & 1 & 2q & q^2 & q^2 \\ q^2 & 1 & q & 2q^2 & q+1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} -1 & 2 & 2q & 2q^2 & 2 \\ q & q & -2q^2 & 1 & q \\ 1 & -2 & q & q^2 & 1 \\ 1 & 1 & q & q^2 & -2 \\ q^2 & q^2 & 1 & -2q & q^2 \end{pmatrix}, \qquad \sigma_5 = \begin{pmatrix} -1 & 2q & 2q^2 & 2 & 2 \\ q^2 & 1 & 2q & q^2 & q^2 \\ q & q^2 & 1 & 2q & q \\ q & q^2 & 1 & q & 2q \\ q^2 & 2 & q & q^2 & q^2 \end{pmatrix},$$

where q is a primitive third root of unity. The groups in the diagram are given in the above generators by:

$$\mathfrak{D}_6 = \langle \sigma_2, \sigma_6 \rangle, \quad C_3 \rtimes C_4 = \langle \sigma_3, \sigma_4 \rangle, \quad \mathfrak{D}_5 = \langle \sigma_4^2, \sigma_5 \rangle, \quad C_9 = \langle \sigma_9 \rangle,$$
$$\mathfrak{S}_3 = \langle \sigma_2, \sigma_6^2 \rangle, \quad C_2^2 = \langle \sigma_2, \sigma_6^3 \rangle, \quad C_6 = \langle \sigma_6 \rangle, \quad C_4 = \langle \sigma_4 \rangle,$$
$$C_5 = \langle \sigma_5 \rangle, \quad C_3 = \langle \sigma_9^3 \rangle, \quad C_3' = \langle \sigma_3 \rangle, \quad C_2 = \langle \sigma_4^2 \rangle,$$

The rest of this section is devoted to a case by case analysis of these actions, organized as follows: First, we recall classical linearization constructions for C_9 and C_5. Then we present a new linearization construction for C_6, which also gives linearization of C_2 and C_3, and use this construction to create a $\mathfrak{D}_6$-equivariant birational map from

X_4 to a smooth quadric threefold $X_2 \subset \mathbb{P}^4$, which gives a linearization for C_2^2. Finally, we present a linearization construction for C_4. The remaining four subgroups are

$$\mathfrak{S}_3,\ \mathfrak{D}_5,\ \mathfrak{D}_6,\ C_3 \rtimes C_4.$$

We do not know whether or not they are linearizable.

To study the linearization problem for $\mathfrak{D}_6$ and $\mathfrak{S}_3$, one can use their actions on the quadric X_2. For $\mathfrak{D}_5$, recall from [13, Example 5.8] that there exists a $\mathfrak{D}_5$-equivariant commutative diagram (3.2) such that Z is the smooth Del Pezzo surface of degree 5, and φ is a conic bundle, whose discriminant curve is the union of all (-1)-curves in Z. This might be a good model for the study of the linearization problem for $\mathfrak{D}_5$. Before showing linearization constructions for abelian groups, we present an explicit $C_3 \rtimes C_4$-equivariant birational map from X_4 to a fibration into quartic Del Pezzo surfaces, which could hopefully be used to study the linearization problem $C_3 \rtimes C_4$.

Example 7.5. Let $G = \langle \sigma_3, \sigma_4 \rangle \simeq C_3 \rtimes C_4$. The defining equation of $X_4 \subset \mathbb{P}^4$ can be rewritten as $f_2 r_2 = g_2 h_2$, where

$$\begin{aligned}
f_2 =& y_1y_2 + q^2y_1y_3 - y_1y_4 - q^2y_1y_5 - q^2y_2^2 + qy_2y_3 + (q-1)y_2y_4 + \\
& (-q-2)y_2y_5 - y_3^2 + (-q-2)y_3y_4 + (-2q-1)y_3y_5 + q^2y_4^2 - qy_4y_5 + y_5^2, \\
g_2 =& y_1^2 - q^2y_1y_4 - qy_1y_5 - qy_2^2 + y_2y_3 - q^2y_2y_4 - q^2y_2y_5 - q^2y_3^2 - q^2y_3y_4 - y_3y_5, \\
r_2 =& qy_1y_4 + y_1y_5 + y_2^2 - q^2y_2y_3 + qy_3^2, \\
h_2 =& y_1^2 + q^2y_1y_4 + qy_1y_5 + qy_2^2 - y_2y_3 - y_2y_4 - qy_2y_5 + q^2y_3^2 - qy_3y_4 - qy_3y_5.
\end{aligned}$$

The surfaces $\{f_2 = h_2 = 0\}$ and $\{g_2 = r_2 = 0\}$ generate a G-invariant pencil, and there exists a G-equivariant birational morphism $\widetilde{X}_4 \to X_4$, where

$$\widetilde{X}_4 = \{f_2u + g_2v = h_2u + r_2v = 0\} \subset \mathbb{P}^4 \times \mathbb{P}^1_{u,v},$$

and the G-action on $\mathbb{P}^4 \times \mathbb{P}^1$ is generated by

$$\sigma_4 \times \begin{pmatrix} 1 & 0 \\ q^2 & -1 \end{pmatrix}, \quad \sigma_3 \times \begin{pmatrix} 1 & -2q-1 \\ 0 & q \end{pmatrix}.$$

The birational morphism $\widetilde{X}_4 \to X_4$ is given by the projection to the first factor of $\mathbb{P}^4 \times \mathbb{P}^1$, while the projection to the second factor $\widetilde{X}_4 \to \mathbb{P}^1$ is a G-equivariant fibration into Del Pezzo surfaces of degree 4.

Linearization of C_9. By [9, Remark 4.3], the classical parametrization of the Burkhardt quartic by Baker is C_9-equivariant, i.e., the action of C_9 pulls back to an action on $\mathbb{P}^3$. An explicit description of the base locus of the birational map $\phi : \mathbb{P}^3 \dashrightarrow X_4$ can be found in [29]:

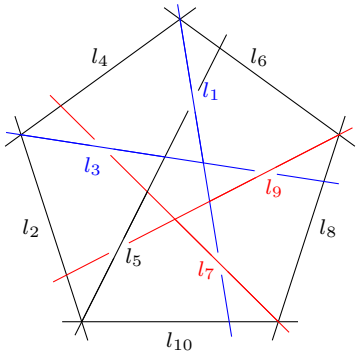
Consider a configuration in $\mathbb{P}^3$ consisting of 9 lines $l_1, ..., l_9$ with l_i meeting l_{i+1} in a point p_i, and l_9 meeting l_1 in p_9. The points $\{p_1, p_4, p_7\}, \{p_2, p_5, p_8\}$ and $\{p_3, p_6, p_9\}$ define lines that intersect in a common point p_{10} and

$$l_1 \cap l_4 \cap l_7, \quad l_2 \cap l_5 \cap l_8, \quad l_3 \cap l_6 \cap l_9$$

define a further 3 points. There is a unique such configuration, modulo PGL_4, and the linear system of all quartic surfaces in $\mathbb{P}^3$ containing these 9 lines gives ϕ. The symmetry group of this configuration is indeed C_9.

Linearization of C_5. There is also another parametrization, due to Todd [57], and described in detail in [45, Section 5.1]. There is a rigid configuration of 10 lines and 15 points in $\mathbb{P}^3$, and the linear system of quartic surfaces passing through these 10 lines gives a birational map from $\mathbb{P}^3$ to the Burkhardt quartic [45, Figure 5.1]. For example, the 10 lines can be given by equations and visualized as follows

$$\begin{aligned}
l_1 &= \{x_0 - x_3 = x_1 + q^2x_2 + qx_3 = 0\}, & l_2 &= \{x_2 = x_3 = 0\},\\
l_3 &= \{x_1 = x_2 + q^2x_3 = 0\}, & l_4 &= \{x_1 = x_2 = x_3\},\\
l_5 &= \{x_0 + q^2x_3 = x_2 + q^2x_3\}, & l_6 &= \{x_0 = x_1 = x_2\},\\
l_7 &= \{x_0 + q^2x_3 = x_1 - x_3\}, & l_8 &= \{x_0 = x_1 = 0\},\\
l_9 &= \{x_0 + q^2x_1 = x_2\}, & l_{10} &= \{x_0 = x_3 = 0\}.
\end{aligned}$$



The symmetry group of this configuration is C_5, and there are exactly two such configurations in $\mathbb{P}^3$, swapped by a $\mathfrak{D}_5$-action on $\mathbb{P}^3$.

Linearization of C_6. Let $G = \langle \sigma_6 \rangle \simeq C_6 \subset \mathrm{Aut}(X_4)$. Fix the following eight Jacobi planes:

$$\begin{aligned}
\Pi_1 &= \{y_1 = y_5 = 0\},\\
\Pi_2 &= \{y_1 = y_2 = 0\},\\
\Pi_3 &= \{(-q+1)y_1 + (-2q-1)y_5 = qy_1 - q^2y_2 - y_3 - y_4 - q^2y_5 = 0\},\\
\Pi_{16} &= \{qy_1 - y_2 - y_3 - qy_4 - y_5 = qy_1 - q^2y_2 - y_3 - y_4 - q^2y_5 = 0\},\\
\Pi_{18} &= \{qy_1 - y_2 - qy_3 - y_4 - y_5 = qy_1 - q^2y_2 - y_3 - y_4 - q^2y_5 = 0\},\\
\Pi_{22} &= \{q^2y_1 - q^2y_2 - y_3 - y_4 - y_5 = (q+2)y_1 + (-q-2)y_2 = 0\},\\
\Pi_{25} &= \{y_1 - y_5 = y_2 + q^2y_3 + qy_4 = 0\},\\
\Pi_{34} &= \{q^2y_1 - y_2 - qy_3 - q^2y_4 - q^2y_5 = qy_1 - qy_2 - y_3 - y_4 - y_5 = 0\},
\end{aligned}$$

and let $D_8 = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_{16} + \Pi_{18} + \Pi_{22} + \Pi_{25} + \Pi_{34}$. Here we keep the enumeration of Jacobi planes in X_4 as set in `Magma`, and recorded in [21]. Let

$$\mathcal{M}_4 = |4(-K_{X_4}) - D_8|,$$

it is a G-invariant four-dimensional mobile linear system, since the class $[D_8] \in \mathrm{Cl}(X_4)$ is G-invariant. Choosing an appropriate basis $f_1, \dots, f_5$ of $\mathcal{M}_4$ (see [21] for explicit equations of polynomials of the choice), we obtain an explicit rational map $X_4 \dashrightarrow \mathbb{P}^4$ given by

$$[y_1 : y_2 : y_3 : y_4 : y_5] \mapsto [f_1 : f_2 : f_3 : f_4 : f_5],$$

whose image is a quadric threefold $Q \subset \mathbb{P}^4$ with equation

$$\begin{aligned}
&3y_1^2 - 3y_1y_2 + 3y_2^2 + (2q-2)y_1y_3 + (-4q-2)y_2y_3 +\\
&(5q+4)y_1y_4 + (-q-5)y_2y_4 + (-q-2)y_3y_4 + (2q+1)y_4^2 +\\
&(-q+1)y_1y_5 + (2q+1)y_2y_5 + (q-1)y_3y_5 + 3y_4y_5 + (-2q-1)y_5^2 = 0.
\end{aligned}$$

Note that Q is a quadric cone with vertex at $[0 : 0 : q^2 : q : 1]$.

We have constructed a G-equivariant rational map $\chi\colon X_4 \dashrightarrow Q$, where the induced G-action on Q is given by the projective transformation

$$\begin{pmatrix}
-3-3q & 0 & q-1 & q+2 & -2q-1\\
3 & 0 & q+2 & 4q+2 & 4q+5\\
1 & q+1 & -2q+2 & -2q-1 & q+2\\
2q+2 & -q-3 & 2q-2 & 2q-2 & -q+1\\
q+3 & 3q-1 & 3q+3 & 6q+3 & 3
\end{pmatrix}.$$

Using `Magma`, one can check that χ is birational. Note that G fixes the point

$$[-2+q : -3q-8 : 3q+1 : 3q+1 : 7] \in Q \tag{7.1}$$

Thus, composing χ with the projection $Q \dashrightarrow \mathbb{P}^3$ from this point, we obtain a G-equivariant birational map $X_4 \dashrightarrow \mathbb{P}^3$, which gives a linearization of the subgroup $G \simeq C_6$.

The linearization of G can also be proved as follows. Consider the following six lines in $\mathbb{P}^4$:

$$
\begin{aligned}
L_1 &= \{y_4 - y_5 = y_3 - y_5 = y_1 + qy_2 = 0\},\\
L_2 &= \{2y_3 - y_4 - y_5 = 2y_2 - y_4 + y_5 = 2y_1 - q^2y_4 + q^2y_5 = 0\},\\
L_3 &= \{y_4 - qy_5 = 2y_3 + q^2y_5 = y_1 + q^2y_2 - (q+2)y_5 = 0\},\\
L_4 &= \{y_4 + q^2y_5 = y_3 = y_1 + q^2y_2 + qy_5 = 0\},\\
L_5 &= \{y_3 + qy_4 + y_5 = y_2 - y_4 - q^2y_5 = y_1 + 2qy_4 + (q+2)y_5 = 0\},\\
L_6 &= \{y_4 - qy_5 = y_3 - qy_5 = y_1 + qy_2 = 0\}.
\end{aligned}
$$

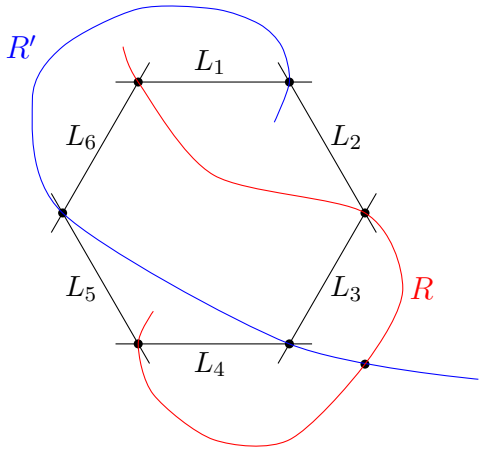
They are contained in the quadric cone Q, and they form a hexagon. Now, consider the following two conics in Q:

$$
\begin{aligned}
R &= \{2y_3 - q^2y_4 - qy_5 = 2y_1 + 2qy_2 - q^2y_4 + qy_5 = 0\} \cap Q,\\
R' &= \{y_3 - y_4 = y_1 + q^2y_2 + qy_4 - qy_5 = 0\} \cap Q.
\end{aligned}
$$

Both R and R' are smooth, they intersect transversally at the G-fixed point (7.1), and they do not contain the singular point of the cone Q. Note that

- R contains the intersection points $L_2 \cap L_3$, $L_4 \cap L_5$, $L_6 \cap L_1$,
- R' contains the intersection points $L_1 \cap L_2$, $L_3 \cap L_4$, $L_5 \cap L_6$.

Let Z be the curve $L_1 + \cdots + L_6 + R + R'$. Then Z is a (singular) G-invariant curve of degree 10 and arithmetic genus 8, which can be visualized as follows:



Let $\mathcal{M}_3$ be the linear subsystem in $|\mathcal{O}_Q(3)|$ consisting of all surfaces that contain Z. Then $\mathcal{M}_3$ is a G-invariant mobile four-dimensional linear system, and it gives a G-equivariant rational map $Q \dashrightarrow \mathbb{P}^4$ whose image is an irreducible quartic threefold X_4' that has 45 nodes. This gives $X_4' \simeq X_4$ [27], so choosing a suitable basis for $\mathcal{M}_3$ (see [21] for equations of the choice), we get $X_4' = X_4$.

This gives us a G-equivariant rational map $\rho\colon Q \dashrightarrow X_4$. One can check that ρ is birational. Moreover, choosing a suitable basis of $\mathcal{M}_3$, we get $\rho = \chi^{-1}$. We conclude that G is linearizable; $\mathsf{PSp}_4(\mathbb{F}_3)$ contains four subgroups isomorphic to C_6 (up to conjugation), and we already proved that three of them are not linearizable.

Note that the indeterminacy of ρ can be resolved via the following G-equivariant commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \pi\ } & V \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\phi} \\ X_4 & \overset{\rho}{\dashleftarrow} & Q \end{array} \tag{7.2}$$

where ϕ is a blow up of the singular curve $R + R'$, π is a blow up of a nodal curve of arithmetic genus 1 that is a union of the proper transforms of the lines $L_1, \dots, L_6$ and the fibers of the morphism ϕ over the points $L_1 \cap L_2$, $L_2 \cap L_3$, $L_3 \cap L_4$, $L_4 \cap L_5$, $L_5 \cap L_6$, $L_6 \cap L_1$, and η is a birational morphism that contracts 31 disjoint curves to 31 nodes of the quartic X_4. The threefold V has two nodes, and U has 14 nodes, since the curves blown up by π form a dodecagon.

Linearization of C_2^2. In the previous subsection, we presented an explicit C_6-equivariant birational map $\chi\colon X_4 \dashrightarrow Q$, where Q is a quadric cone in $\mathbb{P}^4$. Since C_6 fixes a point in Q, this gave us a linearization of C_6. Let us use χ to construct a $\mathfrak{D}_6$-birational map from X_4 to a smooth quadric threefold in $\mathbb{P}^4$, which will give us a linearization of $C_2^2 \subset \mathfrak{D}_6$. Set $G = \langle \sigma_6, \sigma_2 \rangle \simeq \mathfrak{D}_6$. Recall that our birational map χ is $\langle \sigma_6 \rangle$-equivariant, but not G-equivariant, since the involution σ_2 acts birationally on the quadric cone Q via

$$[y_1 : y_2 : y_3 : y_4 : y_5] \mapsto [t_1 : t_2 : t_3 : t_4 : t_5],$$

where

$$\begin{aligned} t_1 = {} & 19y_1y_3 + (18q+7)y_1y_4 + (-6q+4)y_1y_5 + (-5q-3)y_2y_3 + \\ & (-5q-3)y_2y_4 + (4q-9)y_2y_5 + (2q-14)y_3^2 + (2q+5)y_3y_4 + \end{aligned}$$

$$(11q+18)y_3y_5+(3q-2)y_4^2+(-9q+6)y_4y_5+(-9q-13)y_5^2,$$

$$t_2=(3q+17)y_1y_3+(6q-4)y_1y_4+(-3q+2)y_1y_5+(-q-12)y_2y_3+$$
$$(-10q-6)y_2y_4+(-q-12)y_2y_5+(4q-28)y_3^2+(-5q+16)y_3y_4+$$
$$19qy_3y_5+(12q+11)y_4^2+(12-18q)y_4y_5+(-12q-11)y_5^2,$$

$$t_3=(12q+11)y_1y_3+(-2q-5)y_1y_4+(5q+3)y_1y_5+$$
$$(-q-12)y_2y_3+(-3q+2)y_2y_4+(-2q-5)y_2y_5+(8q+1)y_3^2+$$
$$(-2q-5)y_3y_4+(2q+5)y_3y_5+(-3q+2)y_4^2+(-5q-3)y_5^2,$$

$$t_4=(20q+12)y_1y_3+(-9q-13)y_1y_4+(4q+10)y_1y_5+$$
$$(-2q-5)y_2y_3+(5q+3)y_2y_4+(-9q-13)y_2y_5+(8q+1)y_3^2+(-9q-13)y_3y_4+$$
$$(12q-8)y_3y_5+(-4q+9)y_4^2+(6q+15)y_4y_5+(-13q-4)y_5^2,$$

$$t_5=(q+1)y_1y_3+(-q-12)y_1y_4+(-3q+2)y_1y_5+(-9q-13)y_2y_3+$$
$$(4q+10)y_2y_4+(-q-12)y_2y_5+(-10q+13)y_3^2+(23q-9)y_3y_4+$$
$$(10q+6)y_3y_5+(-17q-14)y_4^2+(6q+15)y_4y_5+(-12q-11)y_5^2.$$

However, the linear system $\mathcal{M}_3\subset|\mathcal{O}_Q(3)|$ constructed in the previous subsection is G-invariant, and G acts biregularly on the threefold V. Thus, the birational map $\pi\circ\eta^{-1}\colon X_4\dashrightarrow V$ in the diagram (7.2) is also G-equivariant. Observe that $\operatorname{rk}\operatorname{Cl}(V)^G=\operatorname{rk}\operatorname{Pic}(V)^G=1$, so V is a G-Mori fiber space, cf. [48, Theorem 6.5(iii)].

Now, let $\operatorname{pr}\colon Q\dashrightarrow\mathbb{P}^2$ be the projection of the quadric cone Q from the line passing through the vertex of Q and the G-fixed point (7.1). Choosing appropriate coordinates on $\mathbb{P}^2$, the projection map pr is given by

$$\begin{pmatrix} 3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\\ q+2 & 5q+4 & -6q\\ -q+1 & q+5 & 3q^2 \end{pmatrix}.$$

One can check that

$$\operatorname{pr}\times(\operatorname{pr}\circ\iota)\colon Q\dashrightarrow\mathbb{P}^2_{x_0,x_1,x_2}\times\mathbb{P}^2_{z_0,z_1,z_2}$$

gives a birational map $\varrho\colon Q \to W$, where $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth divisor of bidegree $(1,1)$ that is given by

$$\begin{aligned}&6x_0z_0 - 3x_0z_1 + 2(q-1)x_0z_2 - 3x_1z_0 - 3x_1z_1 + (5q+4)x_1z_2\\ &+2(q-1)x_2z_0 + (5q+4)x_2z_1 + (-2q+5)x_2z_2 = 0.\end{aligned}$$

The birational map ϱ is G-equivariant, and $\operatorname{rk}\operatorname{Cl}(W)^G = 1$, so W is a G-Mori fiber space. This gives the G-equivariant commutative diagram:

$$\begin{array}{ccc} & \widetilde{V} \dashrightarrow \widetilde{W} & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ V & \xrightarrow{\varrho} & W \end{array}$$

where α is the blowup of both singular points of V, $\widetilde{V} \dashrightarrow \widetilde{W}$ is a flop in the strict transform of the line in Q passing through it vertex and the point (7.1), and β is a blow up of a G-irreducible smooth curve consisting of four irreducible components such that one of them is the fiber of the projection to the second factor $W \to \mathbb{P}^2$ over $[1 : 3q+2 : 3]$.

The group G leaves invariant the curve

$$\{3x_1 - (3q+2)x_2 = 3z_1 - (3q+2)z_2 = 0\} \subset W,$$

which is a curve of degree $(1,1)$. Blowing up this curve $\gamma\colon \widehat{W} \to W$, we obtain the following (classical) G-Sarkisov link:

$$\begin{array}{ccc} & \widehat{W} \dashrightarrow \widehat{X}_2 & \\ {\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle\delta} \\ W & & X_2 \end{array}$$

where X_2 is a smooth quadric 3-fold in $\mathbb{P}^4$, and δ is a blow up of two disjoint lines in X_2.

To describe the map $W \dashrightarrow X_2$ explicitly, observe that G leaves invariant the affine chart of W given by

$$3x_1 - (3q+2)x_2 \neq 0 \quad \text{and} \quad 3z_1 - (3q+2)z_2 \neq 0,$$

which is an affine quadric 3-fold, whose G-equivariant compactification is X_2. Hence, choosing appropriate coordinates on $\mathbb{P}^4$, we may assume that X_2 is given by

$$y_1y_4 - y_2y_3 + y_5^2 = 0,$$

and the induced G-action on X_2 is generated by

$$[y_1 : y_2 : y_3 : y_4 : y_5] \mapsto [-q^2y_1 : q^2y_2 : qy_3 : -qy_4 : -y_5],$$
$$[y_1 : y_2 : y_3 : y_4 : y_5] \mapsto [y_4 : y_3 : y_2 : y_1 : y_5].$$

In particular, we see that the subgroup $\langle \sigma_2, \sigma_6^3 \rangle \simeq C_2^2$ is linearizable, because the corresponding C_2^2-action on X_2 has a fixed point.

Linearization of C_4. Now, we let $G \simeq C_4$ be the subgroup in $\mathsf{PSp}_4(\mathbb{F}_3)$ generated by σ_4. Consider the G-orbit of four planes given by

$$\Pi_3 = \{(-q+1)y_1 + (-2q-1)y_5 = 0, qy_1 - q^2y_2 - y_3 - y_4 - q^2y_5 = 0\},$$
$$\Pi_8 = \{q^2y_1 - qy_2 - q^2y_3 - qy_4 - qy_5 = 0, (q+2)y_1 + (2q+1)y_2 = 0\},$$
$$\Pi_{12} = \{q^2y_1 - qy_2 - qy_3 - q^2y_4 - qy_5 = 0, (q+2)y_1 + (2q+1)y_5 = 0\},$$
$$\Pi_{15} = \{q^2y_1 - qy_2 - y_3 - y_4 - qy_5 = 0, qy_1 - q^2y_2 - qy_3 - q^2y_4 - q^2y_5 = 0\},$$

where q is a primitive cube root of unity, as above. Then projection from each of these planes produces a map $X_4 \dashrightarrow \mathbb{P}^1$. The product of these four projections results in a G-equivariant map

$$\pi : X_4 \dashrightarrow \mathbb{P}^1_{x_0,x_1} \times \mathbb{P}^1_{z_0,z_1} \times \mathbb{P}^1_{u_0,u_1} \times \mathbb{P}^1_{t_0,t_1}.$$

One can check that this map is birational onto its image. Choosing appropriate coordinates, the image V is a divisor in $(\mathbb{P}^1)^4$ given by

$$x_0z_0u_0t_0 + x_0z_0u_0t_1 + x_0z_1u_0t_1 + x_1z_1u_0t_1 + x_1z_1u_1t_1 = 0.$$

Notice that V has 5 singular points, which are ordinary double points. Observe also that $\operatorname{rk}\operatorname{Pic}(V)^G = 1$, but $\operatorname{rk}\operatorname{Cl}(V)^G = 2$, so V is not a G-Mori fiber space.

Let us find a $G\mathbb{Q}$-factorialization of V. To do this, we choose another G-orbit of four planes in X_4:

$$\Pi_1 = \{(-q+1)y_1 + (-2q-1)y_5 = (q+2)y_1 + (2q+1)y_5 = 0\},$$
$$\Pi_5 = \{(-q+1)y_1 + (-2q-1)y_5 = 0, qy_1 + (q+1)y_2 - qy_3 - q^2y_4 - q^2y_5 = 0\},$$
$$\Pi_{14} = \{q^2y_1 - qy_2 - q^2y_3 - qy_4 - qy_5 = 0, (q+2)y_1 + (2q+1)y_5 = 0\},$$
$$\Pi_{20} = \{q^2y_1 - qy_2 - q^2y_3 - qy_4 - qy_5 = 0, qy_1 - q^2y_2 - qy_3 - q^2y_4 - q^2y_5 = 0\}.$$

One can check they are not contracted under the map π, and the G-invariant divisor $\pi(\Pi_1) + \pi(\Pi_5) + \pi(\Pi_{14}) + \pi(\Pi_{20})$ is not $\mathbb{Q}$-Cartier. The linear system

$$|-K_V - \pi(\Pi_1) - \pi(\Pi_5) - \pi(\Pi_{14}) - \pi(\Pi_{20})|$$

is G-invariant, and its projective dimension is 5. Moreover, under the choice of basis

$$x_0z_0u_0t_1,\ \ x_0z_1u_0t_0,\ \ x_0z_1u_0t_1,\ \ x_0z_1u_1t_1,\ \ x_1z_1u_0t_1,\ \ x_1z_1u_1t_1,$$

it gives a G-equivariant birational map $\rho : V \dashrightarrow X_{2,2}$, where $X_{2,2}$ is a complete intersection in $\mathbb{P}^5$, with equations:

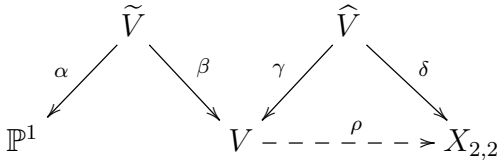
$$v_4v_5 - v_3v_6 = v_1v_2 + v_1v_3 + v_3^2 + v_3v_5 + v_4v_5 = 0.$$

The induced G-action on $X_{2,2}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -q^2 & -1 & 1 & 1 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

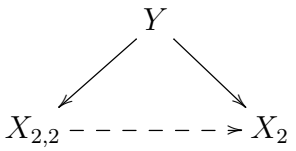
Note that this $X_{2,2}$ has 5 singular points and is not toric, so it is different from the toric intersection of two quadrics with 6 singular points that appeared in Section 5 and 6.

The birational map ρ fits the following G-equivariant commutative diagram:



where β and γ are small G-equivariant birational morphisms that resolve 4 singular points of V forming one G-orbit, δ is a blow up of 4 singular points of V forming one G-orbit, and α is a fibration into Del Pezzo surfaces of degree 4. The small birational morphisms β and γ are $G\mathbb{Q}$-factorializations of V, and the composition $\alpha \circ \beta^{-1} \circ \gamma \circ \delta^{-1}$ is given by the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^1$ from the three-dimensional linear subspace in $\mathbb{P}^5$ that contains 4 singular points of $X_{2,2}$ blown up by δ. A similar G-Sarkisov link appeared in the proof of [2, Lemma 2.16].

Now, we let $P = [0 : 0 : 0 : 0 : 1]$. Then P is a G-fixed singular point of $X_{2,2}$. Projection from P gives a G-birational map from $X_{2,2}$ to a smooth quadric threefold $X_2 \subset \mathbb{P}^4$ that fits the following G-equivariant commutative diagram:



where $Y \to X_{2,2}$ is the blowup of the point P, and $Y \to X_2$ is the blow up of a singular connected curve of arithmetic genus 1 and degree 4,

which is a union of four lines. Note that Y is a singular Fano threefold in the deformation family №2.23, and the constructed G-Sarkisov link is a degeneration of a classical Sarkisov link that blows up a smooth quadric threefold along a smooth quartic elliptic curve.

Since the G-action on a smooth quadric threefold has a fixed point, the action of G is linearizable.

8. Equivariant birational rigidity

Let $X_4 \subset \mathbb{P}^4$ be the Burkhardt quartic and $G \subseteq \mathrm{Aut}(X_4)$ be such that $\mathrm{rk}\,\mathrm{Cl}^G(X_4) = 1$. In this section we prove Proposition 7.4, i.e., we show that X_4 is G-birationally super-rigid. We start by recalling several well-known geometric facts about X_4, and proving three technical lemmas.

The quartic X_4 has 45 isolated ordinary double points (nodes). One can also check that

- a line in $\mathbb{P}^4$ can contain 1, 2 or 3 nodes,
- a plane in $\mathbb{P}^4$ can contain 1, 2, 3, 4, 6 or 9 nodes,
- a hyperplane in $\mathbb{P}^4$ can contain 1, 2, 3, 4, 7, 10, 12 or 18 nodes.

Planes in $\mathbb{P}^4$ containing 9 nodes of X_4 are called Jacobi planes — these planes are contained in X_4. The threefold X_4 contains 40 Jacobi planes, each of these 40 planes contains exactly 9 nodes of X_4, and there are exactly 8 planes in X_4 that pass through a given node. The union of all planes in X_4 is a divisor in $|10(-K_{X_4})|$, which we denote by $\mathbf{J}$. Similarly, hyperplanes in $\mathbb{P}^4$ containing 18 nodes of X_4 are called Steiner hyperplanes — their intersections with X_4 split into unions of 4 Jacobi planes. We will call such unions of 4 Jacobi planes *tetrahedra*. There are 40 Steiner hyperplanes, so X_4 contains 40 tetrahedra.

Lemma 8.1. *Let Σ be a subset of the singular locus $\mathrm{Sing}(X_4)$, of cardinality $s = |\Sigma| \geq 1$. Suppose that at least one of the following conditions is satisfied:*

(1) $s \leqslant 4$,
(2) $s \in \{5, 6\}$, *the set Σ is contained in a plane in $\mathbb{P}^4$, no 3 points in Σ are collinear,*
(3) $s = 7$, *the set Σ is contained in a hyperplane in $\mathbb{P}^4$, no 4 points of the set Σ are contained in a plane in $\mathbb{P}^4$.*

Then X_4 contains a tetrahedron that is disjoint from Σ.

Proof. Computer computations. □

Lemma 8.2. *Let C be an irreducible curve in X_4 of degree $d \leqslant 3$, let $\Sigma = C \cap \mathrm{Sing}(X_4)$, and let $s = |\Sigma|$. If C is a twisted cubic curve, we also suppose that $s \notin \{5, 6\}$. Then there is a Jacobi plane $\Lambda \subset X_4$ such that $\Lambda \cap C$ contains a smooth point of X_4.*

Proof. Since the locus $\mathrm{Sing}(X_4)$ is an intersection of cubics in $\mathbb{P}^4$, we see that $s \leqslant 3d$. Thus, if $d \leqslant 2$, then X_4 contains a tetrahedron T that is disjoint from Σ by Lemma 8.1, so that $T \cap C$ contains a smooth point of X_4, since $T \cap C \neq \varnothing$. This proves the lemma in the case $d \leqslant 2$. Hence, we may assume that C is either a plane cubic or a twisted cubic.

Suppose that C is a plane cubic. Let Π be the plane in $\mathbb{P}^4$ that contains C. If $\Pi \subset X_4$, we are done. If $\Pi \not\subset X_4$, then $X_4|_{\Pi} = C + \ell$ for some line ℓ, which gives $s \leqslant 4$, since $\Sigma \subset \mathrm{Sing}(C + \ell)$. Now, the required assertion follows from Lemma 8.1.

To complete the proof, we may assume that C is a twisted cubic curve. If $s \leqslant 4$ or $s = 7$, the assertion follows from Lemma 8.1. Thus, we may assume that $s \in \{8, 9\}$. Let $f\colon \widetilde{X}_4 \to X_4$ be the blow up of all singular points of X_4, let $\widetilde{\mathbf{J}}$ be the strict transform on $\widetilde{X}_4$ of the divisor $\mathbf{J}$, let E be the union of all f-exceptional prime divisors, and let $\widetilde{C}$ be the strict transform on $\widetilde{X}_4$ of the curve C. Then

$$0 \leqslant \widetilde{\mathbf{J}} \cdot \widetilde{C} = \Big(f^*\big(-10K_{X_4}\big) - 4E\Big) \cdot \widetilde{C} = 30 - 4E \cdot \widetilde{C} \leqslant 30 - 4s \leqslant -2,$$

which is absurd. This completes the proof of the lemma. □

Lemma 8.3. *Let $C \subset X_4$ be a twisted cubic curve and $\Sigma = C \cap \mathrm{Sing}(X_4)$. Set $s = |\Sigma|$ and let $\mathcal{M}$ be a non-empty mobile linear subsystem in $|-nK_{X_4}|$ for some positive integer n. Suppose that $s \leqslant 6$. Then $\mathrm{mult}_C(\mathcal{M}) \leqslant n$.*

Proof. If $s \geqslant 1$, let $g\colon \overline{X}_4 \to X_4$ be the blow up of Σ. If $s = 0$, we let $\overline{X}_4 = X_4$ and $g = \mathrm{Id}_{X_4}$. Let $f\colon \widetilde{X}_4 \to \overline{X}_4$ be the blow up of the strict transform on $\overline{X}_4$ of the twisted cubic curve C, let F be the f-exceptional surface, let $E_1, \ldots, E_s$ be the $(f \circ g)$-exceptional prime divisors that are mapped to the subset Σ, and let $\widetilde{\mathcal{M}}$ be the strict transform on the threefold $\widetilde{X}_4$. Set $m = \mathrm{mult}_C(\mathcal{M})$. Then

$$\widetilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*\big(-nK_{X_4}\big) - mF - \sum_{i=1}^{s} a_i E,$$

where $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}$. We have to show that $m \leqslant n$.

If $s \geqslant 1$, then each E_i is a Del Pezzo surface of degree 7 and

$$\widetilde{\mathcal{M}}\big|_{E_i} \sim -mF\big|_{E_i} - nE_i\big|_{E_i},$$

which implies that $m \leqslant 2a_i$. So, if $s \geqslant 1$, then $a_i \geqslant \frac{m}{2}$ for every i.

Set $H = (f \circ g)^*(-K_{X_4})$. Then $|2H - F - \sum_{i=1}^{s} E_i|$ does not have base curves, because the curve C is cut out by quadrics in $\mathbb{P}^4$. Let D be a general surface in $|2H - F - \sum_{i=1}^{s} E_i|$. Then D is nef. Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be general surfaces in $\widetilde{\mathcal{M}}$. Then $D \cdot \widetilde{M}_1 \cdot \widetilde{M}_2 \geqslant 0$.

Let us compute $D \cdot \widetilde{M}_1 \cdot \widetilde{M}_2$. We have

$$H^3 = 4, \quad E_i \cdot H^2 = 0, \quad F \cdot E_i^2 = 0, \quad F^3 = s-1, \quad E_i \cdot F^2 = -s,$$
$$H \cdot F^2 = -3, \quad F \cdot H^2 = 0, \quad E_i \cdot F \cdot H = 0, \quad E_i^3 = 2s, \quad H \cdot E_i^2 = 0.$$

Then

$$0 \leqslant D \cdot \widetilde{M}_1 \cdot \widetilde{M}_2 = 8 - 5m^2 + 2\Big(\sum_{i=1}^{s} a_i - 3\Big)m - 2\sum_{i=1}^{s} a_i^2.$$

This gives $m \leqslant n$, since $s \leqslant 6$ and $a_i \geqslant \frac{m}{2}$ for $i \in \{1, \ldots, s\}$. □

Now, we are ready to prove that X_4 is G-birationally super-rigid. Suppose it is not. By Corollary 3.3, there is a non-empty G-invariant mobile linear subsystem $\mathcal{M} \subset |-nK_{X_4}|$, for some positive integer n, such that the singularities of the pair $(X_4, \frac{1}{n}\mathcal{M})$ are not canonical. Let us seek a contradiction.

Set $\lambda = \frac{1}{n}$. Let Z be a center of non-canonical singularities of the log pair $(X_4, \lambda\mathcal{M})$, let M_1 and M_2 be two general surfaces in the linear system $\mathcal{M}$. If Z is a smooth point of X_4, then it follows from [52] or [24, Corollary 3.4] that

$$\Big(M_1 \cdot M_2\Big)_Z > \frac{4}{\lambda^2} = 4n^2,$$

which leads to a contradiction:

$$4n^2 = \frac{4}{\lambda^2} = H \cdot M_1 \cdot M_2 \geqslant \Big(M_1 \cdot M_2\Big)_Z > \frac{4}{\lambda^2} = 4n^2,$$

where H is a general hyperplane section of X_4 passing through P. Thus, either Z is a singular point of X_4, or Z is an irreducible curve.

Suppose that Z a singular point of X_4. Let $f\colon \widetilde{X}_4 \to X_4$ be the blow up of this point, let E be the f-exceptional surface, let $\widetilde{\mathcal{M}}$ be the strict transform on $\widetilde{X}_4$ of the linear system $\mathcal{M}$, and let $\widetilde{M}$ be a general surface in $\widetilde{\mathcal{M}}$. Then

$$\widetilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*\big(-nK_{X_4}\big) - aE,$$

for some integer $a > n$, by [22, Theorem 1.7.20] or [24, Theorem 3.10]. Now, let Π be a Jacobi plane in X_4 that contains Z, let L be a general

line in Π that contains Z, and let $\widetilde{L}$ be its strict transform on $\widetilde{X}_4$. Then $\widetilde{L} \not\subset \widetilde{M}$, so that $0 \leqslant \widetilde{M} \cdot \widetilde{L} = n - a < 0$, which is absurd.

Thus, Z is an irreducible curve. Then $\mathrm{mult}_Z(\mathcal{M}) > \frac{1}{\lambda} = n$. Write

$$M_1 \cdot M_2 = mZ + \Delta,$$

where m is a positive integer such that $m > n^2$, and Δ is an effective one-cycle whose support does not contain Z. Then

$$4n^2 = \frac{4}{\lambda^2} = -K_{X_4} \cdot M_1 \cdot M_2 = m\deg(Z) - K_{X_4} \cdot \Delta \geqslant m\deg(Z) > n^2\deg(Z),$$

which gives $\deg(Z) \in \{1, 2, 3\}$. As in Lemma 8.2, let $\Sigma = Z \cap \mathrm{Sing}(X_4)$, and set $s = |\Sigma|$. If Z is a twisted cubic, then $s \notin \{5, 6\}$ by Lemma 8.3. Thus, it follows from Lemma 8.2 that X_4 contains a Jacobi plane Π such that $\Pi \cap Z$ contains a smooth point P of X_4. Let ℓ be a general line in this plane that contains P. Then $\ell \not\subset M_1$, so

$$n = \frac{1}{\lambda} = M_1 \cdot \ell \geqslant \mathrm{mult}_P(M_1) \geqslant \mathrm{mult}_Z(M_1) = \mathrm{mult}_Z(\mathcal{M}) > \frac{1}{\lambda} = n,$$

which is absurd. This completes the proof of Proposition 7.4.

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