EQUIVARIANT BIRATIONAL GEOMETRY OF LINEAR ACTIONS

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Abstract. We study linear actions of finite groups in small dimensions, up to equivariant birationality.

1. Introduction

The classification of actions of finite groups on rational surfaces, up to equivariant birationality, has a rich past and an active present. It goes back at least to the classical work of Bertini, Castelnuovo, Kantor, Segre, with the focus on involutions and their fixed loci, and to the work of Manin, Iskovskikh, and Sarkisov, with an emphasis on the group action on the Picard group [28], classification of elementary birational transformations [17], and equivariant birational rigidity [36]. The fundamental work of Dolgachev–Iskovskikh [13] summarizes and completes this vast program, to a certain extent: it gives a list of finite groups that can act on rational surfaces, and presents an algorithm that allows to distinguish different birational actions of a group, in many cases.

More precisely, the equivariant Minimal Model Program (MMP) shows that an action of a finite group G on a rational surface can be realized as a regular action either on a Del Pezzo surface or conic bundle over $\mathbb{P}^1$, see [31]. One can assume that the surface is *minimal*, i.e., no equivariant blow downs are possible. Finite group actions on minimal Del Pezzo surfaces of low degree are *rigid*, and visible via induced actions on the primitive Picard lattice, i.e., as subgroups of the respective Weyl group.

The most significant *"What is left?"* [13, Section 9] was the classification, up to birationality, of actions on Del Pezzo surfaces of high degree, e.g., linear and projectively linear actions on the projective plane.

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Definition 1.1. A regular generically free action of a finite group G on projective space $\mathbb{P}^n$ is called *linear*, respectively, *projectively linear* if it arises via projectivizations $\mathbb{P}(V)$ of an $(n+1)$-dimensional representation V of G, respectively, of a central extension of G.

Finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ and $\mathrm{PGL}_4(\mathbb{C})$ has been classified by Blichfeldt [3], where he defined a G-actions on $\mathbb{P}^n$ to be:

- *intransitive*: if the representation V is reducible,
- *transitive but imprimitive*: if the action is not intransitive, but there is a nontrivial normal subgroup of G acting intransitively;
- *primitive*: neither of the above.

Geometrically, one can interpret the definition as follows: a finite group action on $\mathbb{P}^2$ is intransitive if it fixes a point, and is imprimitive if it has an orbit of points of length 3; the action on $\mathbb{P}^3$ is intransitive if it fixes a point, or leaves a line invariant, and is imprimitive if it has an orbit of points of length 4, or leaves the union of two disjoint lines invariant. Equivariant birational geometry of *primitive* actions was essentially settled, via equivariant MMP, in [34]. On the other extreme, the birational classification of linear actions of *abelian* groups has been settled, in all dimensions, in [33, Theorem 7.1]. In general, the classification of regular actions on $\mathbb{P}^2$, up to birationality, is still an open problem.

The case of threefolds is much more involved. As in dimension 2, the birational classification of linear actions on $\mathbb{P}^3$ is an open problem. Significant progress has been achieved in analyzing *primitive* actions [11], [7], or involutions in the Cremona group Cr_3 (see [29]).

New equivariant birational invariants were defined in [20] and [23, Definitions 4.2 and 4.4]. The definitions assume that the ground field is of characteristic zero and contains roots of unity of order dividing the order of G. The invariants are computed on an appropriate birational model X (standard form) and take values in the *Burnside group*

$$\mathrm{Burn}_n(G),$$

which is defined as a quotient of a *symbols group* by explicit relations. The symbols encode information about loci with nontrivial abelian stabilizers, the weights of the induced action in the normal bundle to these loci, as well as the induced action on the corresponding function fields, see [14, Section 7] for definitions and [14, Sections 6 and 7.6] for examples. The paper [24] applied this formalism to the study of actions on $\mathbb{P}^2$ and produced new examples of non-birational *intransitive* actions.

In this paper, we work over an algebraically closed field k of characteristic zero. We apply the formalism of Burnside groups to the study of linear actions in dimensions ≤ 3. We make extensive use of the algorithm developed in [24], which allows to recursively compute the class in $\mathrm{Burn}_n(G)$ of a (projectively) linear action of a finite group G on $\mathbb{P}^n$. We have implemented this algorithm in `magma` and compiled tables of classes of such actions on $\mathbb{P}^2$ and $\mathbb{P}^3$, see [39]. Among our results are:

- In dimension 2, the Burnside formalism does not allow to distinguish primitive actions but does yield many new examples of non-birational linear and projectively linear actions, see Section 7.
- In dimension 3, we exhibit new types of non-birational linear actions on $\mathbb{P}^3$ as well as nonlinearizable actions on smooth quadrics, see Section 8 and 9.

In essence, the Burnside formalism complements birational rigidity techniques as in [34], [11], [7].

Here is the roadmap of the paper: In Section 2 we recall basic facts concerning equivariant birational geometry and relevant classical invariants used to distinguish actions up to birationality. In Section 3, we recall the definition of the Burnside group $\mathrm{Burn}_n(G)$ introduced in [20]; this group receives birational invariants of generically free actions of a finite group G on n-dimensional varieties. We tabulate the groups in small dimensions and for small G, and develop new tools for working with these groups. In Section 4 we explain how to compute the class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$$

of a generically free G-action on an n-dimensional variety X. In Section 5 we apply the formalism to curves. In Section 6 we give examples of computations of classes of linear actions, using the algorithm in [24]. In Sections 7 and 8 we investigate linear actions on $\mathbb{P}^2$ and $\mathbb{P}^3$, providing new examples of non-birational actions, not distinguishable with previous tools. In Section 9 we study quadrics of dimension ≤ 3.

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2. Generalities

We recall basic terminology and notation. We consider *generically free*, regular actions of finite groups G on smooth projective algebraic varieties X over an algebraically closed field k of characteristic zero. Generically free means that there exists a Zariski open subset of X where G acts freely. By convention, the action is from the right, and it will be denoted by

$$X \circlearrowleft G.$$

The induced left G-action on the function field $k(X)$ is denoted by $G \circlearrowright k(X)$. We let

$$X^G := \{\mathfrak{p} \in X, \mathfrak{p} \cdot g = \mathfrak{p}\}$$

be the set of G-fixed points on X.

We write

$$X \sim_G X',$$

if there exists a G-equivariant birational map $X \dashrightarrow X'$. This means that there exists a G-equivariant isomorphism of field extensions

$$k(X)/k \xrightarrow{\sim} k(X')/k.$$

We say that X, X' are *stably* equivariantly birational if

$$X \times \mathbb{P}^m \sim_G X' \times \mathbb{P}^m,$$

for some m, with trivial action on the second factor. Of particular interest is the study of (conjugacy classes of) finite subgroups of the *Cremona group*

$$\mathrm{Cr}_n = \mathrm{BirAut}(\mathbb{P}^n),$$

the group of birational automorpisms of projective space, and the study of equivariant birationalities

$$X \sim_G \mathbb{P}(V).$$

We say that the G-action on X is:

- *linearizable* if V is a faithful representation of G, i.e., the action arises from an injective homomorphism $G \to \mathrm{GL}(V^\vee)$, e.g., any cyclic group action on $\mathbb{P}^n$.
- *projectively linearizable* if the G-action on $\mathbb{P}(V)$ arises from a *projective* representation $G \to \mathrm{PGL}_{n+1}$, i.e., a linear representation $\tilde{G} \to \mathrm{GL}(V^\vee)$ of a central extension

$$1 \to \mu_{n+1} \to \tilde{G} \to G \to 1,$$

where μ_{n+1} denotes cyclic group of order $n+1$. As an example, dihedral group $\mathfrak{D}_n$-actions on $\mathbb{P}^1$ with even n are projectively linearizable but not linearizable as such actions necessarily come from 2-dimensional representations of $\mathfrak{D}_{2n}$.

Note that a linearizable action is projectively linearizable, but the converse need not hold. We call the corresponding actions on $\mathbb{P}(V)$ *linear*, respectively, *projectively linear*. Projectively linear actions on $\mathbb{P}^n$ with a fixed point are linear.

Among general approaches to the (stable) linearizability problem are:

- *birational rigidity*, see, e.g., [30], [10],
- *intermediate Jacobians*, see [15],
- group cohomology, such as *Amitsur invariant* (see [2, Section 6], [35, Theorem 2.14]) or invariance of $\mathrm{H}^1(G, \mathrm{Pic}(X))$ under equivariant blowups of smooth projective G-varieties X, see [4].

We list technical tools that are ubiquitous in equivariant birational geometry:

- If X is rationally connected and G is cyclic then $X^G \neq \emptyset$.
- If G is abelian and $\pi : \tilde{X} \dashrightarrow X$ is a G-equivariant birational map between smooth projective varieties then
$$X^G \neq \emptyset \quad \Leftrightarrow \quad \tilde{X}^G \neq \emptyset.$$
Note that abelian group actions on a smooth projective variety do not always have fixed points, e.g., the generically free C_2^2-action on $\mathbb{P}^1$, or translation actions on abelian varieties.
- **(RY)**: Assume that a finite *abelian* group G acts regularly and generically freely on a smooth projective variety X of dimension n. Let $\mathfrak{p} \in X^G$ be a G-fixed point and
$$(a_1, \ldots, a_n), \quad a_j \in G^\vee$$
the collection of characters of G occurring in the tangent space at $\mathfrak{p}$. Let
$$\det(\mathfrak{p}) := a_1 \wedge \cdots \wedge a_n \in \wedge^n(G^\vee)$$
be the determinant. Let $\pi : \tilde{X} \to X$ be a G-equivariant birational morphism. Then, by [33], there exists a G-fixed point $\mathfrak{q} \in \pi^{-1}(\mathfrak{p}) \subset \tilde{X}$ such that
$$\det(\mathfrak{p}) = \pm \det(\mathfrak{q}).$$

- **(No-name lemma):** If G acts generically freely on X and $\mathcal{E} \to X$ is a G-vector bundle of rank m then
$$\mathcal{E} \sim_G X \times \mathbb{P}^m,$$
with trivial action on the second factor.
- **(MRC):** Let $r = r(X)$ be the dimension of the Maximal Rationally Connected (MRC) quotient of an algebraic variety X. This is a well-defined equivariant birational invariant, by the functoriality of MRC quotients (see, e.g., [19, IV.5.5]).
- **(H1):** Let X be a smooth projective variety with a generically free, stably linearizable, action of G. Then, for all $H \subseteq G$, one has
$$\mathrm{H}^1(H, \mathrm{Pic}(X)) = 0.$$
A G-variety satisfying this property will be called H^1-trivial. This is a stable birational property.

In the next sections, we discuss G-birational invariants introduced in [20] and [23]. They are based on an analysis of the geometry of subvarieties of X with nontrivial stabilizers, together with the induced representation in the normal bundle, and can be viewed as a generalization of the **(RY)** invariant.

3. Equivariant Burnside groups

Throughout, G is a finite group and H a finite abelian group. When $H \subseteq G$ is a subgroup, we write $Z_G(H)$ (resp. $N_G(H)$) for its centralizer (resp. normalizer) in G. We write
$$H^\vee := \mathrm{Hom}(H, k^\times)$$
for the group of characters of H.

There are three versions of symbols groups, corresponding to the kind of data we attach to loci with nontrivial stabilizers (on a standard model, see Section 4). We recall the definitions, following [20] and [23].

3.1. **Maximal stabilizers.** This version addresses (generically free, regular) actions of finite abelian groups H on smooth projective varieties X, of dimension n; one records the weights of H in the tangent space at H-fixed points. In detail, for $n \in \mathbb{N}$, let
$$\mathcal{S}_n(H),$$
be the free abelian group generated by *symbols*
$$\beta = (b_1, \dots, b_n), \quad b_1, \dots, b_n \in H^\vee, \quad \langle b_1, \dots, b_n \rangle = H^\vee,$$

subject to the reordering relation

(O) $\beta = (b_1, \dots, b_n) \sim \beta' = (b'_1, \dots, b'_n)$ if there is a permutation $\sigma \in \mathfrak{S}_n$, with $b'_i = b_{\sigma(i)}$ for $i = 1, \dots, n$.

Consider the quotient

$$\mathcal{S}_n(H) \to \mathcal{B}_n(H)$$

by the blow-up relation

(B) For $\beta = (b_1, \dots, b_n)$, $n \geq 2$, β is identified with the symbol

$$(0, b_2, \dots, b_n) \quad \text{when} \quad b_1 = b_2,$$

and with

$$\beta_1 + \beta_2 \quad \text{otherwise}$$

where

$$\beta_1 := (b_1 - b_2, b_2, b_3, \dots, b_n), \qquad \beta_2 := (b_1, b_2 - b_1, b_3, \dots, b_n).$$

3.2. **Combinatorial Burnside group.** This version takes into account arbitrary stabilizers for actions of general finite groups, but ignores the induced action on function fields of strata with nontrivial stabilizers. For $n \in \mathbb{N}$, let

$$\mathcal{SC}_n(G)$$

be the abelian group generated by *symbols*

$$(H, Y, \beta), \tag{3.1}$$

where

- $H \subseteq G$ is an abelian subgroup (the *stabilizer* of the symbol),
- Y is a subgroup of $Z_G(H)/H$, and
- $\beta = (b_1, \dots, b_{n-d})$ is a sequence of *nontrivial* characters of H, generating $H^\vee$, and d runs over all integers in $[0, \dots, n]$,

Symbols with $d = 0$ are called *point symbols* and those with $d = n - 1$ *divisorial symbols*.

Symbols (3.1) are subject to reordering and conjugation relations:

(O) $(H, Y, \beta) = (H, Y, \beta')$ if $\beta \sim \beta'$, as in Section 3.1.

(C) $(H, Y, \beta) = (H', Y', \beta')$ if there exists some $g \in G$ such that

$$H' = gHg^{-1}, \quad Y' = gYg^{-1},$$

and the characters in β' arise from those in β via conjugation by g.

Consider the quotient

$$\mathcal{SC}_n(G) \to \mathcal{BC}_n(G)$$

by the vanishing and blowup relations:

(V) $(H, Y, \beta) = 0$ when $b_1 + b_2 = 0$.

(B) $(H, Y, \beta) = \Theta_1 + \Theta_2$, where:

$$\Theta_1 := \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y, \beta_1) + (H, Y, \beta_2), & \text{if } b_1 \neq b_2, \end{cases}$$

with β_1, β_2 as above, and

$$\Theta_2 := \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{Y}, \bar{\beta}), & \text{otherwise.} \end{cases}$$

Here,

$$\overline{H} := \mathrm{Ker}(b_1 - b_2) \subseteq H,$$

with

$$H/\overline{H} \subseteq \overline{Y} \subseteq Z_G(H)/\overline{H},$$

and $\overline{Y}$ is the preimage of Y in $Z_G(H)/\overline{H}$. The corresponding character $\bar{\beta}$ consists of restrictions of characters of β:

$$\bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots), \quad \bar{b}_i \in H^\vee/\langle b_1 - b_2 \rangle, i \neq 1.$$

The images of point symbols, respectively, divisorial symbols, will be called *point classes*, respectively, *divisorial classes*.

3.3. **Equivariant Burnside group.** The most refined version records both the action of the stabilizer in the normal bundle and the induced action on the function fields of strata.

For $n \in \mathbb{N}$, let

$$\mathrm{Symb}_n(G),$$

be the free abelian group generated by symbols

$$(H, Y \circlearrowright K, \beta), \tag{3.2}$$

where

- $H \subseteq G$ is an abelian subgroup,
- $Y \subseteq Z_G(H)/H$ is a subgroup,
- K is a finitely generated extension of k, of transcendence degree $d \le n$, with faithful action by Y, and
- $\beta = (b_1, \ldots, b_{n-d})$ is a sequence of *nontrivial* characters of H, generating $H^\vee$.

As in the case of combinatorial Burnside groups, we call a symbol in $\mathrm{Symb}_n(G)$ *divisorial* if $d = (n-1)$, i.e., $\beta = (b)$, for some generator b of $H^\vee$. We call a symbol a *point* symbol if $d = 0$. Generally, we call $(n-d)$ the *codimension* of the symbol.

Symbols (3.2) are subject to reordering and conjugation relations:

(O) $(H, Y \circlearrowright K, \beta) = (H, Y \circlearrowright K, \beta')$ if $\beta \sim \beta'$.

(C) $(H, Y \circlearrowright K, \beta) = (H', Y' \circlearrowright K', \beta')$ if, for some $g \in G$, we have $H' = gHg^{-1}$, $Y' = gYg^{-1}$, there is an isomorphism $K \cong K'$, trivial on k, that is compatible with the respective actions, and β' obtained from β via conjugation by g.

We consider the quotient

$$\mathrm{Symb}_n(G) \to \mathrm{Burn}_n(G)$$

by the vanishing and blowup relations on the symbols which are not divisorial:

(V) $(H, Y \circlearrowright K, \beta) = 0$ when $b_1 + b_2 = 0$.

(B) $(H, Y \circlearrowright K, \beta) = \Theta_1 + \Theta_2$, where:

$$\Theta_1 := \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2), & \text{if } b_1 \neq b_2, \end{cases}$$

$$\Theta_2 := \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{Y} \circlearrowright K(x), \bar{\beta}), & \text{otherwise.} \end{cases}$$

Here $\overline{H} := \mathrm{Ker}(b_1 - b_2) \subset H$ and $\bar{\beta}$ is the image of characters of β in $\overline{H}^\vee$; there is also a recipe to produce a $\overline{Y}$-action on $K(x)$, extending the given action of Y (via the canonical homomorphism $\overline{Y} \to Y$) on K, see the **Action construction** in [23, Section 2].

3.4. **Computations.** Let G be abelian. The groups $\mathcal{B}_n(G)$ are defined by finitely many generators and relations and are thus effectively computable. In practice, this is doable for $n \leq 4$ and $|G| < 300$. Such computations allowed to recognize interesting arithmetic and combinatorial structures of $\mathcal{B}_n(G)$: these groups are related to cohomology of congruence subgroups of $\mathrm{GL}_n(\mathbb{Z})$, they carry Hecke operators, admit multiplication and comultiplication, see [20], [21], [22]. Tables for cyclic groups C_m of small order can be found in [20, Section 5].

The groups $\mathcal{BC}_n(G)$ are also finitely generated, with finitely many relations, and thus computable. A structure theorem, [40, Theorem 5.2], provides simplifications in computations of $\mathcal{BC}_n(G)$, by reduction to *modified* $\mathcal{B}_n(H)$, for *abelian* subgroups $H \subseteq G$. For example, for G *abelian*, we proved in [40] that

$$\mathcal{BC}_n(G) = \bigoplus_{H' \subseteq G} \bigoplus_{H'' \subseteq H'} \mathcal{B}_n(H''),$$

and, in particular, there is a surjective homomorphism

$$\mathcal{BC}_n(G) \to \mathcal{B}_n(G).$$

We list $\mathcal{B}_2$, $\mathcal{BC}_2$ and $\mathcal{BC}_3$ for small groups. We start with $G := C_m$.

m	$\mathcal{B}_2(G)$	$\mathcal{BC}_2(G)$	$\mathcal{BC}_3(G)$
2	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}$	0
4	$\mathbb{Z}$	$\mathbb{Z}$	0
5	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
6	$\mathbb{Z}^2 \oplus \mathbb{Z}/2$	$\mathbb{Z}^4 \oplus \mathbb{Z}/2$	0
7	$\mathbb{Z}^3 \oplus \mathbb{Z}/2$	$\mathbb{Z}^3 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$
8	$\mathbb{Z}^3 \oplus \mathbb{Z}/4$	$\mathbb{Z}^5 \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$
9	$\mathbb{Z}^5 \oplus \mathbb{Z}/3$	$\mathbb{Z}^7 \oplus \mathbb{Z}/3$	$\mathbb{Z}$
10	$\mathbb{Z}^4 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/6$	$\mathbb{Z}^8 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/6$	$(\mathbb{Z}/2)^2$
11	$\mathbb{Z}^6 \oplus \mathbb{Z}/5$	$\mathbb{Z}^6 \oplus \mathbb{Z}/5$	$\mathbb{Z} \oplus \mathbb{Z}/5$
12	$\mathbb{Z}^7 \oplus \mathbb{Z}/8$	$\mathbb{Z}^{16} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$	$\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$
13	$\mathbb{Z}^8 \oplus \mathbb{Z}/7$	$\mathbb{Z}^8 \oplus \mathbb{Z}/7$	$\mathbb{Z}^2 \oplus \mathbb{Z}/7$
14	$\mathbb{Z}^7 \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/12$	$\mathbb{Z}^{13} \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/12$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^6$

15	$\mathbb{Z}^{13} \oplus \mathbb{Z}/8$	$\mathbb{Z}^{19} \oplus \mathbb{Z}/8$	$\mathbb{Z}^{5} \oplus \mathbb{Z}/2$
16	$\mathbb{Z}^{10} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/16$	$\mathbb{Z}^{19} \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}/16$	$\mathbb{Z}^{3} \oplus (\mathbb{Z}/2)^7$

The next table concerns $G := C_n \oplus C_m$.

(n,m)	$\mathcal{B}_2(G)$	$\mathcal{BC}_2(G)$	$\mathcal{BC}_3(G)$
$(2,2)$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	0
$(2,4)$	$\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^3$	$\mathbb{Z}^6 \oplus (\mathbb{Z}/2)^7$	$(\mathbb{Z}/2)^3$
$(2,6)$	$\mathbb{Z}^3 \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/4$	$\mathbb{Z}^{20} \oplus (\mathbb{Z}/2)^{14} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2)^9$
$(2,8)$	$\mathbb{Z}^6 \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/8$	$\mathbb{Z}^{30} \oplus (\mathbb{Z}/2)^{18} \oplus (\mathbb{Z}/4)^4 \oplus \mathbb{Z}/8$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^{24}$
$(4,4)$	$\mathbb{Z}^{11} \oplus \mathbb{Z}/2$	$\mathbb{Z}^{41} \oplus (\mathbb{Z}/2)^{29}$	$\mathbb{Z}^5 \oplus (\mathbb{Z}/2)^{31}$
$(3,3)$	$\mathbb{Z}^7$	$\mathbb{Z}^{15}$	$\mathbb{Z}^3$

We also record results for small nonabelian G.

G	$\mathcal{BC}_2(G)$	$\mathcal{BC}_3(G)$
Q_8	$(\mathbb{Z}/2)^3$	0
$\mathfrak{D}_4$	$(\mathbb{Z}/2)^3$	0
$\mathfrak{D}_5$	$(\mathbb{Z}/2)^2$	0
$\mathfrak{A}_5$	$(\mathbb{Z}/2)^3$	0
$\mathfrak{S}_5$	$(\mathbb{Z}/2)^6 \oplus \mathbb{Z}/4$	0
$\mathfrak{D}_6$	$(\mathbb{Z}/2)^5 \oplus \mathbb{Z}/4$	0
$\mathfrak{A}_6$	$(\mathbb{Z}/2)^7 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}$
$\mathfrak{S}_6$	$(\mathbb{Z}/2)^{31} \oplus (\mathbb{Z}/4)^3 \oplus \mathbb{Z}/8$	$(\mathbb{Z}/2)^5 \oplus \mathbb{Z}/4$
$\mathfrak{A}_7$	$(\mathbb{Z}/2)^{12} \oplus (\mathbb{Z}/4)^3 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}^2$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}$
$\mathrm{PSL}_2(\mathbb{F}_7)$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}$	$\mathbb{Z}/2$
$\mathfrak{D}_5 \times \mathfrak{D}_4$	$(\mathbb{Z}/2)^{118} \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/12)^{11} \oplus (\mathbb{Z}/24) \oplus \mathbb{Z}$	$(\mathbb{Z}/2)^{63} \oplus \mathbb{Z}$

In contrast to $\mathcal{B}_n(G)$ and $\mathcal{BC}_n(G)$, the computation of $\mathrm{Burn}_n(G)$ is more difficult. One of the reasons is that the symbols depend on function fields, i.e., algebraic varieties, which have *moduli*. For example, there are 3 types of nonlinearizable involutions in the plane Cremona group Cr_2 (de Jonquières, Geiser, Bertini), fixing curves C of genus ≥ 1, and contributing symbols

$$\mathfrak{s} = (C_2, 1 \circlearrowright k(C), (1)) \in \mathrm{Burn}_2(C_2).$$

Since the conjugacy class of an involution in Cr_2 is uniquely determined by $k(C)$, the symbols $\mathfrak{s}$ parametrize all conjugacy classes of involutions.

In the following sections, we will discuss various approaches to working with $\mathrm{Burn}_n(G)$. There is a natural homomorphism

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G), \tag{3.3}$$

defined by forgetting the field information in each symbol (see [22, Section 8]). Note that it is *not* necessarily surjective (indeed, all point classes in $\mathrm{Burn}_n(G)$ always have trivial Y, while in $\mathcal{BC}_n(G)$, such classes may very well be nontrivial). However, sometimes, this homomorphism allows to distinguish actions by comparing their classes under the homomorphism (3.3), see Section 7, 8 and 9.

3.5. **Tools.** In small dimensions and for small G, we can arrive at simplifications via simple manipulations with defining relations. For reference, we list several such standard operations with symbols, which are independent of the ambient group and will be frequently used.

We consider symbols

$$\mathfrak{s} = (H, Y \circlearrowright K, \beta), \quad \beta = (b_1, \dots, b_{n-d}), \quad K = k(F), \tag{3.4}$$

with small H and Y; geometrically, F is a stratum of dimension d, with generic stabilizer H, see (4.1).

Reduction to point classes: Relation (**B**) implies that if $d \neq n-1$ and $b_1 = b_2$ then

$$\mathfrak{s} = (H, Y \circlearrowright K(x), (b_2, \dots, b_{n-d})), \tag{3.5}$$

with trivial Y-action on x. In particular, every symbol as in (3.4) with $Y = 1$ and $F = \mathbb{P}^d$ can be reduced to a point symbol.

Vanishing: Relation (**V**) implies (see, [23, Proposition 4.7]) that $\mathfrak{s}$ vanishes, provided

$$\sum_{i \in I} b_i = 0 \in H^\vee, \text{ for some } I \subseteq [1, \dots, n-d]. \tag{3.6}$$

Cyclic stabilizers: First recall that the weight r of a character χ_r of a cyclic group $H = C_m$ is defined up to modulo m. Namely, one has $\chi_r = \chi_{r-m}$ for $0 \le r \le m-1$. We now explain many situations where the symbol $\mathfrak{s}$ defined in (3.4) with small cyclic H vanishes:

- $H = C_2$: If β contains more than one entry, $\mathfrak{s} = 0 \in \mathrm{Burn}_n(G)$, by (**V**). Now assume that $\mathfrak{s}$ is a divisorial symbol, and
$$F \sim_Y F' \times \mathbb{P}^1,$$
with F as in (3.4), some F', and with trivial action on the second factor. By (3.5) and (**V**),
$$\mathfrak{s} = (C_2, Y \circlearrowright k(F'), (1,1)) = 0.$$
- $H = C_3$: The symbol $\mathfrak{s}$ vanishes, if its codimension is ≥ 3, by (3.6). Together with (**B**) this implies
$$(C_3, 1 \circlearrowright K, \beta) = 0 \in \mathrm{Burn}_n, \quad \text{for } n \geq 3.$$
For some G, the symbol can be nontrivial, i.e., in $\mathrm{Burn}_2(C_3)$. On the other hand, if there is a $C_6 \subset G$ centralizing H, then it supplies additional relations, leading to additional vanishing. For example, we have
$$\begin{aligned}(C_3, C_2 \circlearrowright k(\mathbb{P}^1), (1,1)) =& (C_6, 1 \circlearrowright k, (1,4,1)) - (C_6, 1 \circlearrowright k, (1,3,1)) \\ & - (C_6, 1 \circlearrowright k, (3,4,1)) \\ =& - (C_6, 1 \circlearrowright k, (1,3,1)) - (C_6, 1 \circlearrowright k, (5,4,1)) \\ & - (C_6, 1 \circlearrowright k, (3,1,1)) \\ =& - 2(C_6, 1 \circlearrowright k, (3,1,1)) \\ =& - 2(C_6, 1 \circlearrowright k(\mathbb{P}^1), (1,3)) \quad \text{by } \mathbf{(B)} \\ =& - 2(C_6, 1 \circlearrowright k, (1,3,3)) = 0 \in \mathrm{Burn}_3(G).\end{aligned}$$
Similarly,
$$\begin{aligned}(C_3, C_2 \circlearrowright k(\mathbb{P}^1), (2,2)) =& (C_6, 1 \circlearrowright k, (2,5,5)) - (C_6, 1 \circlearrowright k, (2,3,5)) \\ & - (C_6, 1 \circlearrowright k, (3,5,5)) \\ =& - (C_6, 1 \circlearrowright k, (2,1,5)) - (C_6, 1 \circlearrowright k, (5,3,5)) \\ & - (C_6, 1 \circlearrowright k, (3,5,5)) = 0 \\ =& - 2(C_6, 1 \circlearrowright k, (5,3,3)) = 0 \in \mathrm{Burn}_3(G).\end{aligned}$$
- $H := C_4$: Consider point symbols for $n = 3$. There are only two potentially nontrivial symbols
$$(C_4, 1 \circlearrowright k, (1,1,1)), \quad (C_4, 1 \circlearrowright k, (3,3,3)), \tag{3.7}$$
indeed, in all other cases, we obtain vanishing of the symbol from relation (**V**). Using (3.6), we derive
$$0 = (C_4, 1 \circlearrowright k, (1,2,1)) = (C_4, 1 \circlearrowright k, (3,2,1)) + (C_4, 1 \circlearrowright k, (1,1,1)),$$

$$0 = (C_4, 1 \circlearrowright k, (3,2,3)) = (C_4, 1 \circlearrowright k, (1,2,3)) + (C_4, 1 \circlearrowright k, (3,3,3)),$$

and thus the two symbols in (3.7) vanishes.

- $H = C_5$: All symbols

$$(C_5, 1 \circlearrowright k(\mathbb{P}^d), \beta) \in \mathrm{Burn}_n(G), \quad n \geq 2,$$

reduce to point classes. Let $n = 3$ and order $b_1 \leq b_2 \leq b_3$, using **(O)**. Potentially nonvanishing generators are:

$$(C_5, 1 \circlearrowright k, (i,i,i)), i = 1, \dots, 4, \quad (C_5, 1 \circlearrowright k, (1,2,1)),$$

and turn to relations:

$$(C_5, 1 \circlearrowright k, (1,2,1)) = (C_5, 1 \circlearrowright k, (4,2,1)) + (C_5, 1 \circlearrowright k, (1,1,1)).$$

Relation **(V)** then implies

$$(C_5, 1 \circlearrowright k, (1,2,1)) = (C_5, 1 \circlearrowright k, (1,1,1)).$$

On the other hand, we have

$$(C_5, 1 \circlearrowright k, (1,2,1)) = (C_5, 1 \circlearrowright k(\mathbb{P}^1), (1,2)) = (C_5, 1 \circlearrowright k, (1,2,2)) = 0.$$

The same argument shows the vanishing of all other generators.

To summarize, we have:

Lemma 3.1. *Let G be a finite group and $n \geq 3$. Every point class in* $\mathrm{Burn}_n(G)$*, with stabilizer $H = C_m \subset G$ and $m \leq 6$ is trivial.*

Proof. It suffices to prove this for $n = 3$. We already dealt with $m = 2, 3, 4, 5$. When $m = 6$, Θ_2-terms in the blow-up relations come from:

$$(C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1,1)) = 0,$$
$$(C_3, C_2 \circlearrowright k(\mathbb{P}^1), (1,2)) = 0,$$
$$(C_3, C_2 \circlearrowright k(\mathbb{P}^1), (\pm 1, \pm 1)).$$

We prove that the last symbols are also zero in $\mathrm{Burn}_3(G)$. By the blowup relation **(B)**, we know

$$\begin{aligned} 0 &= (C_3, C_2 \circlearrowright k(\mathbb{P}^1), (1,2)) \\ &= (C_3, C_2 \circlearrowright k(\mathbb{P}^1), (2,2)) + (C_3, C_2 \circlearrowright k(\mathbb{P}^1), (1,1)). \end{aligned}$$

The vanishing of the last two symbols was showed above in the case of cyclic stabilizer with $H = C_3$.

For compactness, for point classes, we will use the notation

$$(b_1, b_2, b_3) = (C_6, 1 \circlearrowright k(\mathbb{P}^1), (b_1, b_2, b_3)).$$

Applying (**B**), we obtain

$$0 = (1,4,1) = (3,4,1) + (1,3,1) + (C_3, C_2 \circlearrowright k(\mathbb{P}^1), (1,1)).$$

Similarly,

$$(3,4,1) \stackrel{\textbf{(O)}}{=} (1,4,3) = (3,4,3) + (1,3,3) + \Theta_2 = 0,$$

since all terms on the right vanish, by (**V**) and the fact that

$$b_3 \in \langle b_1 - b_2 \rangle, \quad b_3 = 3, b_1 = 1, b_2 = 4.$$

We now have

$$0 = (3,4,1) = (5,4,1) + (3,1,1). \tag{3.8}$$

Thus, all Θ_2 terms vanish.

Next, note that once we know that $(b_1, b_2, b_3) = 0$ then the same relations, applied to negatives, yield $(-b_1, -b_2, -b_3) = 0$ as well. Thus we need to prove the vanishing of the non-boldface symbols in the following sequence of relations, which we apply in the given sequence; in bold we have indicated the terms that vanish by **(V)**, by previous identities, or by sign change on previously obtained vanishing symbols:

$$\begin{aligned}
(\mathbf{1},\mathbf{2},\mathbf{3}) &= (\mathbf{2},\mathbf{3},\mathbf{1}) = (\mathbf{5},\mathbf{3},\mathbf{1}) + (2,1,1)\\
(\mathbf{1},\mathbf{1},\mathbf{2}) &= (\mathbf{2},\mathbf{1},\mathbf{1}) = (\mathbf{2},\mathbf{5},\mathbf{1}) + (1,1,1)\\
(\mathbf{1},\mathbf{2},\mathbf{3}) &= (\mathbf{1},\mathbf{3},\mathbf{2}) = (\mathbf{4},\mathbf{3},\mathbf{2}) + (1,2,2)\\
(1,3,4) &= (1,4,3) = (\mathbf{3},\mathbf{4},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{3})\\
(1,1,3) &= (1,3,1) = (\mathbf{4},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1})\\
(2,2,3) &= (2,3,2) = (\mathbf{5},\mathbf{3},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{2})\\
(1,4,4) &= (\mathbf{3},\mathbf{4},\mathbf{4}) + (\mathbf{1},\mathbf{3},\mathbf{4})
\end{aligned}$$

□

3.6. **Incompressibles.** For $n = 1$, there are no relations, with the exception of the conjugation relation (**C**), i.e., $\mathrm{Burn}_1(G)$ is the free abelian group spanned by symbols

$$(H, 1 \circlearrowright k, (b_1)),$$

where $H \subseteq G$ is a *cyclic* subgroup (up to conjugation).

In dimensions $n \geq 2$, we call a divisorial symbol *incompressible* if it *does not* appear in the Θ_2-term of any relation (**B**). Geometrically, this means that the corresponding divisor F in (3.4), with the indicated Y-action, is not equivariantly birational to what could arise, as an exceptional divisor, via a blowup of a *standard* model from smaller-dimensional strata with nontrivial stabilizer. In particular, such divisorial symbols do not participate in *any* relations, except in the conjugation relation **(C)**. We have

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_n^{\mathrm{triv}}(G) \oplus \mathrm{Burn}_n^{\mathrm{inc}}(G) \oplus \mathrm{Burn}_n^{\mathrm{comp}}(G), \tag{3.9}$$

where

- $\mathrm{Burn}_n^{\mathrm{triv}}(G)$ is freely spanned by symbols
$$(1, G \circlearrowright K, ()),$$
where K is a field of transcendence degree n, with a generically free action of G;
- $\mathrm{Burn}_n^{\mathrm{inc}}(G)$ is freely spanned by *incompressible divisorial symbols*, modulo conjugation; and
- the third summand is generated by all other symbols, subject to relations in Section 3.3 (see [24, Proposition 3.4]).

In some examples, the presence of incompressible symbols already allows to distinguish birational types of actions, greatly simplifying the arguments (see Section 7). In other examples, one has to perform computations in $\mathrm{Burn}_n^{\mathrm{comp}}(G)$.

Recall that, for $n = 2$, we have

- *point* classes, i.e., $K = k$ and $\beta = (b_1, b_2)$,
- *divisorial classes*:
  - classes of rational curves, i.e., $K = k(x)$, $\beta = (b_1)$, and Y cyclic,
  - classes of rational curves, with $\beta = (b_1)$, and Y noncyclic,
  - classes of curves of genus ≥ 1, i.e., those where $K = k(C)$, and C is a curve of genus ≥ 1.

The *incompressible* divisorial symbols correspond to the last two cases, see [24, Proposition 3.6]

Example 3.2. For $G = C_3$, we have

$$\mathrm{Burn}_2^{\mathrm{comp}}(C_3) = \mathbb{Z}.$$

Indeed, the generators are symbols

$$(C_3, 1 \circlearrowleft k, (1,2)),$$
$$(C_3, 1 \circlearrowleft k, (1,1)),$$
$$(C_3, 1 \circlearrowleft k, (2,2)),$$
$$(C_3, 1 \circlearrowleft k(\mathbb{P}^1), (1)),$$
$$(C_3, 1 \circlearrowleft k(\mathbb{P}^1), (2)),$$

subject to relations

$$(C_3, 1 \circlearrowleft k, (1,2)) \stackrel{\textbf{(V)}}{=} 0,$$
$$(C_3, 1 \circlearrowleft k, (1,2)) \stackrel{\textbf{(B)}}{=} (C_3, 1 \circlearrowleft k, (1,1)) + (C_3, 1 \circlearrowleft k, (2,2)),$$
$$(C_3, 1 \circlearrowleft k, (2,2)) \stackrel{\textbf{(B)}}{=} (C_3, 1 \circlearrowleft k(\mathbb{P}^1), (2)),$$
$$(C_3, 1 \circlearrowleft k, (1,1)) \stackrel{\textbf{(B)}}{=} (C_3, 1 \circlearrowleft k(\mathbb{P}^1), (1)).$$

Thus, $\mathrm{Burn}_2^{\mathrm{comp}}(C_3) = \mathbb{Z}$ is freely generated by $(C_3, 1 \circlearrowleft k, (1,1))$.

The table below shows the structure of $\mathrm{Burn}_2^{\mathrm{comp}}(G)$ for $G = C_m$:

m	$\mathrm{Burn}_2^{\mathrm{comp}}(G)$
2	0
3	$\mathbb{Z}$
4	$\mathbb{Z}^2$
5	$\mathbb{Z}^2$
6	$\mathbb{Z}^6$
7	$\mathbb{Z}^3 \oplus \mathbb{Z}/2$
8	$\mathbb{Z}^8 \oplus \mathbb{Z}/2$
9	$\mathbb{Z}^8 \oplus \mathbb{Z}/3$
10	$\mathbb{Z}^{11} \oplus \mathbb{Z}/3$
11	$\mathbb{Z}^6 \oplus \mathbb{Z}/5$
12	$\mathbb{Z}^{22} \oplus \mathbb{Z}/4$
13	$\mathbb{Z}^8 \oplus \mathbb{Z}/7$
14	$\mathbb{Z}^{17} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6$
15	$\mathbb{Z}^{22} \oplus \mathbb{Z}/8$
16	$\mathbb{Z}^{25} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$

The analysis of incompressible divisorial symbols

$$\bar{\mathfrak{s}} = (\bar{H}, \bar{Y} \circlearrowleft k(D), (\bar{b}_1) \in \mathrm{Burn}_3(G),$$

in dimensions $n \geq 3$ is more involved. We have not attempted a full classification. The definition of "incompressibility" of a symbol depends on the group theoretical information of the ambient group G, see Example 3.3. But in some cases, the geometric information of D governs the incompressibility – for example, in the following cases, $\bar{\mathfrak{s}}$ is an incompressible symbol regardless of what is the ambient group G:

- D is not uniruled,
- D is $\bar{Y}$-solid, i.e., not $\bar{Y}$-birational to a $\bar{Y}$-equivariant Mori fiber space over a positive-dimensional base (see [6] for a detailed study of solid *toric* varieties in dimension ≤ 3),
- $n = 3$ and D is a rational surface which is not $\bar{Y}$-equivariantly birational to a Hirzebruch surface, see [13] for a classification of such actions.

How to tell whether or not a symbol

$$\bar{\mathfrak{s}} := (\bar{H}, \bar{Y} \circlearrowright \bar{K}, \beta) \in \mathrm{Burn}_n(G) \tag{3.10}$$

is incompressible, in practice? A *necessary* condition is that

$$\bar{K} \not\sim_{\bar{Y}} \bar{K}'(x),$$

for some function field $\bar{K}'$, with trivial action of $\bar{Y}$ on x; such symbols arise via blowup relations from symbols where some characters in β have multiplicity ≥ 2. The next steps, after verifying this condition, are:

(1) List all conjugacy classes of abelian subgroups $H \subseteq G$, together with their centralizers $Z_G(H)$.
(2) For each H enumerate all nontrivial proper subgroups $H' \subsetneq H$. List all subgroups
$$Y' \subseteq Z_G(H)/H'.$$
(3) If there is no (H', Y') conjugated to $(\bar{H}, \bar{Y})$ then $\bar{\mathfrak{s}}$ is incompressible.
(4) If there is such a pair, one needs to analyze in detail whether or not the **Action construction** in [23, Section 2] can produce, birationally, the given action $\bar{Y} \circlearrowright \bar{K}$.

Example 3.3. Let $n = 3$ and G a finite group. Consider a symbol

$$\bar{\mathfrak{s}} = (C_2, C_2 \times C_2 \circlearrowright \bar{K}, (1)) \in \mathrm{Burn}_3(G)$$

where $\bar{K} = k(\mathbb{P}^2)$ and $\bar{Y} = C_2 \times C_2$ acts *linearly* on $\mathbb{P}^2$, in particular, with fixed points.

When $G = Q_8$, $\bar{\mathfrak{s}}$ is incompressible. There are 4 conjugacy classes of nontrivial abelian subgroups of G, one C_2, with centralizer G, and three C_4, with centralizer itself. Such an action is not birational to an action of $\bar{Y}$ on $\mathbb{P}^1 \times \mathbb{P}^1$, with trivial action on the second factor. By Step 2 above, such $\bar{Y} = C_2 \times C_2$ does not arise.

On the other hand, when $G = C_2^3$, the symbol $\bar{\mathfrak{s}}$ is compressible – applying the blowup relation **(B)** to the symbol

$$(C_2^2, C_2 \circlearrowright k(\mathbb{P}^1), ((1,0),(0,1))), \tag{3.11}$$

the Θ_2 symbol arising is

$$\Theta_2 = (C_2, C_2^2 \circlearrowright k(\mathbb{P}^1 \times \mathbb{P}^1), ((1,0),(0,1))),$$

with each factor of C_2^2 acts faithfully on each factor of $\mathbb{P}^1 \times \mathbb{P}^1$. This action has a fixed point on $\mathbb{P}^1 \times \mathbb{P}^1$ and is thus linearizable, i.e., birational to the linear C_2^2-action on $\mathbb{P}^2$. We then conclude $\bar{\mathfrak{s}} = \Theta_2$ and $\bar{\mathfrak{s}}$ is a compressibble symbol.

Remark 3.4. When the stabilizer group H is an abelian group with k generators, we identify the character group $H^\vee$ with the group H itself, and denote a character by a k-tuple of weights. For example, in the symbol (3.11),

$$(1,0), \quad (0,1)$$

represents two characters of C_2^2. We will use this notation frequently when displaying computations of symbols in the remaining sections.

3.7. **MRC quotients.** In the key relation **(B)**

$$\mathfrak{s} = (H, Y \circlearrowright k(F), \beta) = \Theta_1 + \Theta_2,$$

the term Θ_2 is a symbol with the function field $k(F)(x)$. Note that $k(F)$ is the function field of a rationally connected (RC) variety iff this holds for $K(F)(x)$. In fact, in any given relation, all appearing terms have the same dimension of the MRC quotient $r = r(F)$. This yields a direct sum decomposition

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_n^{\mathrm{triv}}(G) \oplus \mathrm{Burn}_n^{\mathrm{rc}}(G) \oplus \bigoplus_{r=1}^{n-1} \mathrm{Burn}_n^{\mathrm{nrc,r}}(G), \tag{3.12}$$

where

- $\mathrm{Burn}_n^{\mathrm{triv}}(G)$ is freely spanned by symbols with $H = 1$,
- $\mathrm{Burn}_n^{\mathrm{rc}}(G)$ is generated by symbols $\mathfrak{s}$ with $H \neq 1$, and fields $K = k(F)$, where F is a rationally connected variety, and

- $\mathrm{Burn}_n^{\mathrm{nrc,r}}(G)$ is generated by symbols with $H \neq 1$ and $K = k(F)$ the function field of a variety whose MRC quotient has dimension r.

Different summands in this decomposition could have nontrivial intersection with $\mathrm{Burn}_n^{\mathrm{inc}}(G)$, the incompressible divisorial symbols.

3.8. H^1**-triviality.** Further decompositions of $\mathrm{Burn}_n(G)$ can be obtained by realizing that relation (**B**) preserves

$$\mathrm{H}^1(Y', \mathrm{Pic}(F)), \quad Y' \subseteq Y,$$

where F is a smooth projective model of the function field in the symbol $\mathfrak{s}$. In particular, we have

$$\mathrm{Burn}_n^{\mathrm{rc}}(G) = \mathrm{Burn}_n^{\mathrm{rc,H1=0}}(G) \oplus \mathrm{Burn}_n^{\mathrm{rc,H1\neq 0}}(G),$$

depending on the (non)triviality of the H^1-condition (see Section 3).

Lemma 3.5. *If* $\bar{\mathfrak{s}} \in \mathrm{Burn}_3^{\mathrm{rc}}(G)$ *is a compressible divisorial symbol then*

$$\bar{\mathfrak{s}} \in \mathrm{Burn}_3^{\mathrm{rc,H1=0}}(G).$$

Proof. Indeed, it can only arise from a symbol

$$\mathfrak{s} = (H, Y \circlearrowright k(\mathbb{P}^1), \beta)$$

which is H^1-trivial. □

4. Computing the classes

We recall the definition of the class of a generically free G-action on a smooth projective variety X.

We assume that X is in *standard form*, i.e., there is an open subset $U \subset X$ where the G-action is free, with complement $X \setminus U$ a normal crossings divisor such that for its every component D and all $g \in G$, we have $(D \cdot g) \cap D$ is either empty or all of D, see [14, Section 7.2] for more details. Such a model of the function field K can always be obtained via equivariant blowups with smooth centers, and every further blowup of such a model is also in standard form. One of its features is that all stabilizers are *abelian*, see [14, Section 7.2] and [32, Theorem 4.1]. By definition, the class of such an action

$$[X \circlearrowleft G] := \sum_H \sum_F (H, Y \circlearrowright k(F), \beta_F) \in \mathrm{Burn}_n(G) \tag{4.1}$$

is a sum over conjugacy classes of stabilizers H of maximal strata $F \subseteq X$ with these stabilizers, with the induced action of a subgroup $Y \subset Z_G(H)/H$ on the corresponding function field; β_F is the sequence of weights of H in the normal bundle to F. In other words, the symbol records *one* representative of a G-orbit of a (maximal) stratum with stabilizer H: changing a component in this G-orbit conjugates the stabilizer by an element $g \in G$, the action on that component, and the induced action in the normal bundle to that component; this is reflected in the conjugation relation (**C**).

The sum (4.1) contains a distinguished summand,

$$(1, G \circlearrowright k(X), ()) \in \mathrm{Burn}_n^{\mathrm{triv}}(G)$$

reflecting the G-action on the generic point of X. Of course, there can be actions where there are no other summands in (4.1), e.g., a translation action on an elliptic curve. In such cases, the Burnside group formalism provides *no* information about the G-action. On the other hand, we will exhibit many examples, where the actions can be distinguished via images of the corresponding classes under projections to $\mathrm{Burn}_n^{\mathrm{inc}}(G)$ or $\mathrm{Burn}_n^{\mathrm{comp}}(G)$.

We note that incompressible divisorial symbols can be read off from *any* equivariant birational model, even one which is not in standard form. It is typically a nontrivial task to find a standard model. Indeed, a linear representation V of a nonabelian group G, and its equivariant compactification $\mathbb{P}(1 \oplus V)$, where 1 is the trivial representation, by definition have strata with nonabelian stabilizers, e.g., the origin of V; and one may have to blow up several times to reach abelian stabilizers. In [24] it was shown that a G-equivariant version of De Concini–Procesi compactifications of subspace arrangements provides a standard model for the G-action on $\mathbb{P}(V)$; here the relevant subspaces in $\mathbb{P}(V)$ correspond to loci with nontrivial stabilizers. We illustrate this in Section 6. A similar algorithm for actions on toric varieties was presented in [25].

Next, assume we are given different G-actions, presented on X and X', which are both in standard form. To distinguish these, one expresses the classes as in (4.1), and considers the projection of the difference

$$[X \circlearrowleft G] - [X' \circlearrowleft G]$$

to

$$\mathrm{Burn}_n^{\mathrm{inc}}(G).$$

Since there are no blowup relations between symbols in that group, it is easy to see whether or not this difference vanishes; see Corollary 7.7.

If the difference does vanish in this group, we can consider projections to other direct summands introduced in Sections 3.6, 3.7, and 3.8

$$\mathrm{Burn}_n^{\mathrm{comp}}(G), \quad \mathrm{Burn}_n^{\mathrm{rc}}(G), \ldots$$

As mentioned in Section 3, these groups are harder to compute, in general. One of the main difficulties is that one has to keep track of infinitely many generating symbols, and of relations that are implied by (often nontrivial) stable equivariant birationalities. For example, by the No-name Lemma, any two faithful G-representations are stably equivariantly birational, but not necessarily equivariantly birational. Further examples of such stable equivariant birationalities can be found in [16]. In some cases, we are able to overcome this intrinsic difficulty by passing to the combinatorial Burnside group $\mathcal{BC}_n(G)$, via (3.3). We have implemented algorithms checking nonvanishing of any given class in $\mathcal{BC}_n(G)$, for all $n \geq 2$; however, these are practical only for small n.

In the following sections, we will apply this machinery to

- (projectively) linear actions on $\mathbb{P}^n$, with $G \subset \mathrm{PGL}_{n+1}$, $n \leq 3$,
- smooth quadric hypersurfaces $X \subset \mathbb{P}^n$, $n \leq 4$.

5. Linear actions in dimension one

We recall the well-known list of finite $G \subset \mathrm{PGL}_2$:

$$C_m, \mathfrak{D}_m, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5,$$

where C_m is the cyclic group of order m and $\mathfrak{D}_m$ is the dihedral group of order $2m$. The corresponding actions on $\mathbb{P}^1$ are linear if and only if G is cyclic, or dihedral with m odd.

The classification of birational actions on $\mathbb{P}^1$ is straightforward: two G-actions on $\mathbb{P}^1$ are equivariantly birational if and only if the corresponding representations V are projectively equivalent, i.e., conjugated in PGL_2. In detail:

- $G = C_m$: the action arises via a representations of the form $\mathbb{P}(1 \oplus \epsilon)$, where ϵ is a primitive character of G; given ϵ, ϵ', birationality of the corresponding G-actions holds if and only if $\epsilon = \pm\epsilon'$.
- $G = \mathfrak{D}_m$: when m is odd, G acts on $\mathbb{P}^1$ via a faithful two-dimensional representations of $\mathfrak{D}_m$; when m is even, G acts via a faithful two-dimensional representations of $\mathfrak{D}_{2m}$. Two such actions are birational if and only if their restrictions to the subgroup $C_m \subseteq \mathfrak{D}_m$ induce birational actions of C_m on $\mathbb{P}^1$.
- $G = \mathfrak{A}_4$: the actions arise from faithful two-dimensional representations of $\mathrm{SL}_2(\mathbb{F}_3)$, all of which are projectively equivalent. So $\mathfrak{A}_4$ admits a unique action on $\mathbb{P}^1$.

- $G = \mathfrak{S}_4$: the actions arise from faithful two dimensional representations of $\mathrm{GL}_2(\mathbb{F}_3)$, all of which are projectively equivalent. So $\mathfrak{S}_4$ also admits a unique action on $\mathbb{P}^1$.
- $G = \mathfrak{A}_5$: the actions arise from faithful two-dimensional representations of $\mathrm{SL}_2(\mathbb{F}_5)$. There are two such representations, inducing two non-isomorphic actions of $\mathfrak{A}_5$ on $\mathbb{P}^1$ after projectivization. So $\mathfrak{A}_5$ admits two non-birational actions.

Note that in dimension 1, non-birational actions of cyclic groups can be distinguished by the Reichstein-Youssin invariant **(RY)** [33]: when C_m acts on $\mathbb{P}^1$ via a character χ, the action is determined by $\pm\chi$.

In applications to nonabelian groups, we can consider determinants of actions upon restrictions to their abelian subgroups, e.g., for G dihedral. For $G = \mathfrak{A}_5$, the two non-birational actions can also be distinguished already via restriction to $C_5 \subset G$: in one case, we obtain

$$(C_5, 1 \circlearrowright k, (1)) + (C_5, 1 \circlearrowright k, (4))$$

and in the other

$$(C_5, 1 \circlearrowright k, (2)) + (C_5, 1 \circlearrowright k, (3)),$$

and these are different in $\mathrm{Burn}_1(G)$.

Proposition 5.1. *The birational type of the action of a finite group G on $\mathbb{P}^1$ is uniquely determined by*

$$[\mathbb{P}^1 \circlearrowleft G] \in \mathrm{Burn}_1(G).$$

6. Computing the classes of linear actions

The computation of classes in the Burnside group of (projectively) linear actions in dimensions ≥ 2 is more involved. Given a faithful linear representation $G \to \mathrm{GL}(V^\vee)$ we obtain a faithful projective representation $G/C \to \mathrm{PGL}(V^\vee)$, where $C \subset G$ is the maximal (cyclic) subgroup acting via scalar matrices. An algorithm to compute the class

$$[\mathbb{P}(V) \circlearrowleft G/C] \in \mathrm{Burn}_n(G/C)$$

of the induced action of G/C on $\mathbb{P}^n$ was developed in [24], and implemented in [39]. It is based on an equivariant version of the De Concini–Procesi approach to wonderful compactifications of subspace arrangements [12], which provides a systematic way of turning any given projectively linear action into a standard form. We note that

- all symbols produced and appearing as summands in
$$[\mathbb{P}(V) \circlearrowleft G] = \sum_H \sum_F (H, Y \curvearrowright k(F), \beta_F),$$
are in
$$\mathrm{Burn}_n^{\mathrm{rc}}(G),$$
see (3.12), and
- all actions $Y \curvearrowright k(F)$ are equivariantly birational to products of projectively linear actions on projective spaces, without permutation of the factors (see Corollary 6.1).

We explain the main ideas below, supplemented with two examples (our notation follows the one in [24]). First, consider pairs

$$(\Gamma, \epsilon), \quad C \subseteq \Gamma \subseteq G, \quad \epsilon \in \mathrm{Hom}(\Gamma, k^\times), \tag{6.1}$$

where Γ is the generic stabilizer group of some one-dimensional subspace $\ell \subset V$ and ϵ is the character of Γ given by its action on ℓ; we are using the identification $\Gamma \to \mathrm{GL}(\ell^\vee) \simeq k^\times$. Then Γ/C stabilizes the point $\mathbb{P}(\ell) \in \mathbb{P}(V)$. The set

$$\bar{\mathcal{L}} = \bar{\mathcal{L}}(V) := \{\text{pairs } (\Gamma, \epsilon) \text{ as above}\} \cup \{\infty\}$$

carries information about the subspace arrangement. In particular, we associate to every pair $(\Gamma, \epsilon) \in \bar{\mathcal{L}}$ the subspace

$$V_{\Gamma,\epsilon} := \{v \in V \mid v \cdot g = \epsilon(g)v, \text{ for all } g \in \Gamma\}.$$

The De Concini–Procesi model $\mathbb{P}(V)_{\bar{\mathcal{L}}}$ is defined as the closure of the image of the natural map

$$\mathbb{P}(V)^\circ \to \mathbb{P}(V) \times \prod_{\substack{(\Gamma,\epsilon)\in\bar{\mathcal{L}} \\ \Gamma \neq C}} \mathbb{P}(V/V_{\Gamma,\epsilon}),$$

where the $\mathbb{P}(V)^\circ$ is the complement in $\mathbb{P}(V)$ of the union of all proper subspaces of the form $\mathbb{P}(V_{\Gamma,\epsilon})$. The natural projection

$$\mathbb{P}(V)_{\bar{\mathcal{L}}} \to \mathbb{P}(V)$$

is an isomorphism on $\mathbb{P}(V)^\circ$, whose complement in $\mathbb{P}(V)_{\bar{\mathcal{L}}}$ is a normal crossings divisor. It is shown in [24, Proposition 7.2] that the G-action on $\mathbb{P}(V)_{\bar{\mathcal{L}}}$ is in standard form with respect to this divisor. We now describe the main steps of the algorithm.

Input. A faithful linear representation $G \to \mathrm{GL}(V^\vee)$.

Step 1. Find C and $\bar{\mathcal{L}} = \bar{\mathcal{L}}(V)$, i.e., all possible pairs (Γ, ϵ) as in (6.1).

Step 2. Find all chains of subspaces, up to conjugation by G,

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_t \subsetneq V$$

such that

- $V_i = V_{\Gamma^i,\epsilon}$ for some pair $(\Gamma^i, \epsilon) \in \bar{\mathcal{L}}$ with $\Gamma^i \neq C$, for every $i = 1, \ldots, t$ and a common character ϵ,
- Γ^i is the (maximal) stabilizer group of V_i.

Associated with each chain of subspaces is a chain of stabilizer groups,

$$\Lambda := \Gamma^1 \supsetneq \Gamma^2 \supsetneq \cdots \supsetneq \Gamma^t,$$

and a character ϵ of Γ^1.

Step 3. For each conjugacy class of chains of subspaces $V_1 \subsetneq \cdots \subsetneq V_t$ and the corresponding chain of stabilizers Λ, find

- $N_G(\Lambda) \subseteq G$, the intersection of normalizers of Γ^i in G which stabilize ϵ, this is the stabilizer of Λ.
- Δ^t_Λ, the maximal subgroup of $N_G(\Lambda)$ acting via scalars on all V_{i+1}/V_i.

The input representation induces a faithful representation of $N_G(\Lambda)$ on

$$V_1^\vee \times (V_2/V_1)^\vee \times (V_3/V_2)^\vee \times \cdots \times (V/V_t)^\vee,$$

where Δ^t_Λ acts via scalars on each factor; we record characters ϵ^i of Δ^t_Λ on V_{i+1}/V_i, $i = 0, \ldots, t$. By convention, $V_0 = 0$ and $V_{t+1} = V$.

Step 4. For each conjugacy class of chains, compute the *intermediate class*

$$[\mathbb{P}(V_1) \times \mathbb{P}(V_2/V_1) \times \ldots \times \mathbb{P}(V/V_t) \circlearrowleft N_G(\Lambda)]_{(\mathcal{O}(-1))}$$

of the induced action of $N_G(\Lambda)$, with respect to $(\mathcal{O}(-1))$, a sequence of line bundles below

$$\mathcal{O}_{\mathbb{P}(V_1)}(-1), \mathcal{O}_{\mathbb{P}(V_1)}(1) \otimes \mathcal{O}_{\mathbb{P}(V_2/V_1)}(-1), \mathcal{O}_{\mathbb{P}(V_2/V_1)}(1) \otimes \mathcal{O}_{\mathbb{P}(V_3/V_2)}(-1), \ldots.$$

This intermediate class takes values in

$$\mathrm{Burn}_{n,\{0,\ldots,t\}}(N_G(\Lambda), \Delta^t_\Lambda),$$

the equivariant indexed Burnside group with respect to line bundles $(\mathcal{O}(-1))$, defined in [24, Section 4 and Section 5] (the definition is of this group is notationally heavy, it involves *two* abelian subgroups $H \subseteq H'$ of G and *two* sequences of characters). Since the De Concini–Procesi model satisfies the conditions in [24, Lemma 5.1], we can compute the intermediate class by [24, Definition 5.3].

Step 5. A recursive formula [24, Proposition 8.3 and Theorem 8.4] allows to compute the class

$$[\mathbb{P}(V) \circlearrowleft G]_{(\mathcal{O}_{\mathbb{P}(V)}(-1))} \in \mathrm{Burn}_{n,\{0\}}(G, C)$$

using all intermediate classes of chains found in Step 2. Essentially, this formula reflects contributions to the class from the intersections of various strata with nontrivial stabilizers, on an appropriate model. Apply this recursion to obtain the class, taking values in the equivariant indexed Burnside group with respect to the line bundles $(\mathcal{O}_{\mathbb{P}(V)}(-1))$.

Step 6. Apply the surjective homomorphism

$$\eta_{\{0\}} : \mathrm{Burn}_{n,\{0\}}(G, C) \to \mathrm{Burn}_n(G/C),$$

defined by

$$(C \subseteq H', Z' \curvearrowright K, \beta, \gamma) \mapsto (H'/C, Z' \curvearrowright K, \beta).$$

By [24, Theorem 8.5], we have

$$[\mathbb{P}(V) \circlearrowleft G/C] = \eta_{\{0\}} \left([\mathbb{P}(V) \circlearrowleft G]_{(\mathcal{O}_{\mathbb{P}(V)}(-1))} \right).$$

Output. The class

$$[\mathbb{P}(V) \circlearrowleft G/C] \in \mathrm{Burn}_n(G/C)$$

is presented as a finite sum of symbols in $\mathrm{Symb}_n(G)$.

As already noted, an important observation is:

Corollary 6.1. *Every symbol $\mathfrak{s}$ appearing as a summand in the class*

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_n(G),$$

via the algorithm from [23] *is of the shape*

$$\mathfrak{s} = (H, Y \curvearrowright k(F), \beta),$$

where

- *F is birational to $\prod_j \mathbb{P}(W_j)$,*
- *$Y \subseteq Z_G(H)/H$ acts without interchanging the factors, and*
- *the action on each factor is (birational) to a (projectively) linear action.*

In particular,

$$[\mathbb{P}(V) \circlearrowleft G] \in \mathrm{Burn}_n^{\mathrm{rc,H1=0}}(G),$$

(see Section 3.8).

An example computation, for $G = \mathfrak{S}_4$, acting on $\mathbb{P}^2 = \mathbb{P}(V)$, where V is the standard 3-dimensional representation of $\mathfrak{S}_4$, can be found in [24, Section 9]. Here, we provide new examples, in dimensions 2 and 3.

Example 6.2. Let $G = C_3 \times \mathfrak{D}_5$ acting on $\mathbb{P}^2 = \mathbb{P}(1 \oplus V_\epsilon)$; here

$$V_\epsilon := \epsilon \otimes V$$

is the twist by a nontrivial character of C_3 of the standard 2-dimensional representation of $\mathfrak{D}_5$, with generators acting via

$$\begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We tabulate the relevant information for conjugacy classes of chains of stabilizer groups from Steps 1, 2 and 3.

t	Λ	$N_G(\Lambda)$	Δ_Λ^t	ϵ^i
1	$C_3 \times \mathfrak{D}_5$	$C_3 \times \mathfrak{D}_5$	C_3	0
1	C_{15}	C_{15}	trivial	–
1	C_6	C_6	C_2	1
1	C_6	C_6	trivial	–
1	C_3	$C_3 \times \mathfrak{D}_5$	C_3	1
1	C_2	C_6	C_2	0
2	$C_3 \times \mathfrak{D}_5 \supset C_2$	C_6	C_6	$0, 4$
2	$C_{15} \supset C_3$	C_{15}	C_{15}	$4, 1$
2	$C_6 \supset C_3$	C_6	C_6	$1, 4$
2	$C_6 \supset C_3$	C_6	C_6	$4, 1$
2	$C_6 \supset C_2$	C_6	C_6	$4, 0$

Each chain Λ contributes to $[\mathbb{P}^2 \circlearrowleft G]$ via its intermediate class, obtained in Step 4. We record these classes:

- $\Lambda = C_3 \times \mathfrak{D}_5$:

$$\begin{aligned} &(C_3 \subseteq C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (), (0,2)) + (C_3 \subseteq C_6, 1 \circlearrowright k, (3), (0,2)) \\ &+ (C_3 \subseteq C_6, 1 \circlearrowright k, (3), (0,5)) + (C_3 \subseteq C_{15}, 1 \circlearrowright k, (9), (0,8)) \\ &\qquad \in \mathrm{Burn}_{3,\{0,1\}}(C_3 \times \mathfrak{D}_5, C_3) \end{aligned}$$

- $\Lambda = C_{15}$:

$$(1 \subseteq 1, C_{15} \circlearrowright k(\mathbb{P}^1), (), (0,0)) + (1 \subseteq C_{15}, 1 \circlearrowright k, (7), (13,2)) \\ + (1 \subseteq C_{15}, 1 \circlearrowright k, (8), (13,9)) \in \mathrm{Burn}_{3,\{0,1\}}(C_{15}, 1)$$

- $\Lambda = C_6$ with $\Delta^t_\Lambda = 1$:

$$(1 \subseteq 1, C_6 \circlearrowright k(\mathbb{P}^1), (), (0,0)) + (1 \subseteq C_6, 1 \circlearrowright k, (1), (2,3)) \\ + (1 \subseteq C_6, 1 \circlearrowright k, (5), (2,4)) \in \mathrm{Burn}_{3,\{0,1\}}(C_6, 1)$$

- $\Lambda = C_6$ with $\Delta^t_\Lambda = C_2$:

$$(C_2 \subseteq C_2, C_3 \circlearrowright k(\mathbb{P}^1), (), (1,1)) + (C_2 \subseteq C_6, 1 \circlearrowright k, (4), (5,3)) \\ + (C_2 \subseteq C_6, 1 \circlearrowright k, (2), (5,1)) \in \mathrm{Burn}_{3,\{0,1\}}(C_6, C_2)$$

- $\Lambda = C_3$:

$$(C_3 \subseteq C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (), (2,1)) + (C_3 \subseteq C_6, 1 \circlearrowright k, (3), (2,4)) \\ + (C_3 \subseteq C_6, 1 \circlearrowright k, (3), (5,1)) + (C_3 \subseteq C_{15}, 1 \circlearrowright k, (9), (8,7)) \\ \in \mathrm{Burn}_{3,\{0,1\}}(C_3, C_3).$$

- $\Lambda = C_2$:

$$(C_2 \subseteq C_2, C_3 \circlearrowright k(\mathbb{P}^1), (), (0,1)) + (C_2 \subseteq C_6, 1 \circlearrowright k, (2), (4,5)) \\ + (C_2 \subseteq C_6, 1 \circlearrowright k, (4), (0,3)) \\ \in \mathrm{Burn}_{3,\{0,1\}}(C_2, C_2).$$

Our algorithm records the action on function fields in each symbol, e.g., the action of $\mathfrak{D}_5$ on $k(\mathbb{P}^1)$ in the last expression, but we omit it from the notation.

When $t = 2$, each graded piece is a one-dimensional vector space, with $N_G(\Lambda)$ acting via scalars. We will obtain classes

$$(N_G(\Lambda) \subseteq N_G(\Lambda), 1 \circlearrowright k, (), (\epsilon, \epsilon^1 - \epsilon, \epsilon^2 - \epsilon^1)).$$

Then we use the recursion in Step 5 to compute

$$[\mathbb{P}(V) \circlearrowleft G]_{(\mathcal{O}_{\mathbb{P}(V)}(-1))} \in \mathrm{Burn}_{n,\{0\}}(G, C).$$

In this example, G acts generically freely on $\mathbb{P}^2$, so that $C = 1$. After applying the map $\eta_{\{0\}}$ in Step 6 and cancellations by relations, we have

$$\begin{aligned}[\mathbb{P}(V) \circlearrowleft G] = &(1, G \circlearrowright k(\mathbb{P}^2), ()) + 2(C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (2)) + (C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_6, 1 \circlearrowright k, (3, 2)) + (C_6, 1 \circlearrowright k, (3, 4)) \\ &+ (C_6, 1 \circlearrowright k, (3, 5)) + (C_6, 1 \circlearrowright k, (2, 1) \\ &+ (C_6, 1 \circlearrowright k, (3, 1)) + (C_6, 1 \circlearrowright k, (4, 5)) \\ &+ (C_{15}, 1 \circlearrowright k, (1, 11)) + (C_{15}, 1 \circlearrowright k, (3, 11)) \\ &+ (C_{15}, 1 \circlearrowright k, (12, 4)).\end{aligned}$$

There is an alternative method to compute the class $[\mathbb{P}(V) \circlearrowleft G]$ [22, Section 5]: First, consider the action of $\mathfrak{D}_5$ on $\mathbb{P}^1$ via its two-dimensional representation V. Let L_1 be $\mathcal{O}_{\mathbb{P}^1}(1)$ twisted by the non-trivial character ϵ of C_3, and L_0 be the trivial line bundle on $\mathbb{P}^1$. Then

$$\mathbb{P}(1 \oplus V_\epsilon) \sim_G \mathbb{P}(L_0 \oplus L_1),$$

equivariantly. Using [22, Proposition 5.2], we obtain

$$\begin{aligned}[\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] = &(1, G \circlearrowright k(\mathbb{P}^2), ()) + (C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (2)) + (C_3, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_6, 1 \circlearrowright k, (3, 2)) + (C_6, 1 \circlearrowright k, (3, 4)) \\ &+ (C_6, 1 \circlearrowright k, (3, 5)) + (C_6, 1 \circlearrowright k, (3, 1)) \\ &+ (C_{15}, 1 \circlearrowright k, (3, 11)) + (C_{15}, 1 \circlearrowright k, (3, 4)).\end{aligned}$$

Here we specify the subgroups and their representations:

$$C_3 = \left\langle \begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \right\rangle, \quad C_6 = \left\langle \begin{pmatrix} 0 & \zeta_3 \\ \zeta_3 & 0 \end{pmatrix} \right\rangle \quad C_{15} = \left\langle \begin{pmatrix} \zeta_3\zeta_5 & 0 \\ 0 & \zeta_3\zeta_5^4 \end{pmatrix} \right\rangle.$$

Note that

$$\begin{aligned}&[\mathbb{P}(V) \circlearrowleft G] - [\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] \\ &= (C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1)) + (C_6, 1 \circlearrowright k, (2, 1)) + (C_6, 1 \circlearrowright k, (4, 5)) \\ &+ (C_{15}, 1 \circlearrowright k, (1, 11)) + (C_{15}, 1 \circlearrowright k, (12, 4)) - (C_{15}, 1 \circlearrowright k, (3, 4)).\end{aligned}$$

By conjugation relations **(C)**,

$$(C_{15}, 1 \circlearrowright k, (3, 4)) = (C_{15}, 1 \circlearrowright k, (12, 1))$$

The blowup relations **(B)** yield

$$(C_{15}, 1 \circlearrowright k, (12, 1)) = (C_{15}, 1 \circlearrowright k, (11, 1) + (C_{15}, 1 \circlearrowright k, (12, 4)),$$

$$(C_6, 1 \circlearrowright k, (2,3)) = (C_6, 1 \circlearrowright k, (5,3)) + (C_6, 1 \circlearrowright k, (2,1)),$$

$$(C_6, 1 \circlearrowright k, (3,5)) = \\ (C_6, 1 \circlearrowright k, (3,2)) + (C_6, 1 \circlearrowright k, (4,5)) + (C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1)).$$

Summing up the last two equalities, we obtain

$$(C_6, 1 \circlearrowright k, (2,1)) + (C_6, 1 \circlearrowright k, (4,5)) + (C_2, C_3 \circlearrowright k(\mathbb{P}^1), (1)) = 0$$

and conclude

$$[\mathbb{P}(V) \circlearrowleft G] - [\mathbb{P}(L_0 \oplus L_1) \circlearrowleft G] = 0 \in \mathrm{Burn}_2(G),$$

as expected.

Example 6.3. Consider the action of $G = \mathfrak{D}_7$ on $\mathbb{P}^3$, given by

$$G = \left\langle \begin{pmatrix} \zeta_7 & 0 & 0 & 0 \\ 0 & \zeta_7^{-1} & 0 & 0 \\ 0 & 0 & \zeta_7^2 & 0 \\ 0 & 0 & 0 & \zeta_7^{-2} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \subset \mathrm{PGL}_4.$$

The stabilizer chains are

t	Λ	$N_G(\Lambda)$	Δ^t_Λ	ϵ
1	C_2	C_2	C_2	0
1	C_2	C_2	C_2	1
1	C_7	C_7	C_1	2
1	C_7	C_7	C_1	3

The intermediate classes in the equivariant indexed Burnside groups are:

- $\Lambda = C_2$ with $N_G(\Lambda) = C_2$:
$$(C_2 \subseteq C_2, 1 \circlearrowright k(\mathbb{P}^2), (), (0,1)) \in \mathrm{Burn}_{3,\{0,1\}}(C_2, C_2)$$
- $\Lambda = C_2$ with $N_G(\Lambda) = C_2$:
$$(C_2 \subseteq C_2, 1 \circlearrowright k(\mathbb{P}^2), (), (1,1)) \in \mathrm{Burn}_{3,\{0,1\}}(C_2, C_2)$$
- $\Lambda = C_7$:

$$(C_1 \subseteq C_1, C_7 \circlearrowright k(\mathbb{P}^2), (), (0,0)) + (C_1 \subset C_7, 1 \circlearrowright k, (5,6), (2,3)) \\ + (C_1 \subseteq C_7, 1 \circlearrowright k, (1,2), (2,1)) + (C_1 \subseteq C_7, 1 \circlearrowright k, (1,6), (2,2)) \\ \in \mathrm{Burn}_{3,\{0,1\}}(C_7, 1)$$

- $\Lambda = C_7$:

$$(C_1 \subseteq C_1, C_7 \curvearrowright k(\mathbb{P}^2), (), (0,0)) + (C_1 \subseteq C_7, 1 \curvearrowright k, (4,6), (3,2))$$
$$+ (C_1 \subseteq C_7, 1 \curvearrowright k, (2,3), (3,6)) + (C_1 \subseteq C_7, 1 \curvearrowright k, (1,5), (3,1))$$
$$\in \mathrm{Burn}_{3,\{0,1\}}(C_7, 1).$$

These classes are combined to obtain

$$[\mathbb{P}(V) \circlearrowleft G]_{(\mathcal{O}_{\mathbb{P}(V)}(-1))} = (C_1 \subseteq C_1, G \curvearrowright k(\mathbb{P}^3), (), (0))$$
$$+ (C_1 \subset C_2, 1 \curvearrowright k(\mathbb{P}^2), (1), (0)) + (C_1 \subseteq C_2, 1 \curvearrowright k(\mathbb{P}^2), (1), (1))$$
$$+ (C_1 \subseteq C_7, 1 \curvearrowright k, (3,5,6), (2)) + (C_1 \subset C_7, 1 \curvearrowright k, (1,1,2), (2))$$
$$+ (C_1 \subseteq C_7, 1 \curvearrowright k, (1,2,6), (2)) + (C_1 \subseteq C_7, 1 \curvearrowright k, (2,4,6), (3))$$
$$+ (C_1 \subseteq C_7, 1 \curvearrowright k, (2,3,6), (3)) + (C_1 \subseteq C_7, 1 \curvearrowright k, (1,1,5), (3))$$

Applying $\eta_{\{0\}}$ and using relations **(V)** and **(B)**, we obtain the nonzero class

$$\begin{aligned}[\mathbb{P}(V) \circlearrowleft G] =& (1, G \curvearrowright k(\mathbb{P}^3), ()) \\ &+ (C_7, 1 \curvearrowright k, (1,1,2)) + (C_7, 1 \curvearrowright k, (2,4,6)) \\ &+ (C_7, 1 \curvearrowright k, (2,3,6)) \in \mathrm{Burn}_3(G);\end{aligned}$$

in fact, the point classes in this formula are equal, and nonzero, in $\mathcal{BC}_3(G) = \mathbb{Z}/2$. The action is birational to an action on $\mathbb{P}^1 \times \mathbb{P}^2$, with trivial action on the second factor and faithful action on the first factor, by the No-name Lemma.

7. Automorphisms of $\mathbb{P}^2$

In this section, we apply the Burnside group formalism to the problem of classification of actions of finite subgroups of PGL_3 up to conjugation in the plane Cremona group Cr_2 (see [13]).

For $n = 2$, the classification of actions up to conjugation in PGL_3 takes the form (we follow [13, Section 4.2] and [24, Section 10]):

- *intransitive*: $G = C_m \times G'$, with $G' \subset \mathrm{GL}_2$,
- *transitive but imprimitive*: certain extensions of C_3 or $\mathfrak{S}_3$ by bi-cyclic groups,
- *primitive*: $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathrm{PSL}_2(\mathbb{F}_7)$, the Hessian group $3^2 : \mathrm{SL}_2(\mathbb{F}_3)$, and two of its subgroups.

Primitive actions. These are completely understood via birational (super)rigidity techniques [34]. E.g., $\mathfrak{A}_5$ admits one, $\mathfrak{A}_6$ admits four, and $\mathrm{PSL}_2(\mathbb{F}_7)$ admits two non-birational actions on $\mathbb{P}^2$ (see [5, Theorem B.2]).

Proposition 7.1. *The Burnside group formalism does not distinguish primitive actions on* $\mathbb{P}^2$.

The proof proceeds via a computation of all classes involved and comparisons of the resulting expressions in the respective Burnside groups. Here is a representative example:

Example 7.2. The action of $G := \mathrm{PSL}_2(\mathbb{F}_7)$ on $\mathbb{P}^2$ is *super-rigid*, and there are non-isomorphic 3-dimensional representations V and V' of G, giving rise to non-birational G-actions on $\mathbb{P}^2 = \mathbb{P}(V)$ and $\mathbb{P}(V')$. The characters of the corresponding representations differ on elements of order 7. We compute the classes

$$\begin{aligned}[\mathbb{P}(V) \circlearrowleft G] = {} & (1, G \circlearrowright k(\mathbb{P}^2), ()) + 2(C_2, \mathfrak{D}_2 \circlearrowright k(\mathbb{P}^1), (1)) \\ & + (C_3, 1 \circlearrowright k, (1,1)) + (C_4, 1 \circlearrowright k, (1,1)) + 2(C_4, 1 \circlearrowright k, (1,2)) \\ & + (C_7, 1 \circlearrowright k, (6,5)) + (C_7, 1 \circlearrowright k, (1,4)) \\ & + (C_2^2, 1 \circlearrowright k, ((0,1),(1,0))) + ((C_2')^2, 1 \circlearrowright k, ((0,1),(1,0)))\end{aligned}$$

$$\begin{aligned}[\mathbb{P}(V') \circlearrowleft G] = {} & (1, G \circlearrowright k(\mathbb{P}^2), ()) + 2(C_2, \mathfrak{D}_2 \circlearrowright k(\mathbb{P}^1), (1)) \\ & + (C_3, 1 \circlearrowright k, (1,1)) + (C_4, 1 \circlearrowright k, (1,1)) + 2(C_4, 1 \circlearrowright k, (2,3)) \\ & + (C_7, 1 \circlearrowright k, (6,3)) + (C_7, 1 \circlearrowright k, (1,2)) \\ & + (C_2^2, 1 \circlearrowright k, ((0,1),(1,0))) + ((C_2')^2, 1 \circlearrowright k, ((1,1),(1,0))).\end{aligned}$$

The representations V and V' differ by $\zeta_7 \mapsto \zeta_7^3$. Conjugation relations imply that

$$[\mathbb{P}(V) \circlearrowleft G] = [\mathbb{P}(V') \circlearrowleft G].$$

We record useful method to produce incompressible classes in dimension 3 (see Section 3.6).

Proposition 7.3. *Let* G *be a finite group and*

$$\bar{s} = (\bar{H}, \bar{Y} \circlearrowright k(\mathbb{P}^1)(t), (\bar{b})) \in \mathrm{Burn}_3(G)$$

a symbol appearing in a Θ_2*-relation. Then* $\bar{Y}$ *does not admit a primitive action on* $\mathbb{P}^2$.

Proof. By [34, Theorem 1.3], $\mathbb{P}^2$ are $\bar{Y}$-rigid (and thus $\bar{Y}$-solid) for a primitive action from $\bar{Y}$. Therefore the corresponding actions do not appear in Θ_2 relations, see Section 3.6. □

Transitive Imprimitive actions. There are four types of such actions, two types with G an extension of C_3 and two additional types when G is an extension of $\mathfrak{S}_3$, see [13, Theorem 4.7].

Proposition 7.4. *The Burnside group formalism allows to distinguish transitive imprimitive actions, indistinguishable by the* (**RY**) *invariant.*

We do not claim that we can distinguish *all* such actions. In each of the four types there is a bi-cyclic group $H \subset G$; restricting to H and applying the Reichstein-Youssin determinant invariant **(RY)** to H gives non-birational actions in some cases. Our examples focus on the simpler types in [13, Theorem 4.7], as it is more difficult to distinguish smaller actions.

We consider:

(1) extensions
$$1 \to C_n \oplus C_n \to G \to C_3 \to 1$$
with the action on $\mathbb{P}^2 = \mathbb{P}^2(s,t)$ generated by
$$(x:y:z) \mapsto (\zeta_n^s x : y : z),\ (x : \zeta_n^t y : z),\ (z:x:y), \tag{7.1}$$
where $s,t \in (\mathbb{Z}/n)^\times$, and ζ_n is a primitive n-th root of unity.

(2) extensions
$$1 \to C_n \oplus C_m \to G \to C_3 \to 1,$$
with $m = n/d$, with $d > 1$, $d|n$, $s^2 - s + 1 = 0 \pmod d$, and with the action on $\mathbb{P}^2 = \mathbb{P}^2(r,s,t)$ generated by
$$(x:y:z) \mapsto (\zeta_m^r x : y : z),\ (\zeta_n^s x : \zeta_n^t y : z),\ (z:x:y). \tag{7.2}$$

Example 7.5. Let G be a group of type (1), with $n = 8$. Consider actions as in (7.1) with and
$$s = 1, \quad t = 7,$$
respectively,
$$s' = 3, \quad t' = 5.$$
The **(RY)** invariant is inconclusive in this case. Computing the Burnside symbols as in Section 6, we obtain
$$\begin{aligned}[\mathbb{P}^2(s,t) \circlearrowleft G] = {} & (1, G \circlearrowright k(\mathbb{P}^2), ()) \\ & + (C_8, C_8 \circlearrowright k(\mathbb{P}^1), (3)) + (C_8, C_8 \circlearrowright k(\mathbb{P}^1), (5)) \\ & + (C_8^2, 1 \circlearrowright k, ((1,2),(6,7))) + (C_8^2, 1 \circlearrowright k, ((7,6),(7,1))).\end{aligned}$$

$$\begin{aligned}[\mathbb{P}^2(s',t') \circlearrowleft G] = (1, G &\curvearrowright k(\mathbb{P}^2), ()) \\ &+ (C_8, C_8 \curvearrowright k(\mathbb{P}^1), (1)) + (C_8, C_8 \curvearrowright k(\mathbb{P}^1), (7)) \\ &+ (C_8^2, 1 \curvearrowright k, ((3,6),(2,5))) + (C_8^2, 1 \curvearrowright k, ((5,2),(5,3))).\end{aligned}$$

(As before, we omit to specify the action of C_8 on $k(\mathbb{P}^1)$ from our notation.) There are no incompressible symbols in the expressions above, however we are still able to distinguish the actions in the combinatorial Burnside group, after applying map (3.3) to the difference

$$[\mathbb{P}^2(s,t) \circlearrowleft G] - [\mathbb{P}^2(s',t') \circlearrowleft G]. \tag{7.3}$$

Performing computations in `magma`, we found the group $\mathcal{BC}_2(C_8^2)$ has rank 733, and the image of (7.3) in $\mathcal{BC}_2(G)$ is nonzero. This allows us to conclude that the given two C_8^2-actions on $\mathbb{P}^2$ are not birational to each other.

The same argument applies to $n = 5$, $s = 1, t = 2, s' = 3$, and $t' = 4$; or $n = 9$, $s = 2, t = 3, s' = 4$, and $t' = 6$.

The non-birationality of actions in the following two examples follows from [34, Theorem 1.3], indeed, these actions are birationally rigid. However, they serve as an illustration of the symbols formalism, in the case of imprimitive actions. In the case of intransitive actions, discussed below, birational rigidity techniques not apply.

Example 7.6. Let G act via type (2) with $n = 14$ and $m = 2$. Consider actions as in (7.2) with

$$r = t = 1, s = 3,$$

respectively,

$$r' = t' = 1, s' = 5.$$

Again, the **(RY)** invariant is inconclusive. We have

$$\begin{aligned}[\mathbb{P}^2(r,s,t) \circlearrowleft G] = (1, G &\curvearrowright k(\mathbb{P}^2), ()) \\ &+ (C_2, C_{14} \curvearrowright k(\mathbb{P}^1), (1)) + (C_2, C_{14} \curvearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_2 \times C_{14}, 1 \curvearrowright k, ((0,3),(1,5))) \\ &+ (C_2 \times C_{14}, 1 \curvearrowright k, ((0,11),(1,8))),\end{aligned}$$

$$\begin{aligned}[\mathbb{P}^2(r',s',t') \circlearrowleft G] = (1, G &\curvearrowright k(\mathbb{P}^2), ()) \\ &+ (C_2, C_{14} \curvearrowright k(\mathbb{P}^1), (1)) + (C_2, C_{14} \curvearrowright k(\mathbb{P}^1), (1)) \\ &+ (C_2 \times C_{14}, 1 \curvearrowright k, ((1,11),(0,1))) \\ &+ (C_2 \times C_{14}, 1 \curvearrowright k, ((1,3),(1,12))).\end{aligned}$$

Applying map (3.3) to the difference and computing in $\mathcal{BC}_2(G)$ we find that the actions are non-birational.

Intransitive actions. Existence of G-fixed points makes it more difficult to classify intransitive actions using birational rigidity techniques. However, it is well-suited for the Burnside group formalism. Recall that intransitive actions take the form of

$$G = C_n \times G', \quad n \geq 2,$$

where $G' \subset \mathrm{GL}_2$ is a lift of a subgroup $\bar{G}' \subset \mathrm{PGL}_2$. We are again in the situation of Section 5:

- $\bar{G}' = C_m$ for some $m \geq 2$. Then G' is also a cyclic group, i.e., G is a rank 2 abelian group. The **(RY)** invariant determines equivariant birationality of such actions [33, Theorem 7.1].
- $\bar{G}' = \mathfrak{D}_m, \mathfrak{A}_4, \mathfrak{S}_4$ or $\mathfrak{A}_5$. By [24, Section 10], we know that G admits non-birational actions when the Euler function $\varphi(n) \geq 3$. Here we modify the proof to cover more cases when $n \geq 2$.

Let ϵ be a primitive character of C_m, V a faithful two-dimensional linear representation of G', and $V_\epsilon := \epsilon \otimes V$ its twist by ϵ. This yields generically free action G-action on $\mathbb{P}^2 = \mathbb{P}(1 \oplus V_\epsilon)$. To put the action in standard form, we first need to blow up the point $(1:0:0)$ as it has nonabelian generic stabilizer. The action on the exceptional divisor is given by $\mathbb{P}(V_\epsilon)$. On the standard model, there are two divisors with generic stabilizer H, where H is the maximal subgroup of G acting via scalars on V_ϵ. For example, when $\bar{G}' = \mathfrak{A}_5$, we can choose the lift $G' = \mathrm{SL}_2(\mathbb{F}_5)$ and in this case,

$$H = \begin{cases} C_n & \text{when } n \text{ is even,} \\ C_{2n} & \text{when } n \text{ is odd.} \end{cases}$$

Let χ_ϵ be the character of H corresponding to the action, which depends on choice of ϵ. The two divisors contribute

$$(H, \bar{G}' \circlearrowright k(\mathbb{P}(V)), (\chi_\epsilon)) + (H, \bar{G}' \circlearrowright k(\mathbb{P}(V)), (-\chi_\epsilon)) \tag{7.4}$$

to the class $[\mathbb{P}^2 \circlearrowleft G]$; these symbols are incompressible, as explained in Section 3.6, or see [24, Proposition 3.6]. When $\varphi(n) \geq 3$, we can produce non-birational actions by choosing characters $\epsilon \neq \pm\epsilon'$. But one can do better:

Corollary 7.7. *For $\bar{G}' = \mathfrak{D}_m$, with $m \neq 1, 2, 3, 4, 6, 8, 12$, or $\bar{G}' = \mathfrak{A}_5$, and all $n \geq 2$, the group $G = C_n \times G'$ admits non-birational linear actions on $\mathbb{P}^2$.*

Proof. From Section 5, we know that $\mathfrak{D}_m$, with m as in the statement, and $\mathfrak{A}_5$ admit non-birational actions on $\mathbb{P}^1$. This will contribute different incompressible symbols to (7.4). □

Now we consider the case $\bar{G}' = \mathfrak{D}_m$ in more detail. Recall that for m odd, a generically free action of $G' = \mathfrak{D}_m$ on $\mathbb{P}^1$ is linear; for m even, it is projectively linear—it arises from a 2-dimensional faithful representation of $G' = \mathfrak{D}_{2m}$. In both cases, the representation is determined by a primitive character ψ of $H = C_m$, respectively C_{2m}, we denote it by V_ψ. Explicitly, it is give by

We obtain an action of $G = C_n \times \mathfrak{D}_m$ on

$$\mathbb{P}^2 = \mathbb{P}^2(\epsilon, \psi) := \mathbb{P}(1 \oplus V_{\epsilon,\psi}), \quad V_{\epsilon,\psi} := \epsilon \otimes V_\psi, \quad V_\psi = \mathrm{Ind}_H^{G'}(\psi).$$

Lemma 7.8. *We have*

$$\mathbb{P}^2(\epsilon, \psi) \sim_G \mathbb{P}^2(-\epsilon, \psi) \sim_G \mathbb{P}^2(\epsilon, -\psi) \sim_G \mathbb{P}^2(-\epsilon, -\psi).$$

Proof. Indeed, equivariant birationality from the G-action on $\mathbb{P}^2(\epsilon, \psi)$ to the other actions is realized by

$$(x : y : z) \dashrightarrow (\frac{1}{x} : \frac{1}{z} : \frac{1}{y}), \ (x : z : y), \text{ and } (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}),$$

respectively. □

Suppose m is odd: The following sum of incompressible symbols

$$(C_n, \mathfrak{D}_m \circlearrowright k(\mathbb{P}(V_\psi)), \epsilon) + (C_n, \mathfrak{D}_m \circlearrowright k(\mathbb{P}(V_\psi)), -\epsilon) \tag{7.5}$$

contributes to the class $[\mathbb{P}^2(\epsilon, \psi) \circlearrowleft G]$; we obtain similar expressions for the G-action on $\mathbb{P}^2(\epsilon', \psi')$. We observe:

- when $\epsilon \neq \pm\epsilon'$, the symbols in (7.5) have different weights;
- when $\psi \neq \pm\psi'$, the actions of $\mathfrak{D}_m$ on $\mathbb{P}^1$ is not birational to each other.

Lemma 7.8 implies that the Burnside group formalism determines equivariant birationality in this case.

On the other hand, when m is *even*, the classification of equivariant birational types remains open:

Example 7.9. Consider $G = C_3 \times \mathfrak{D}_8$, and put $\psi' := \psi^3$. Then

$$[\mathbb{P}^2(\epsilon,\psi) \circlearrowleft G] - [\mathbb{P}^2(\epsilon,\psi') \circlearrowleft G] = 0 \in \mathrm{Burn}_2(G).$$

However, we cannot tell whether or not

$$\mathbb{P}^2(\epsilon,\psi) \overset{?}{\sim}_G \mathbb{P}^2(\epsilon,\psi').$$

In detail,

$$\begin{aligned}
&[\mathbb{P}^2(\epsilon,\psi) \circlearrowleft G] = (1, G \circlearrowright k(\mathbb{P}^2), ())\\
&+ 2(C_2, C_6 \circlearrowright k(\mathbb{P}^1), (1)) + 2(C_2', C_6 \circlearrowright k(\mathbb{P}^1), (1))\\
&+ (C_6, \mathfrak{D}_4 \circlearrowright k(\mathbb{P}^1), (1)) + (C_6, \mathfrak{D}_4 \circlearrowright k(\mathbb{P}^1), (5))\\
&+ (C_2'' \times C_6, 1 \circlearrowright k, ((0,3),(1,5))) + (C_2'' \times C_6, 1 \circlearrowright k, ((1,2),(1,1)))\\
&+ (C_2'' \times C_6, 1 \circlearrowright k, ((1,4),(0,3))) + (C_2''' \times C_6, 1 \circlearrowright k, ((1,2),(0,3)))\\
&+ (C_2''' \times C_6, 1 \circlearrowright k, ((1,5),(1,4))) + (C_2''' \times C_6, 1 \circlearrowright k, ((1,1),(0,3)))\\
&+ (C_{24}, 1 \circlearrowright k, (19,11)) + (C_{24}, 1 \circlearrowright k, (5,6)) + (C_{24}, 1 \circlearrowright k, (19,18)),
\end{aligned}$$

while

$$\begin{aligned}
&[\mathbb{P}^2(\epsilon,\psi') \circlearrowleft G]\\
&= (1, G \circlearrowright k(\mathbb{P}^2), ()) + \dots\\
&+ (C_{24}, 1 \circlearrowright k, (6,17)) + (C_{24}, 1 \circlearrowright k, (7,23)) + (C_{24}, 1 \circlearrowright k, (7,18)),
\end{aligned}$$

with the only difference in the sum of terms with stabilizer C_{24}, and these expressions are equal in $\mathrm{Burn}_2(G)$.

8. Automorphisms of $\mathbb{P}^3$

In this section, we give new examples of non-birational imprimitive linear actions on $\mathbb{P}^3$. The basic terminology is as follows:

$$\text{actions}\begin{cases}\text{intransitive: invariant point or line}\\ \text{transitive:} \quad \begin{cases}\text{imprimitive:} \quad \begin{cases}\text{2 skew lines}\\ \text{orbit of length 4 (monomial)}\end{cases}\\ \text{primitive: none of the above}\end{cases}\end{cases}$$

Primitive actions. We follow [11]. There are 30 conjugacy classes of finite subgroups $G \subset \mathrm{PGL}_4$ yielding primitive actions. They are listed, with inclusions, in [11, Appendix A]. These actions can be analyzed by birational (super)rigidity techniques, see [8] or [11]. By [11, Theorem 1.1], the action is birationally *rigid* iff $G \neq \mathfrak{A}_5$ or $\mathfrak{S}_5$. This means that applying G-MMP to any G-birational model one is reduced to

$\mathbb{P}^3$; but this does not imply that different actions on $\mathbb{P}^3$ are equivariantly birational. We now list representative computations of Burnside classes:

- $G := \mathfrak{A}_5$: Let V be its irreducible 4-dimensional representation. Consider the induced action on $\mathbb{P}^3 = \mathbb{P}(V)$. Then

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] &= (1, \mathfrak{A}_5 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^3), ()) + 2(C_2, C_2 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) \\ &+ (C_3, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^1), (2,2)) + (C_3, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^1), (1,1)) \\ &+ (C_5, \mathrel{\reflectbox{$\circlearrowleft$}} k, (1,1,2)) + (C_5, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, (2,2,4))\end{aligned}$$

By Lemma 3.1, the point classes are trivial; furthermore,

$$(C_2, C_2 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) = (C_2, C_2 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^1), (1,1)) = 0 \in \mathrm{Burn}_3(G),$$

$$(C_3, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^1), (b,b)) = (C_3, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, (b,b,b)) = 0 \in \mathrm{Burn}_3(G),$$

by (**B**) and the vanishing relation (**V**).

- $G = \mathrm{PSL}_2(\mathbb{F}_7)$: The G-action on $\mathbb{P}^3$ is rigid [11, Theorem 1.3], but every faithful action gives

$$[\mathbb{P}^3 \circlearrowleft G] = (1, G \mathrel{\reflectbox{$\circlearrowleft$}} \mathbb{P}^3, ()) \in \mathrm{Burn}_3(G).$$

- $G = \mathfrak{A}_6$: There are only two actions; they are rigid but equivariantly birational. The corresponding classes are

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = (1, G \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^3), ()) &+ (C_3, C_3 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (2)), \\ &+ (C_3^2, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, ((2,2),(0,1),(2,1))) \\ &+ (C_3^2, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, ((0,2),(2,1),(2,2))).\end{aligned}$$

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = (1, G \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^3), ()) &+ (C_3', C_3 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (2)), \\ &+ (C_3^2, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, ((0,2),(1,1),(1,0))) \\ &+ (C_3^2, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, ((0,2),(1,0),(2,2))),\end{aligned}$$

and the nontrivial contributions to their classes in $\mathcal{BC}_3(G)$ are equal, as expected. But they are nontrivial in this group.

- $G = \mathfrak{S}_6$: There are two actions, with Burnside classes

$$\begin{aligned}[\mathbb{P}^3 \circlearrowleft G] = (C_1, \mathfrak{S}_6 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^3), ()) & \\ + (C_2, \mathfrak{A}_4 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) &+ (C_2', \mathfrak{A}_4 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) \\ + (C_2'', C_2^2 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) &+ (C_3, \mathfrak{S}_3 \mathrel{\reflectbox{$\circlearrowleft$}} k(\mathbb{P}^2), (1)) \\ + (C_3^2, 1 \mathrel{\reflectbox{$\circlearrowleft$}} k, ((1,1),(1,2),(2,0))), &\end{aligned}$$

respectively,

$$
\begin{aligned}
[\mathbb{P}^3 \circlearrowleft G] = {} & (C_1, \mathfrak{S}_6 \circlearrowright k(\mathbb{P}^3), ()) \\
& + (C_2, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) + (C_2', \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (1)) \\
& + (C_2'', C_2^2 \circlearrowright k(\mathbb{P}^2), (1)) + (C_3', \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^2), (2)) \\
& + (C_3^2, 1 \circlearrowright k, ((0,2),(2,0),(2,2))),
\end{aligned}
$$

here, C_2, C_2', C_2'' are not conjugated in G. These differ in

$$\mathcal{BC}_3(G) = (\mathbb{Z}/2)^5 \oplus \mathbb{Z}/4;$$

thus, the actions are not birational.

- $G = \mathfrak{A}_7$: There are two actions. The actions are super-rigid and thus not birational to each other. The respective classes are:

$$
\begin{aligned}
[\mathbb{P}^3 \circlearrowleft G] = {} & (1, G \circlearrowright k(\mathbb{P}^3), ()) + (C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^2), (1)) \\
& + (C_3, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (2)) \\
& + (C_7, 1 \circlearrowright k, (2,4,4)) + (C_7, 1 \circlearrowright k, (1,3,5)) \\
& + (C_7, 1 \circlearrowright k, (2,3,3)) \\
& + (C_3^2, 1 \circlearrowright k, ((0,1),(1,1),(2,0))) \\
& + (C_3^2, 1 \circlearrowright k, ((0,1),(2,0),(2,2))),
\end{aligned}
$$

$$
\begin{aligned}
[\mathbb{P}^3 \circlearrowleft G] = {} & (1, G \circlearrowright k(\mathbb{P}^3), ()) + (C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^2), (1)) \\
& + (C_3, \mathfrak{A}_4 \circlearrowright k(\mathbb{P}^2), (2)) \\
& + (C_7, 1 \circlearrowright k, (2,4,4)) + (C_7, 1 \circlearrowright k, (1,3,5)) \\
& + (C_7, 1 \circlearrowright k, (2,3,3)) \\
& + (C_3^2, 1 \circlearrowright k, ((0,1),(1,0),(2,1))) \\
& + (C_3^2, 1 \circlearrowright k, ((0,1),(1,0),(1,2))).
\end{aligned}
$$

We have $\mathcal{BC}_3(G) = (\mathbb{Z}/2)^3 \oplus \mathbb{Z}$, the (nontrivial contributions to) combinatorial Burnside classes of the two actions are equal, which in this case implies that the classes are equal in $\mathrm{Burn}_3(G)$.

Transitive imprimitive actions. Recall that these are of two types:

- leaving invariant a union of two skew lines,
- having an orbit of length 4 (monomial subgroups)

The second type was analyzed in [7]; by its main theorem, every imprimitive monomial subgroup, with the exception of (`GAP ID`)

$$G_{48,3}, \quad G_{96,72}, \quad \text{or} \quad G_{324,160},$$

is *G-solid* (i.e., not G-birational to conic bundles or Del Pezzo fibrations). Examples of non-birational actions are given in [7, Example 1.6, 1.7 and 1.8].

Here we present applications of the Burnside group formalism to actions leaving invariant two skew lines.

Example 8.1. Let $G := \mathfrak{D}_5 \times \mathfrak{D}_4$ and write ψ_m for a primitive character of C_m. As in Section 7, let V_ψ be a faithful 2-dimensional representation of $\mathfrak{D}_m$ determined by ψ_m.

We have generically free linear G-actions on

$$\mathbb{P}^3 = \mathbb{P}(V_{\psi_5} \oplus V_{\psi_4}), \quad \text{respectively,} \quad \mathbb{P}^3 = \mathbb{P}(V_{\psi_5^2} \oplus V_{\psi_4}). \tag{8.1}$$

Our algorithm presents the class of each action in (8.1) as a sum of more than 60 symbols; we have listed them at [39]. Again, with `magma`, we find that the projection of the difference of the classes to $\mathcal{BC}_3(G)$ is nonzero and we conclude that the actions are not birational.

This is the smallest such example we could find; the same holds for $G := \mathfrak{D}_7 \times \mathfrak{D}_4$ (and ψ_5 replaced by ψ_7).

Intransitive actions: The discussion is similar to that in Section 7. In dimension 3, intransitive actions take the form of

$$G = C_n \times G', \quad n \geq 2,$$

where $G' \subset \mathrm{GL}_3$ is a lift of $\bar{G}' \subset \mathrm{PGL}_3$. It is shown in [24, Theorem 11.2] that when

$$\bar{G}' = \mathfrak{S}_4, \quad \mathfrak{A}_5, \quad \mathrm{PSL}_2(\mathbb{F}_7), \quad \mathfrak{A}_6 \quad \text{and} \quad \varphi(n) \geq 3,$$

G admits non-birational actions. Here we use the same argument to cover more cases again: Let V be a 3-dimensional faithful representation of G' and ϵ a primitive character of C_n. Let $V_\epsilon := \epsilon \otimes V$ and consider the action $\mathbb{P}(1 \oplus V_\epsilon)$. We need to blow up the fixed point $(1 : 0 : 0 : 0)$ to put the action into standard form and on the blow-up model, there will be two divisors with generic stabilizer H, where H is the maximal subgroup of G acting via scalars. Their contribution to the class is

$$(H, \bar{G}' \circlearrowright k(\mathbb{P}(V)), (\chi_\epsilon)) + (H, \bar{G}' \circlearrowright k(\mathbb{P}(V)), (-\chi_\epsilon)).$$

These symbols are incompressible for our choice of $\bar{G}'$ because $\mathrm{PSL}_2(\mathbb{F}_7)$ and $\mathfrak{A}_6$ are nonabelian and cannot act generically freely on $\mathbb{P}^1$ (see

Proposition 7.3). Actions of $\mathfrak{S}_4$ and $\mathfrak{A}_5$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with trivial action on one factor and generically free actions on the other factor are not linearizable. Similarly to Corollary 7.7, we know that if $\bar{G}'$ admits non-birational actions on $\mathbb{P}^2$, then G admits non-birational actions on $\mathbb{P}^3$. Keeping the notation above, we arrive at:

Corollary 8.2. *For $G = C_n \times G'$, with $\bar{G}' = \mathrm{PSL}_2(\mathbb{F}_7)$ or $\mathfrak{A}_6$, there exist non-birational intransitive G-actions on $\mathbb{P}^3$, for all $n \geq 2$.*

Proof. As in Section 7, these choices of $\bar{G}'$ give non-birational actions on $\mathbb{P}^2$. □

9. Automorphisms of quadrics

There is an extensive literature on birationality of quadrics over nonclosed fields (see, e.g., [37]); of course, this is only interesting in absence of k-rational points. One of the central problems there is the following.

Zariski problem for quadrics: If two smooth quadrics of the same dimension, over a nonclosed field, are stably birational then they are birational.

This is known in dimensions ≤ 7. On the other hand, in the G-equivariant context, there are examples of stably equivariantly birational but not birational quadrics, already in dimension 2. Their equivariant geometry has been addressed in, e.g., [18], [35], [16, Section 7]. In particular, the quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$ admits actions of $G = C_2 \times \mathfrak{D}_n$, for odd n, which are not birational to linear actions but such that the G-action on $Q \times \mathbb{P}^2$, with trivial G-action on the second factor, is birational to a linear action [27], [16]. The existence of such stable birationalities makes the analysis of $\mathrm{Burn}_n^{\mathrm{rc}}(G)$, $n \geq 3$, challenging, as one has to account for all such possibilities.

We are not aware of a systematic study of G-equivariant geometry of quadrics in higher dimensions. In particular, it would be interesting to study systematically constructions of G-equivariant (stable) birationalities to projective spaces which do not rely on existence of G-fixed points.

Assumptions on fixed points: Projection from fixed points gives trivially linearizability of the action, thus we will assume that

- $X^G = \emptyset$.

On the other hand, existence of fixed points on a smooth model is a birational invariant for actions of abelian groups [32], and linear actions of abelian groups have fixed points. Since we are only interested in linear actions, we will assume that

- $X^H \neq \emptyset$, for all abelian $H \subset G$.

In this section we consider the birational classification of automorphisms of quadrics from the perspective of Burnside groups. In particular, we focus on G-actions satisfying the assumptions above.

Conics: Consider $X \subset \mathbb{P}^2$, given by

$$\sum_{j=1}^{3} x_j^2 = 0,$$

with an action of a subgroup G of the Weyl group $W(\mathsf{D}_3) = \mathfrak{S}_4$. The group $W(\mathsf{D}_3)$ has 11 conjugacy classes of subgroups. Only one satisfies the requirements (concerning fixed points), namely $\mathfrak{S}_3 = \langle \sigma, \tau \rangle$, with $\tau^2 = \sigma^3 = 1$, and the natural permutation action on the coordinates; this action is linearizable. We turn to quadric surfaces.

Abelian actions on $\mathbb{P}^1 \times \mathbb{P}^1$: Their birational classification is in [1, Proposition 6.2.4]. In [14, Section 5.5] we noted that the following actions of C_2^2 on $\mathbb{P}^1 \times \mathbb{P}^1$ are not distinguishable with the Burnside formalism: the product action has fixed points, while the diagonal action does not, thus the actions are not birational, but the projections of the classes to the nontrivial part of the Burnside group vanish.

On the other hand, consider the following, nonlinearizable, actions of C_2^3 on $\mathbb{P}^1 \times \mathbb{P}^1$: in the first case, via $\mathfrak{K}_4 = C_2^2$ on one factor and C_2 on the other factor, and in second case via $\mathfrak{K}_4$ on both factors, together with a switch of the factors. In the first case, we record

$$2(C_2, \mathfrak{K}_4 \circlearrowright k(\mathbb{P}^1), (1)),$$

coming from the two fixed points on the second $\mathbb{P}^1$, and in the second case only *one* such class. Since this symbol is incompressible (see [24, Proposition 3.6]), we conclude that the two actions have different classes in the Burnside group.

Nonabelian actions on $\mathbb{P}^1 \times \mathbb{P}^1$: A full list of such actions is given in [13, Theorem 4.9]. Here we consider the quadric surface Q given by

$$\sum_{j=1}^{4} x_j^2 = 0. \tag{9.1}$$

We focus on actions changing signs and permuting the variables. There are 2 conjugacy classes of such groups G satisfying the assumptions on fixed points, namely:

$$\mathfrak{D}_6 \longrightarrow \mathfrak{S}_3,$$

where

$$\mathfrak{D}_6 = C_2 \times \mathfrak{S}_3 = \langle \iota, \sigma, \tau \rangle, \quad \tau^2 = \sigma^3 = 1.$$

Here ι inverts the sign on x_4, $\mathfrak{S}_3 = \langle \sigma, \tau \rangle$ acts via permutation of the first three coordinates, and the specialization is to $\mathfrak{S}_3 = \langle \sigma, \iota \cdot \tau \rangle$.

The fixed-point free $\mathfrak{S}_3$-action is linearizable; it is birational to an action on $\mathbb{P}(1 \oplus V_2)$, where V_2 is the standard 2-dimensional representation of $\mathfrak{S}_3$; in particular, there is a fixed point on $\mathbb{P}^2$.

On the other hand, by [27, Section 9] (see also [16, Section 6]), the $\mathfrak{D}_6$-action on Q is not linearizable but stably linearizable. The proof of nonlinearizability in [18] was based on classification of birational transformations (links) between rational surfaces. An alternative proof, using the Burnside group formalism, is in [14, Section 7.6]; we give a similar argument in the following example.

Example 9.1. Let $G = C_2^2 \times \mathfrak{S}_3$. We analyze whether or not the symbol

$$\bar{\mathfrak{s}} = (C_2, C_2 \times \mathfrak{S}_3 \circlearrowright \bar{K}, (1)) \in \mathrm{Burn}_3(G), \quad \bar{K} = k(Q),$$

is incompressible. There is a candidate symbol

$$\mathfrak{s} = (C_2^2, \mathfrak{S}_3 \circlearrowright K, (e_1, e_2)),$$

that could lead to the given $\bar{\mathfrak{s}}$ via the blowup relation **(B)**. Here e_1, e_2 are nontrivial distinct characters of C_2^2.

Let us specify the action of $\bar{Y} = C_2 \times \mathfrak{S}_3$ on $\bar{K} = k(Q)$, with Q the quadric surface in (9.1): C_2 switches the sign on x_4 and $\mathfrak{S}_3$ permutes the first three coordinates.

Since Q is rational, we must have $K = k(\mathbb{P}^1)$. The Action construction produces Θ_2-terms where the $\bar{Y}$-action is birational to an action on a Hirzebruch surface S, a projectivization of a rank-2 vector bundle on $\mathbb{P}^1$, either with trivial action or a C_2-action on the generic fiber.

In the first case, such an action is birational to an action on $\mathbb{P}^1 \times \mathbb{P}^1$, with $C_2 \times \mathfrak{S}_3 = \mathfrak{D}_6$ acting on one of the factors, and trivial action on the second factor. This action has no fixed points upon restriction to $C_2 \times \mathfrak{S}_2 \subset C_2 \times \mathfrak{S}_3$, which is not the case for the $\bar{Y}$ action on Q. Thus the actions are not birational.

In the second case, we compare the classes in $\mathrm{Burn}_2(\bar{Y})$, for the actions on Q and on S. We find *one* incompressible symbol

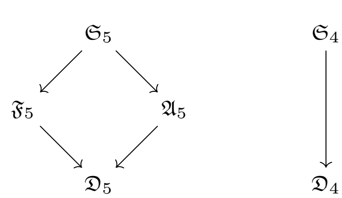
$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \in \mathrm{Burn}_2(C_2 \times \mathfrak{S}_3)$$

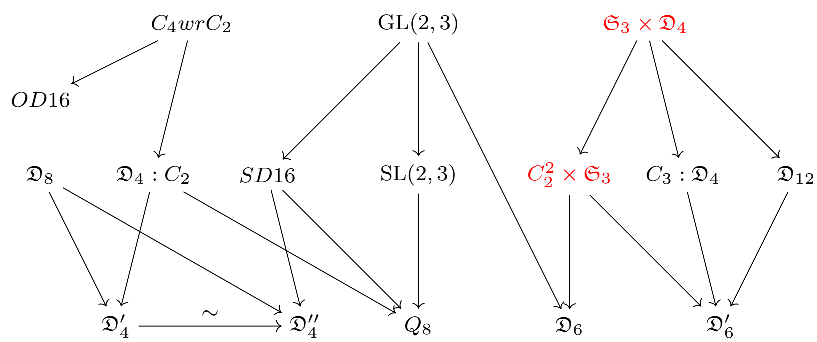
in the class $[Q \circlearrowleft \bar{Y}]$, and *two* such symbols, corresponding to the two sections fixed by C_2, in the class $[S \circlearrowleft \bar{Y}]$ (see Section 3.6). Thus the actions are not birational and $\bar{\mathfrak{s}}$ is incompressible.

Quadric threefolds: We consider first $X \subset \mathbb{P}^4$ given by $\sum_{j=1}^{5} x_j^2 = 0$, with a natural action of the Weyl group $W(\mathsf{D}_5)$. This group has 197 conjugacy classes of subgroups, examined in [26, Section 5] in connection with the analysis of possible Galois actions (or automorphisms) on Picard groups of Del Pezzo surfaces of degree 4; the goal there was to identify potentially rational surfaces over nonclosed fields (see also [38]). There are 112 (conjugacy classes of) subgroups $G \subset W(\mathsf{D}_5)$ which give rise to fixed-point free actions.

We focus on the linearizability problem. Note that the **(RY)** invariant (see Section 3) does not provide any information: $W(\mathsf{D}_5)$ does not contain abelian subgroups of rank 3 that could give a nontrivial obstruction.

We obtain 33 $W(\mathsf{D}_5)$-conjugacy classes of subgroups satisfying our assumptions on fixed points; several of these are conjugated in PGL_5. We list the remaining cases:





Note that $\mathfrak{D}_4'$ and $\mathfrak{D}_4''$ are not conjugated in $W(\mathsf{D}_5)$ but are conjugated in $\mathrm{PGL}_5(k)$, while $\mathfrak{D}_6$ and $\mathfrak{D}_6'$ are not conjugated in $\mathrm{PGL}_5(k)$.

Example 9.2. We consider $G = C_2^2 \times \mathfrak{S}_3 \subset W(\mathsf{D}_5)$. The action is realized via involutions c_4 and c_5 switching signs on x_4 and x_5, and the permutation action by $\mathfrak{S}_3$ on the remaining variables $x_1, \dots, x_3$.

This contributes the symbol

$$\bar{\mathfrak{s}} := (\bar{H}, \bar{Y} \circlearrowright k(Q), (1)) \in \mathrm{Burn}_3(G),$$

to the class $[X \circlearrowleft G]$; here $\bar{H} := \langle c_5 \rangle$, and $\bar{Y} := \langle c_4, \mathfrak{S}_3 \rangle \simeq C_2 \times \mathfrak{S}_3$ is acting on the quadric surface $Q \subset \mathbb{P}^3$, given by

$$\sum_{i=1}^{4} x_i^2 = 0. \tag{9.2}$$

We claim that

(1) $\bar{\mathfrak{s}}$ is an incompressible divisorial symbol in $\mathrm{Burn}_3(G)$,
(2) the $\bar{Y}$ action on Q is not birational to a (projectively) linear action, or products of such actions.

We have addressed (1) in Example 9.1. The same argument shows that the $\bar{Y}$-action on Q is not (projectively) linearizable. Note also that in this case, we do not need to pass to a standard model $\tilde{X}$ for the G-action. Indeed, when the class is computed on $\tilde{X}$, it will be a sum of various classes, with *positive* coefficients, and the incompressible class $\bar{\mathfrak{s}}$ will be among them. Since symbols $\bar{\mathfrak{s}}$ are not produced by the algorithm in Section 6 and since $\bar{\mathfrak{s}}$ is incompressible, we conclude that the G-action on X is not (projectively) linearizable.

This G is contained in $\mathfrak{S}_3 \times \mathfrak{D}_4$, so that the corresponding action on X is therefore also not (projectively) linearizable.

Example 9.3. Consider the quadric threefold X given by

$$\sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} x_i = 0.$$

It carries a natural action of $\mathfrak{S}_6$, by permutation of the coordinates as well as the induced action of $\mathfrak{A}_6$. By [9, Theorem 6.2], the $\mathfrak{A}_6$-action is super-rigid, in particular, it is not equivariantly birational to a projectively linear action.

Here we give an alternative argument, based on the Burnside formalism. First we treat $G = \mathfrak{S}_6$. The involution $x_5 \leftrightarrow x_6$ fixes a quadric surface Q with residual $\mathfrak{S}_4$-action. We have:

- The corresponding symbol

$$\bar{\mathfrak{s}} := (C_2, \mathfrak{S}_4 \circlearrowright k(Q), (1))$$

  is incompressible. Indeed, symbols appearing in the Θ_2-term come from actions on the projectivization of a rank-2 vector bundle over $\mathbb{P}^1$. Since $\mathfrak{S}_4$ does not have normal cyclic subgroups, it has to act trivially on the fibers, and generically freely on the base $\mathbb{P}^1$. In particular, any $\mathfrak{K}_4 \subset \mathfrak{S}_4$ would act without fixed points. On the other hand, the $\mathfrak{K}_4$-action on Q, generated by the transpositions $(1, 2)$ and $(3, 4)$, switching x_1, x_2 and x_3, x_4, respectively, fixes two points. This implies that $\bar{\mathfrak{s}}$ is incompressible.
- There are two *projectively linear* $\mathfrak{S}_6$-actions on $\mathbb{P}^3$, with Burnside classes presented in Section 8. The symbol $\bar{\mathfrak{s}}$ does not appear in these expressions.

We conclude that the $\mathfrak{S}_6$-action on X is not birational to a projectively linear action on $\mathbb{P}^3$.

Now we give a different argument, for $G := \mathfrak{A}_6$, and by extension $\mathfrak{S}_6$. Here, we base the argument on computations in

$$\mathcal{BC}_3(\mathfrak{A}_6) = \mathbb{Z}/2 \oplus \mathbb{Z}.$$

We analyze the fixed loci for (conjugacy classes of) subgroups $H \subset G$:

stabilizer H	$Z_G(H)$	orbit representatives of fixed loci of H
$\mathfrak{A}_4$	1	one point
$\mathfrak{A}'_4$	1	one point
$\mathfrak{S}_3$	1	two points
C_3^2	C_3^2	one point
C_5	C_5	two points
C_4	C_4	two points
C_3	C_3^2	one conic
C'_3	C_3^2	one line
C_2	$\mathfrak{D}_4$	one conic

Note that all symbols in $\mathcal{BC}_3(\mathfrak{A}_6)$ with stabilizer not equal to $H := C_3^2$ are trivial. The group $H = \langle(1,2,3),(4,5,6)\rangle$ has four fixed points, contained in the G-orbit of

$$\mathfrak{p} = (0:0:0:1:\zeta_3:\zeta_3^2).$$

The G-action is not in standard form; however, since $H = C_3^2$ is maximal, in the poset of groups with nontrivial fixed loci, symbols with this stabilizer on a standard form can only arise from blowing up these fixed points. Relation (**B**) implies that contributions from H-fixed points on the blowup equal to those on X. Thus

$$[X \circlearrowleft G] = (H, 1, ((0,2),(1,2),(2,2))) \in \mathcal{BC}_3(G),$$

which vanishes, by relation (**V**). On the other hand, the classes of projectively linear actions of G do not vanish in $\mathcal{BC}_3(G)$, see Section 8.

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