EQUIVARIANT BIRATIONAL TYPES AND DERIVED CATEGORIES

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Abstract. We investigate equivariant birational geometry of rational surfaces and threefolds from the perspective of derived categories.

1. Introduction

Let X be a smooth projective variety over an algebraically closed field k, of characteristic zero. Assume that X is equipped with a regular, generically free, action of a finite group G. A major topic in birational geometry is to understand equivariant birational types, e.g., to decide whether or not X is

- *(projectively) linearizable*, i.e., equivariantly birational to projective space, with a (projectively) linear action of G, or
- *stably (projectively) linearizable*, i.e., (projectively) linearizable after taking a product with $\mathbb{P}^m$, for some m, with trivial action on the second factor.

One of the motivations is the analogy of this theory with birational geometry over nonclosed ground fields and, in particular, with the central problem of (stable) rationality over such fields, where the role of G is taken by the absolute Galois group of the ground field, acting on geometric objects.

Various tools have been developed to distinguish equivariant birational types, e.g., cohomology, derived categories, and more recently, equivariant Burnside groups (see [HT23]). In this note, we investigate the interactions between different perspectives on the (stable) linearizability problem. We focus on low-dimensional examples, in particular, Del Pezzo surfaces, rational Fano threefolds and fourfolds. We explore the compatibility of group actions with standard (stable) rationality

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constructions and conjectures, and produce new examples of stably linearizable but nonlinearizable actions.

In detail, in Section 2, we discuss basic notions of equivariant birational geometry, classical invariants of G-actions on varieties, as well as the recently developed Burnside formalism [KT22b]. We present applications of a multilinear algebra construction, Proposition 4, to exhibit new examples of nonlinearizable but stably linearizable actions, e.g., we show in Example 7 that, for $G = \mathfrak{A}_5$, the G-birationally rigid, and thus not linearizable, quintic Del Pezzo threefold is stably linearizable.

In Section 3, we study exceptional sequences in derived categories, in presence of G-actions, and their connections with classical invariants. In Section 4, we prove

Theorem. *A smooth projective rational G-surface that is linearizable has a full G-equivariant exceptional sequence.*

The proof relies on the classification of finite subgroups in the Cremona group of [DI09], and subsequent developments in equivariant geometry of rational surfaces. Over nonclosed fields the situation was investigated in [AB18] and [BD21], in particular, we view this theorem as an analog of [AB18, Corollary 1]. However, we also give an example, in the equivariant context, where the analog of [AB18, Theorem 1] fails.

In Section 5, we turn to Fano threefolds. For quintic Del Pezzo threefolds, with give examples of nonlinearizable actions of finite groups G with derived categories admitting full exceptional sequences of G-linearized objects. This disproves the equivariant analog of the well-known conjecture that a smooth projective variety with a full exceptional sequence over the ground field should be rational. The corresponding G-actions are stably linearizable.

For Fano threefolds of genus 7, we show that there are nonlinearizable actions in presence of G-invariant semiorthogonal decompositions, with pieces equivalent, as G-categories, to derived categories of G-varieties of codimension ≥ 2.

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2. Equivariant geometry

Terminology. Throughout, G is a finite group. We consider generically free regular actions of G on irreducible algebraic varieties over k, an algebraically closed field of characteristic zero, and refer to such varieties as G-varieties. We write

$$X \sim_G Y$$

if the G-varieties X, Y are equivariantly birational, and introduce subcategories of the category of G-varieties:

- $\mathfrak{Lin}$ - G-linearizable,
- $\mathfrak{PLin}$ - projectively G-linearizable,
- $\mathfrak{SLin}$ - stably G-linearizable.

A basic result in G-birational geometry is equivariant resolution of singularities and weak factorization: G-birational varieties are related via blowups and blowdowns, with centers in smooth G-stable subvarieties. Moreover, after a sequence of such blowups, one can reach a *standard model* $\tilde{X} \to X$ such that on $\tilde{X}$ all stabilizers are abelian, and G-orbits of divisors with nontrivial stabilizers are smooth, i.e., for every such D and $g \in G$, the intersection $(D \cdot g) \cap D$ is either all of D or empty.

Classical invariants. The G-action on X induces actions on cohomology groups, and in particular on the Picard group $\mathrm{Pic}(X)$. There is an exact sequence, see, e.g., [KT22a, §3]

$$\mathrm{Pic}(X, G) \longrightarrow \mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G, k^\times)$$

$$\longrightarrow \mathrm{Br}([X/G]) \longrightarrow \mathrm{H}^1(G, \mathrm{Pic}(X)) \xrightarrow{\delta_3} \mathrm{H}^3(G, k^\times),$$

where:

- $\mathrm{Pic}(X, G)$ is the group of isomorphism classes of G-linearized line bundles, and $\mathrm{Pic}(X)^G$ the group of G-invariant line bundles
- $[X/G]$ is the quotient stack, $\mathrm{Br}([X/G])$ its Brauer group, and

Both δ_2 and δ_3 are zero when there are G-fixed points; these give sections of the map $[X/G] \to BG$. Other frequently studied obstructions to (stable) linearizabilty are:

- $\mathrm{Am}(X, H)$, the Amitsur invariant, i.e., the image of

$$\delta_2 : \mathrm{Pic}(X)^H \to \mathrm{H}^2(H, k^\times), \quad H \subseteq G,$$

- **(H1)**: $\mathrm{H}^1(H, \mathrm{Pic}(X)) = \mathrm{H}^1(H, \mathrm{Pic}(X)^\vee) = 0$, $H \subseteq G$,
- **(SP)**: $\mathrm{Pic}(X)$ is a stable G-permutation module.

If $X \in \mathfrak{PLin}$ or $\mathfrak{Lin}$, then **(H1)** and **(SP)** hold; when $X \in \mathfrak{Lin}$ then the Amitsur invariant vanishes.

Burnside formalism. Let G be a finite group, acting on X, a standard model for the action. On such a model one computes the class of the G-action in the *Burnside group*:

$$[X \circlearrowleft G] = \sum_{F,H} (H, Y \circlearrowright k(F), \beta) \in \mathrm{Burn}_n(G),\ n = \dim(X), \tag{2.1}$$

as a sum of *symbols*, recording (G-orbits of) irreducible subvarieties $F \subset X$ with nontrivial generic stabilizer H, together with the induced action of a subgroup $Y \subseteq Z_G(H)/H$ on the function field $k(F)$ and the collection β of weights of H in the normal bundle to F (all defined up to conjugation in G). In particular, this sum contains the trivial summand

$$(1, G \circlearrowright k(X), ()),$$

The symbols are subject to explicit relations so that the class (2.1) is an equivariant birational invariant (see [KT22b], [HKT21] for definitions and examples). The trivial summand does not participate in relations; we say that the G-action on X has trivial Burnside class if

$$[X \circlearrowleft G] = (1, G \circlearrowright k(X), ())$$

in $\mathrm{Burn}_n(G)$. *Incompressible divisorial symbols* (modulo conjugation relation), in the terminology of [KT22c, Definition 3.3], generate, freely, a direct summand of $\mathrm{Burn}_n(G)$; in many situations, it suffices to compare their contribution to $[X \circlearrowleft G]$ to distinguish G-actions up to equivariant birationality, see [TYZ23, Section 3.6].

The paper [KT22c] provides an algorithm for the computation of $[\mathbb{P}(V) \circlearrowleft G]$ for linear and projective linear actions of a finite group G; this algorithm has been implemented in `Magma`, see [TYZ23]. While the formalism and the computations can be involved, incompressible divisorial symbols allow to quickly show nonlinearizability of some actions. Indirectly, they also lead to constraints on possible actions:

Example 1. Let $X \subset \mathbb{P}^n$ be a prime (smooth) Fano threefold of index 1, in its anticanonical embedding. Let $\sigma \in \mathrm{PGL}_{n+1}$ be an involution preserving a hyperplane. Does σ preserve X?

If so, we would have $X^\sigma = S$, a surface, yielding a symbol

$$(\langle \sigma \rangle, 1 \circlearrowright k(S), (1)) \in \mathrm{Burn}_3(C_2).$$

Generically, S would be a K3 surface, the symbol incompressible, and thus the action not linearizable.

On the other hand, consider smooth Fano threefolds $X = X_{22}$ of genus 12. We know that G-actions on X are linearizable, if there is a (sufficiently general) fixed point; in the arithmetic setup this is discussed in [KP23, Theorem 5.17].

This tension can be reconciled, in fact, X cannot carry such involutions. We sketch an argument: According to Mukai, cf. [Sch01], X can be constructed as follows: start with a 7-dimensional vector space V, a 3-dimensional vector space U, and a linear map $\eta\colon \wedge^2(V) \to U^*$. Dually, this arises from a linear map $\eta^*\colon U \to \wedge^2(V^*)$. Consider the Grassmannian $\mathrm{Gr}(3,V) \subset \mathbb{P}(\wedge^3(V))$, and for the universal subbundle $\mathcal{U}$ notice that $\mathrm{H}^0(\mathrm{Gr}(3,V),\mathcal{U}^*) = \wedge^2(V^*)$. The zeros of the sections in U on $\mathrm{Gr}(3,V)$ yield X. Equivalently, we have a linear map

$$\wedge^3(V) \to V \otimes U^*,$$

induced by wedging elements in $\wedge^3(V)$ with elements in U, and the kernel K is a 14-dimensional subspace of $\wedge^3(V)$ such that

$$X = \mathrm{Gr}(3,V) \cap \mathbb{P}(K).$$

We can view η^* as an element in $\wedge^2(V^*) \otimes U^*$, i.e., a skew-symmetric 7×7-matrix with entries in U^*. The 6×6 Pfaffians of this matrix define an Artinian Gorenstein module of codimension 3 over $k[U^*]$, with dual socle generator a quartic F, see [Sch01, Theorem 2.6 and its proof]. Conversely, the datum of this quartic or Artinian Gorenstein module allows to reconstruct the skew-symmetric 7×7-matrix with entries in U^* by considering the middle map in the Buchsbaum-Eisenbud resolution of the module.

By [Sch01, Theorem 6.1], the Scorza quartic S_F covariantly associated to F is isomorphic to the Hilbert scheme of lines $F_1(X)$ in X, and by [DM22, Appendix, Claim A.1.1., (by Prokhorov)], the automorphisms of X embed injectively into those of $F_1(X)$.

Now assume that $G = C_2$ is acting faithfully on X, through the G-representations U and V and an equivariant map η, as described above. Then G embeds into the automorphisms of the quartic C, and acts faithfully on U with weights $(1,1,0)$ or $(0,0,1)$. Then [Sch01, Theorem 2.6] implies that V can be recovered as the kernel of the linear map

$$\mathrm{Sym}^3(U) \to U$$

given by contracting with the equation of the C_2-invariant quartic F. This map is equivariant (possibly after tensoring the target U with a sign). After that, we recover K as the kernel of the G-equivariant map

$$\wedge^3(V) \to V \otimes U^*,$$

above. Working through all sign combinations, one verifies that G cannot act on K with all weights but one equal to each other.

Example 2. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, with an action of $G = C_m$, with weights $(0,0,0,0,0,b)$; note that only $m = 2,3$ are possible, under this assumption. Assume that the divisor $D \subset X$, given by the vanishing of the last coordinate, is smooth. Then $X \notin \mathfrak{PLin}$.

Indeed, the corresponding symbol $(C_m, 1 \circlearrowleft k(D), (b))$ is incompressible, since D is not birational to $S \times \mathbb{P}^1$, for any surface S, and does not appear in classes of linear actions, (see [TYZ23, Corollary 6.1]).

For $m = 2$, [Mar22, Theorem 1.2(2)] shows that a *very general* X carrying such an involution does not have an associated K3 surface and is expected to be nonrational; and, in particular, the action would not be linearizable. The same argument applies for $m = 3$.

Pfaffians and Grassmannians. In [BvBT23, Section 7], we have used a construction from multilinear algebra, the Pfaffian construction, to exhibit nonlinearizable but stably linearizable actions of finite groups on rational varieties, e.g., rational cubic fourfolds. The starting point is a Pfaffian variety

$$X := \mathrm{Pf}(W) \cap \mathbb{P}(L),$$

where V is a vector space of dimension $n = 2m$ and $L \subset \wedge^2(W)$ a linear subspace of dimension n. Then there is a diagram

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{K}_X) & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & \mathbb{P}(W^*) \end{array}$$

where $\mathcal{K}_X$ is a vector bundle of rank 2 and q is birational. In presence of group actions, choosing a G-representation W and a subrepresentation L, one obtains, under suitable genericity assumptions, an equivariant birationality:

$$X \times \mathbb{P}^1 \sim_G \mathbb{P}(W^*),$$

with trivial action on the second factor.

Example 3. Let $G = C_5 \rtimes \mathfrak{D}_{15} \rtimes C_3$, `GapID`(450,24). It acts generically freely on the singular (toric) cubic fourfold $X \subset \mathbb{P}^5$ with equation

$$x_1x_3x_5 + x_2x_4x_6 = 0,$$

a degeneration of the Pfaffian cubic considered in [BvBT23, Example 14]. The G-action on X is not linearizable, as G does not have faithful representations of dimension < 6. The Pfaffian construction applies: by [BvBT23, Corollary 13], the G-action on $X \times \mathbb{P}^1$, with trivial action on the second factor, is linearizable.

Here, we present another such construction, applicable to subvarieties of Grassmannians. Linear sections of Grassmannians admit tautological stable rationality constructions, that we now describe: Let W be an n-dimensional vector space over k and $\mathrm{Gr}(2, W)$ the Grassmannian of planes in W. Let $V \subset \wedge^2(W)$ be a linear subspace of codimension r. Put

$$X := \mathrm{Gr}(2, W) \cap \mathbb{P}(V)$$

and consider the diagram

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{U}_X) & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & \mathbb{P}(W), \end{array}$$

where $\mathcal{U}_X$ is the restriction of the universal vector bundle over $\mathrm{Gr}(2, W)$ to X. This yields a stable rationality construction, as both p and q are vector bundles, in the indicated range of dimensions.

Proposition 4. *Let k be an algebraically closed field of characteristic zero and G a finite group. Let W be an n-dimensional representation of G over k and $V \subset \wedge^2(W)$ a subrepresentation of codimension $r \leq n-2$ such that the G-actions on $\mathbb{P}(W)$ and $\mathbb{P}(V)$ are generically free. Assume that*

($*$) *X is irreducible of dimension* $\dim(\mathrm{Gr}(2, n)) - r = 2(n-2) - r$.

Then

$$X \times \mathbb{P}^1 \sim_G \mathbb{P}(W) \times \mathbb{P}^{n-2-r},$$

with trivial actions on the second factors.

Proof. By the No-name Lemma, $X \times \mathbb{P}^1 \sim_G \mathbb{P}(\mathcal{U}_X)$. Note that each fiber of q is nonempty: indeed, the fiber over $[w] \in \mathbb{P}(W)$ is $\mathbb{P}(w \wedge W) \cap \mathbb{P}(V)$, which has dimension $\geq n - 2 - r \geq 0$, the last inequality by the assumption $r \leq n-2$. By assumption $(*)$, it follows that for generic $w \in W$, one has

$$\dim(\mathbb{P}(w \wedge W) \cap \mathbb{P}(V)) = n - 2 - r.$$

Thus $\mathbb{P}(\mathcal{U}_X)$, which is irreducible since X is, is generically the projectivization of a G-vector bundle over $\mathbb{P}(W)$ via q. Another application of the No-name Lemma yields the result. □

Remark 5. If we drop assumption $(*)$, but keep assuming $r \leq n-2$, then the construction of Proposition 4 still yields stable linearizability for the unique component of X such that the restriction of $\mathbb{P}(\mathcal{U}_X)$ to it dominates $\mathbb{P}(W)$. But proving nonlinearizability of such a component of X is usually difficult, unless we assume a condition similar to $(*)$, a priori.

This construction works also over nonclosed fields. However, there one does not gain new insights: by [Xu12, Theorem 2.2.1], if $r \leq n-2$ and X is smooth then X is already rational over k. The proof uses the same diagram, restricted to a codimension one linear subspace Π in $\mathbb{P}(W)$, exhibiting X as birational to a vector bundle over Π, thus rational over k. In presence of group actions, this can fail, e.g., if W does not admit a subrepresentation of codimension one! This yields many examples of nonlinearizable but stably linearizable actions.

Example 6. Let $G = \mathfrak{S}_5$, and $W := W_5$ its 5-dimensional representation. We have a decomposition

$$\wedge^2(W) = W_6 \oplus W_4,$$

as representations. When $V = W_6$ is the 6-dimensional subrepresentation,

$$S := \mathrm{Gr}(2, W) \cap \mathbb{P}(V)$$

is the del Pezzo surface of degree 5. It is easy to see that the induced G-action on S is not linearizable, indeed, $\mathfrak{S}_5$ does not admit a linear action on $\mathbb{P}^2$. Even the restriction to $\mathfrak{A}_5 \subset \mathfrak{S}_5$ is not linearizable, see, e.g., [CS16, Theorem 6.6.1].

Note that the assumptions of Proposition 4 are *not* fulfilled, we have $n = 5$ and $r = 4$, rather than $r \leq 3$. Nevertheless, by [Pro10, Proposition 4.7], $S \times \mathbb{P}^1$ is $\mathfrak{S}_5$-equivariantly birational to the Segre cubic threefold, with the action of the *nonstandard* $\mathfrak{S}_5 \subset \mathfrak{S}_6$, which is linearizable. An alternative proof of stable linearizability of S, using the equivariant torsor formalism, is in [HT23, Proposition 20].

Example 7. We modify the previous example, considering $G = \mathfrak{A}_5$. Then

$$\wedge^2(W) = W_3 \oplus W_3' \oplus W_4,$$

and we put $V := W_3 \oplus W_4$. Then

$$X := \mathrm{Gr}(2, W) \cap \mathbb{P}(V)$$

is a smooth threefold [CS16, Lemma 7.1.1], the quintic Del Pezzo threefold. One of the main results of [CS16] is that X is G-birationally rigid.

Here, the construction of Proposition 4 applies, and we obtain

$$X \times \mathbb{P}^1 \sim_G \mathbb{P}(W).$$

Thus, $X \notin \mathfrak{L}\mathrm{in}$ but $X \in \mathfrak{S}\mathfrak{L}\mathrm{in}$.

To check the condition $(*)$ in Proposition 4 and the smoothness of X (independently of [CS16]), one can proceed as follows: we view $\mathfrak{A}_5$ as a subgroup of PSL_2 and proceed in terms of PSL_2-representations, as in [CS16, Section 7]. Put $W = \mathrm{Sym}^4(k^2)$ and consider the decomposition

$$\wedge^2(W) = \mathrm{Sym}^2(k^2) \oplus \mathrm{Sym}^6(k^2). \tag{2.2}$$

Let

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of the Lie algebra of SL_2 satisfying

$$[\mathbf{H}, \mathbf{X}] = 2\mathbf{X},\ [\mathbf{H}, \mathbf{Y}] = -2\mathbf{Y},\ [\mathbf{X}, \mathbf{Y}] = \mathbf{H}.$$

Let w_4 be a highest weight vector in W (subscripts in the sequel indicate the weight). Thus $\mathbf{X}(w_4) = 0$, and

$$w_4,\ w_2 = \mathbf{Y}(w_4),\ w_0 = \mathbf{Y}^2(w_4),\ w_{-2} = \mathbf{Y}^3(w_4),\ w_{-4} = \mathbf{Y}^4(w_4)$$

form a basis for W; using the commutation relations inductively gives

$$\mathbf{X}(w_4) = 0, \mathbf{X}(w_2) = 4w_4, \mathbf{X}(w_0) = 6w_2, \mathbf{X}(w_{-2}) = 6w_0, \mathbf{X}(w_{-4}) = 4w_{-2}.$$

To find a highest weight vector in the subrepresentation $\mathrm{Sym}^2(k^2)$ of $\wedge^2(W)$ one is looking for a linear combination of $w_4 \wedge w_{-2}$ and $w_2 \wedge w_0$ annihilated by $\mathbf{X}$. These are thus multiples of

$$x_2 := 3w_2 \wedge w_0 - 2w_4 \wedge w_{-2},$$

and applying $\mathbf{Y}$ and $\mathbf{Y}^2$ to x_2 we obtain a basis for $\mathrm{Sym}^2(k^2)$ as a submodule of $\wedge^2(W)$ as

$$\begin{aligned} x_0 := \mathbf{Y}(x_2) &= \mathbf{Y}\left(3w_2 \wedge w_0 - 2w_4 \wedge w_{-2}\right) \\ &= 3(\mathbf{Y}(w_2) \wedge w_0 + w_2 \wedge \mathbf{Y}(w_0)) - 2(\mathbf{Y}(w_4) \wedge w_{-2} + w_4 \wedge \mathbf{Y}(w_{-2})) \\ &= 3(w_0 \wedge w_0 + w_2 \wedge w_{-2}) - 2(w_2 \wedge w_{-2} + w_4 \wedge w_{-4}) \\ &= w_2 \wedge w_{-2} - 2w_4 \wedge w_{-4} \end{aligned}$$

and

$$\begin{aligned} x_{-2} := \mathbf{Y}\left(w_2 \wedge w_{-2} - 2w_4 \wedge w_{-4}\right) \\ = w_0 \wedge w_{-2} - w_2 \wedge w_{-4}. \end{aligned}$$

Similarly, one can find a basis of $\mathrm{Sym}^6(k^2) \subset \wedge^2(W)$ by applying $\mathbf{Y}$ successively to the highest weight vector $w_4 \wedge w_2$ in that copy of $\mathrm{Sym}^6(k^2)$.

We have thus explicitly identified both $\mathrm{Sym}^2(k^2)$ and $\mathrm{Sym}^6(k^2)$ with PGL_2-subrepresentations of $\wedge^2(W)$, and can check that

$$X = \mathrm{Gr}(2, W) \cap \mathbb{P}(\mathrm{Sym}^6(k^2))$$

is irreducible, smooth of the expected dimension 3 by computer algebra. The necessary checks were performed using `Macaulay2`[1].

Example 8. Let $G = C_9 \rtimes C_6$, `G:=SmallGroup(54,6)`. Its smallest faithful representation has dimension 6, in particular, G does not admit a linear action on $\mathbb{P}^4$. Let W be its unique irreducible representation of dimension 6, it has character

$$(6, 0, -3, 0, 0, 0, 0, 0, 0, 0).$$

We have a decomposition:

$$\wedge^2(W) = V_1 \oplus V_1' \oplus V_1'' \oplus V_2 \oplus V_2' \oplus V_2'' \oplus V_6 \tag{2.3}$$

into irreducible representations. Choose a suitable subrepresentation

$$V := V_1 \oplus V_2 \oplus V_2' \oplus V_6,$$

more precisely, that with respective characters, for $\zeta = \zeta_3$,

[1] `warwick.ac.uk/fac/sci/maths/people/staff/boehning/m2filesequivariantderived`

$$\begin{aligned}\texttt{X.6} =&(1,-1,1,\zeta^2,\zeta,-\zeta,-\zeta^2,\zeta,1,\zeta^2),\\ \texttt{X.7} =&(2,0,2,2,2,0,0,-1,-1,-1),\\ \texttt{X.8} =&(2,0,2,2\zeta,2\zeta^2,0,0,-\zeta^2,-1,-\zeta),\\ \texttt{X.10} =&(6,0,-3,0,0,0,0,0,0,0).\end{aligned}$$

The complement decomposes as

$$\begin{aligned}\texttt{X.2} =&(1,-1,1,1,1,-1,-1,1,1,1),\\ \texttt{X.3} =&(1,-1,1,\zeta,\zeta^2,-\zeta^2,-\zeta,\zeta^2,1,\zeta),\\ \texttt{X.9} =&(2,0,2,2\zeta^2,2\zeta,0,0,-\zeta,-1,-\zeta^2).\end{aligned}$$

Then, according to `magma`,

$$X := \mathrm{Gr}(2, W) \cap \mathbb{P}(V)$$

is a smooth and irreducible variety of dimension 4 and degree 14. Note that choosing a different 11-dimensional subrepresentation V also yields irreducible fourfolds of degree 14, but some of these are singular. Thus the construction of Proposition 4 applies, and we have $X \notin \mathfrak{L}\mathrm{in}$ and $X \in \mathfrak{S}\mathfrak{L}\mathrm{in}$.

Let $G = C_3^3 \rtimes \mathfrak{S}_3$, `SmallGroup(162,19)`. Its smallest faithful representation has dimension 6, in particular, G does not admit a linear action on $\mathbb{P}^4$. Let W be an irreducible G-representation with character

$$(6,0,-3,0,0,3,-3,0,0,0,0,0,0).$$

We have a decomposition

$$\wedge^2(W) = V_1 \oplus V_2 \oplus V_3 \oplus V_3' \oplus V_6$$

into irreducible representations. We choose

$$V := V_2 \oplus V_3 \oplus V_6.$$

Then

$$X := \mathrm{Gr}(2, W) \cap \mathbb{P}(V)$$

is irreducible (singular) of dimension 4 as can be checked by computer algebra[2]. Therefore this construction satisfies the hypotheses of Proposition 4, thus $X \notin \mathfrak{L}\mathrm{in}$ but $X \in \mathfrak{S}\mathfrak{L}\mathrm{in}$.

[2] `warwick.ac.uk/fac/sci/maths/people/staff/boehning/m2filesequivariantderived`

3. Derived categories

Terminology. Let X be a smooth projective variety (over an algebraically closed field k of characteristic zero) and $\mathsf{D}^b(X)$ its derived category of coherent sheaves. We use freely the following terms; see, e.g., [BvBT23, Section 2], or [Kuz16, Section 1 and 2] for definitions and references:

- admissible subcategories of $\mathsf{D}^b(X)$,
- exceptional objects,
- (full) exceptional sequences,
- (maximal) semiorthogonal decompositions.

G-actions on categories. Let G be an algebraic group, not necessarily finite. Let X be a smooth projective G-variety, i.e., a smooth projective variety with a generically free, regular, action of G. In [BvBT23, Proposition 3] it was remarked that the fundamental reconstruction theorem by Bondal and Orlov [BO01] admits the following equivariant version:

Proposition 9. *Suppose X and Y are smooth projective G-varieties over k, X is Fano, and*

$$\Phi\colon \mathsf{D}^b(X) \simeq \mathsf{D}^b(Y)$$

is an equivalence as k-linear triangulated categories together with the induced G-actions. Then X and Y are isomorphic as G-varieties, i.e., there exists a G-equivariant isomorphism

$$X \xrightarrow{\sim} Y.$$

In practice, this general theorem is not very useful since the derived category contains too much information; in the context of rationality problems, the focus is on trying to extract information about the variety from more accessible data, such as a piece, or several pieces, in a semiorthogonal decomposition of $\mathsf{D}^b(X)$.

We will explore the extent to which these considerations apply in the equivariant context. We investigate, in several representative geometric examples, the effects of G-equivariant birationalities on

- the existence of full exceptional sequences in $\mathsf{D}^b(X)$ that are compatible with G-actions, and
- derived Hom-spaces between objects in $\mathsf{D}^b(X)$.

G-actions and exceptional sequences.

Definition 10. An object $E \in \mathsf{D}^b(X)$ is called *G-invariant* if g^*E is isomorphic to E, for all $g \in G$. It is called *G-linearized* if it is equipped with a G-linearization, i.e. a system of isomorphisms

$$\lambda_g \colon E \to g^*E, \quad \forall g \in G,$$

satisfying the compatibility condition

$$\lambda_1 = \mathrm{id}_E, \quad \lambda_{gh} = h^*(\lambda_g) \circ \lambda_h.$$

Several notions of compatibility of exceptional sequences with G-actions have been studied; we follow [CT20, Definition 2.1].

Definition 11. Let X be a smooth projective G-variety and

$$\mathbf{E} := (E_1, \dots, E_n)$$

a full exceptional sequence in $\mathsf{D}^b(X)$.

(1) $\mathbf{E}$ is *G-invariant* if for every $r \in \{1, \dots, n\}$ and every $g \in G$, there is an s such that $g^*E_r \simeq E_s$.
(2) $\mathbf{E}$ is *G-equivariant* if it is G-invariant and, for all r, E_r is isomorphic to a G_r-linearized object in $\mathsf{D}^b(X)$, where $G_r \subseteq G$ is the stabilizer of the isomorphism class of E_r.
(3) $\mathbf{E}$ is *G-linearized* if it is G-equivariant and, for all r, $G_r = G$, i.e., each E_r is a G-linearized object.

Example 12. Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, and the full exceptional sequence in $\mathsf{D}^b(X)$ from [Kap88]

$$\mathbf{E} = (\mathcal{O}(-1,-1), \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1)).$$

Then

- $\mathbf{E}$ is H-linearized, if we view X as $\mathbb{P}(V) \times \mathbb{P}(V)$ with its natural diagonal H-action, where H is a finite group admitting a two-dimensional faithful linear representation V such that H acts generically freely on $\mathbb{P}(V)$.
- $\mathbf{E}$ is G-equivariant, but not G-linearized, if we let $G = \mathbb{Z}/2 \times H$ act on $X = \mathbb{P}(V) \times \mathbb{P}(V)$, with the first factor $\mathbb{Z}/2$ in G switching the rulings,
- $\mathbf{E}$ is G-invariant, but not G-equivariant, if we let $H \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ act on $\mathbb{P}^1$ via the two-dimensional faithful irreducible representation of its Schur cover $\mathfrak{D}_8$ instead, and then let $G = \mathbb{Z}/2 \times H$ act on $\mathbb{P}^1 \times \mathbb{P}^1$, again with $\mathbb{Z}/2$ switching the factors and H acting diagonally.

Example 13. The moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points has a full $\mathfrak{S}_n$-equivariant exceptional sequence, where $\mathfrak{S}_n$ is the symmetric group permuting the marked points, by the main result of [CT20].

The following observation will be useful in applications.

Lemma 14. *Let X be a smooth projective G-variety, and*

$$\mathbf{E} := (E_1, \dots, E_n)$$

a G-invariant exceptional sequence consisting of line bundles. If X is G-linearizable, then $\mathbf{E}$ is a G-equivariant exceptional sequence.

Proof. If X is G-linearizable, $\mathrm{Am}(X, G)$ is trivial. The same holds for $\mathrm{Am}(X, H)$, for any $H \subseteq G$, since X is also H-linearizable. Therefore, under the assumptions of the Lemma, any line bundle on X is H-linearized for every subgroup H that leaves this line bundle invariant. □

Connections with classical invariants.

Proposition 15. *Let G be a finite group and X a smooth projective G-variety admitting a G-linearized full exceptional sequence. Then*

$$\mathrm{Pic}(X, G) \twoheadrightarrow \mathrm{Pic}(X)^G,$$

in particular,

$$\mathrm{Am}(X, G) = 0.$$

Proof. Taking the first Chern class gives a well-defined homomorphism

$$c_1 \colon \mathsf{D}^b(X) \to \mathrm{Pic}(X).$$

If $\mathsf{D}^b(X)$ is generated by an exceptional sequence $\mathbf{E} = (E_1, \dots, E_n)$ of G-linearized objects, then every class in $\mathrm{Pic}(X)$ is a $\mathbb{Z}$-linear combination of the $c_1(E_r)$, which are G-linearized. Indeed, the first Chern class of a G-linearized complex is G-linearized; it is the alternating sum of the Chern classes of the cohomology sheaves which are G-linearized, so we just need to show invariance of the Chern class for a G-linearized sheaf. Such a sheaf always has a finite locally free resolution by G-linearized vector bundles since there exists a G-linearized ample sheaf on X and the statement is true on projective space. □

Remark 16. Let G be a finite group and X a smooth projective G-variety with ample anticanonical class. By Proposition 9, the derived category $\mathsf{D}^b(X)$ determines X, as a G-variety. In particular, we can extract the G-action on $\mathrm{Pic}(X)$, and determine whether or not it satisfies **(H1)** or **(SP)**. Concretely, we have

$$\mathrm{Aut}(\mathsf{D}^b(X)) = (\mathrm{Pic}(X) \times \mathbb{Z}) \rtimes \mathrm{Aut}(X),$$

and the derived automorphisms acting trivially on point objects can be identified with $\mathrm{Pic}(X)$.

In the following sections we investigate connections between existence of full exceptional sequences with various compatibility properties with the G-action, and (stable) linearizability of X. It turns out that G-linearizability often implies the existence of a full equivariant exceptional sequence, provided such sequences exist in the non-equivariant setting, as for Del Pezzo surfaces.

4. Del Pezzo surfaces

Terminology. By the Minimal Model Program, every rational surface is birational to a conic bundle over $\mathbb{P}^1$ or a Del Pezzo surface, i.e., a smooth projective surface X with ample anticanonical class $-K_X$; we let

$$d = d(X) = (-K_X)^2$$

be its degree. The same holds over nonclosed field, and in presence of group actions.

Here and below *conic bundle* means that X is smooth and all fibers of $f\colon X \to \mathbb{P}^1$ are isomorphic to reduced conics in $\mathbb{P}^2$. We recall the terminology of [DI09, Pro15]: a conic bundle $f\colon X \to \mathbb{P}^1$ is called *exceptional* if for some positive integer g the number of degenerate fibers equals $2g+2$ and there are two disjoint sections C_1 and C_2 with $C_1^2 = C_2^2 = -(g+1)$. Exceptional conic bundles can be constructed explicitly, see [DI09, §5.2].

Nonlinearizable actions. In this section, G is a finite group. The following nonlinearizability results for G-conic bundles are probably known to experts in birational rigidity; here, we rely on the Burnside formalism.

Lemma 17. *Let $X \to \mathbb{P}^1$ be a relatively minimal G-conic bundle with $K_X^2 = 1$. Then X is not linearizable.*

Proof. If X fails **(H1)**, then then $X \notin \mathfrak{Lin}$. If X satisfies **(H1)**, then the classification in [Pro15, §8], Theorem 8.3, shows that G must be the binary dihedral group $\widetilde{\mathfrak{D}}_5$, a nontrivial central C_2-extension of $\mathfrak{D}_5$.

Such X, with the G-action, are given by an explicit construction [Pro15, Construction 8.4]. In particular, there is a distinguished involution $\tau \in G$, generating the center of G and fixing a smooth rational curve C. The residual action of $\mathfrak{D}_5 = G/\langle\tau\rangle$ is generically free on C. Applying the Burnside formalism to this situation, we find a unique, incompressible, symbol

$$(C_2, \mathfrak{D}_5 \circlearrowright k(\mathbb{P}^1), (1)), \tag{4.1}$$

contributing to the class

$$[X \circlearrowleft G] \in \mathrm{Burn}_2(G).$$

A generically free linear action of $\widetilde{\mathfrak{D}}_5$ on $\mathbb{P}^2$ necessarily arises from a representation $V = V_1 \oplus V_2$, where V_1 is 1-dimensional representation which is nontrivial on τ and V_2 is a 2-dimensional representation which is trivial on τ. Then G fixes the point $p_0 := [1 : 0 : 0] \in \mathbb{P}^2$ and stabilizes the line given by $x_0 = 0$. Passing to a standard model, we observe, as in a similar situation in [HKT21, Section 7.6], that the linear action contributes *two* symbols (4.1), one from the exceptional divisor of the blowup of p_0 and the other from $\mathbb{P}^1 = \mathbb{P}(V_2)$. It follows that

$$[X \circlearrowleft G] \neq [\mathbb{P}^2 \circlearrowleft G]$$

in $\mathrm{Burn}_2(G)$, and the G-action on X is not linearizable. □

Lemma 18. *Let X be a minimal rational G-surface that is an exceptional conic bundle with $K_X^2 = 2$ and $g = 2$. Then X is not linearizable.*

Proof. If X fails **(H1)** then X is not linearizable. The other cases have been classified in [Pro15, §8]: consider the representation

$$\varrho \colon G \to \mathrm{Aut}(\mathrm{Pic}(X)),$$

its kernel $\ker(\varrho)$, and the exact sequence

$$1 \to G_F \to G \to G_B \to 1, \tag{4.2}$$

where $G_F \subset G$ is the largest subgroup acting trivially on the base $B = \mathbb{P}^1$. By [Pro15, Theorem 8.3], we have $\ker(\varrho) \neq \{1\}$; by [Pro15, Theorem 8.6(2)] it is cyclic, whereas $G_B \simeq \mathfrak{D}_n$, with $n \geq 3$, or $G_B \simeq \mathfrak{S}_4$. The table in [Pro15, Section 8.7] shows that $G_B = \mathfrak{S}_4$, and [Pro15, Thm. 8.6] shows that $G_F = \ker(\varrho) = C_m$, a nontrivial cyclic group of order m.

Write $C_m = C_{2^r} \times C_{m'}$ with $\gcd(m', 2) = 1$, and consider a 2-Sylow subgroup G_2 of G that contains C_{2^r}. Then G_2 has order $2^r \times 8$ and sits in an extension

$$1 \to C_{2^r} \to G_2 \to \bar{G}_2 \to 1$$

where $\bar{G}_2$ is a subgroup of $\mathfrak{S}_4$ of order 8, hence equal to $\mathfrak{D}_4$. Since the order of a group is divisible by the degree of any of its irreducible representations, every 3-dimensional representations V of G_2 has to decompose into irreducible summands of degrees $1, 1, 1$ or $1, 2$. Only the latter can be generically free. Thus, we may assume that V is of the form

$$V = V_1 \oplus V_2$$

with V_i irreducible of dimension i. Here $V_1 = k_\chi$ is a representation of G_2 by some character χ, and we can assume that V_1 is trivial, and V_2 is a faithful G_2-representation. A standard model for G_2-action is the blowup $\widetilde{\mathbb{P}(V)} \to \mathbb{P}(V)$ of the G_2-fixed point $p_0 = [1 : 0 : 0]$, see [HKT21, Section 7.2]. The only incompressible divisorial symbols might arise from the exceptional divisor, respectively, the preimage of the projectivization $\mathbb{P}^1 = \mathbb{P}(V_2) \subset \mathbb{P}(V)$. The corresponding symbols are

$$(C, G_2/C \curvearrowright k(\mathbb{P}^1), (\chi)), \quad (C, G_2/C \curvearrowright k(\mathbb{P}^1), (\bar{\chi})),$$

where $C \subset G_2$ is a cyclic group and χ is a primitive character of C. Their sum in $\mathrm{Burn}_2(G_2)$ cannot equal to

$$(C_{2^r}, \mathfrak{K}_4 \curvearrowright k(\mathbb{P}^1), (\psi))$$

with ψ some primitive character of C_{2^r}, for any choices of C, χ. Thus

$$[X \circlearrowleft G_2] \neq [\mathbb{P}^2 \circlearrowleft G_2]$$

in $\mathrm{Burn}_2(G_2)$, for any generically free linear action of G_2 on $\mathbb{P}^2$. $\square$

Linearization and derived categories. We consider rational G-surfaces and investigate which pieces and properties of the G-category $\mathsf{D}^b(X)$ are sensitive to geometric, and in particular, G-birational, characteristics of the G-action on X.

Lemma 19. *Let X be a rational G-surface X admitting a full G-invariant exceptional sequence. Then the G-action on* $\mathrm{Pic}(X)$ *satisfies* **(H1)** *and* **(SP)**.

Proof. The classes of the terms of the sequence in the Grothendieck K-group $\mathrm{K}_0(X)$ form a $\mathbb{Z}$-basis that is permuted by G. Thus $\mathrm{K}_0(X)$ is a permutation module, and since

$$\mathrm{K}_0(X) \simeq \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$$

as G-modules, with trivial G-action on the two summands $\mathbb{Z}$, we obtain the claim. □

Incidentally, assuming X is a minimal G-Del Pezzo surface, Theorem 1.2 of [Pro15] shows that **(H1)** is equivalent to the fact that G does not fix a curve of positive genus, and also equivalent to the condition $K_X^2 \geq 5$ or X being a special quartic Del Pezzo surface with a very special action, described in [Pro15, Thm. 1.2, (iii), (b)].

Lemma 20. *Let $X = \mathbb{P}^2$ with a projectively linear but nonlinear action of a finite group G. Then X does admit a full G-invariant exceptional sequence, but no full G-equivariant exceptional sequence.*

Proof. The exceptional sequence $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ is G-invariant. From [KO94] it is known that every exceptional object in $\mathsf{D}^b(X)$ is, up to shift, a vector bundle. Thus assume that $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ is a G-equivariant full exceptional sequence consisting of vector bundles. Since the map

$$\mathsf{D}^b(X) \to \mathrm{K}_0(X)$$

is G-equivariant and the action on $\mathrm{K}_0(X)$ is trivial in this case, we see that every element in G fixes the isomorphism class of each $\mathcal{E}_i$ (because the images of $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ form a $\mathbb{Z}$-basis of $\mathrm{K}_0(X)$). If we compose $\mathsf{D}^b(X) \to \mathrm{K}_0(X)$ with the first Chern class map, we get a surjective map to $\mathrm{Pic}(X)$. In other words, the top exterior powers of the $\mathcal{E}_i$ generate the Picard group, hence at least one of them has to be isomorphic to $\mathcal{O}_{\mathbb{P}^2}(r)$ for some odd integer r. Thus $\mathcal{O}_{\mathbb{P}^2}(1)$ is also G-linearized, contradicting our assumption that the action is nonlinearizable. □

Examples with $\mathrm{Aut}(X)$-equivariant exceptional sequences. We present examples of rational surfaces X such that $\mathsf{D}^b(X)$ admits a full $\mathrm{Aut}(X)$-equivariant exceptional sequence but X is not leariable, for some $G \subseteq \mathrm{Aut}(X)$.

DP6: Let X be a Del Pezzo surface of degree 6. Then X has full $\mathrm{Aut}(X)$-*invariant* exceptional sequence. Indeed, recall that there is an exact sequence

$$0 \to T \to \mathrm{Aut}(X) \to W_X \to 0,$$

where

$$W_X \simeq \mathbb{Z}/2 \times \mathfrak{S}_3, \quad \mathrm{Aut}(X) \simeq N(T) \rtimes \mathbb{Z}/2,$$

and T is the maximal torus of PGL_3, the quotient of $(k^\times)^3$ by the diagonal subgroup $k^\times$, $N(T)$ its normalizer. A generator of $\mathbb{Z}/2$ in $W_X = \mathbb{Z}/2 \times \mathfrak{S}_3$ can be identified with the lift of the standard Cremona involution on $\mathbb{P}^2$ and $\mathfrak{S}_3$ is realized as the group of permutations of the points

$$p_1 = (1:0:0), \quad p_2 = (0:1:0), \quad p_3 = (0:0:1)$$

that are blown up to obtain X. There is always a full invariant exceptional collection for the entire automorphism group of X. Indeed, X has the following (three block) exceptional sequence:

$$\mathcal{O}_X, \quad \mathcal{O}_X(H), \quad \mathcal{O}_X(2H - E_1 - E_2 - E_3),$$
$$\mathcal{O}_X(2H - E_1 - E_2), \quad \mathcal{O}_X(2H - E_2 - E_3), \quad \mathcal{O}_X(2H - E_1 - E_3),$$

where H is the pullback of a hyperplane class and E_i the exceptional divisors. The Cremona involution σ acts as

$$H \mapsto 2H - E_1 - E_2 - E_3, \quad E_i \mapsto H - E_j - E_k, \quad \{i,j,k\} = \{1,2,3\},$$

whereas $\mathfrak{S}_3$ permutes the E_i and fixes H, and T fixes H, E_i. However, this sequence is not always an equivariant exceptional sequence (for example, the normalizer of a maximal torus in PGL_3 is in the stabilizer of $\mathcal{O}_X(H)$, but the line bundle is not linearized since the action does not lift to a linear action on k^3).

DP5: Let X be a Del Pezzo surface of degree 5. We have

$$\mathrm{Aut}(X) \simeq \mathfrak{S}_5.$$

By [CT20, Theorem 1.2 and Example 1.3], X has a full $\mathfrak{S}_5$-equivariant exceptional collection. However, X is G-superrigid for $G = \mathfrak{A}_5$ [DI09].

A special DP4. By [Pro15, Theorem 1.2], there is a unique minimal G-Del Pezzo surface X of degree ≤ 4 that satisfies **(H1)**. It is a Del Pezzo surface of degree 4, an interection of two quadrics in $\mathbb{P}^4$

$$x_1^2 + \zeta x_2^2 + \zeta^2 x_3^2 + x_4^2 = x_1^2 + \zeta^2 x_2^2 + \zeta x_3^2 + x_5^2 = 0, \tag{4.3}$$

with $\zeta = \zeta_3$ a primitive cube root of unity, and $G = \mathbb{Z}/3 \rtimes \mathbb{Z}/4$, with generators

$$\gamma\colon (x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_1, \zeta x_4, \zeta^2 x_5),$$
$$\beta'\colon (x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_3, x_2, -x_5, x_4).$$

Theorem 21. *The derived category $\mathsf{D}^b(X)$ of the minimal G-Del Pezzo surface X given by* (4.3) *does not admit a full G-invariant exceptional sequence.*

Proof. Arguing by contradiction, we assume that such a sequence

$$(\mathcal{E}_1, \dots, \mathcal{E}_8)$$

exists. The group G acts on the terms of the sequence by permutations, decomposing the set of terms into G-orbits, each of which is again an exceptional sequence. Let

$$(\mathcal{F}_1, \dots, \mathcal{F}_r)$$

be one of the orbits. Consider the classes v_i of the $\mathcal{F}_i$ in the Grothendieck group

$$\mathrm{K}_0(X) \simeq \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z} \simeq \mathbb{Z}^8.$$

Let $\chi(-,-)$ be the Euler bilinear pairing on $\mathrm{K}_0(X)$ and $v := v_r = [\mathcal{F}_r]$. Since $(v_1, \dots, v_r)$ is a numerically exceptional sequence with respect to the Euler pairing, we have

$$(*) \qquad \chi\big(v, g(v)\big) = \begin{cases} 0 & \text{if } g(v) \neq v, \\ 1 & \text{if } g(v) = v, \end{cases}$$

for all $g \in G$. These are quadratic equations for the coefficients of v. Let $H \subseteq G$ be the subgroup fixing v. If $H = G$, then $r = 1$ and $v_r = v_1$ is G-invariant. If $H = 1$, then $r = 12$, a contradiction. Let us now assume that H is a nontrivial proper subgroup of G. There are six such subgroups and they are all cyclic. Let $\mathrm{K}_0(X)^H \simeq \mathbb{Z}^r$ be the space of H-invariants and consider the ideal I_H generated by the conditions $(*)$ in $\mathbb{Z}[s_1, \dots, s_r]$. One can show that $I_H = (1)$ mod 3 for all such H, e.g., with `Macaulay2`[3]. Hence only $H = G$ is possible. Since this is the case for all G-orbits, we obtain that all classes $[E_i] \in \mathrm{K}_0(X)$ are G-invariant. However, they also form a $\mathbb{Z}$-basis of $\mathrm{K}_0(X)$. But $\mathrm{K}_0(X)^G \neq \mathrm{K}_0(X)$.

More precisely, we proceed as follows. Since X is a DP4, there are 16 lines on X. To determine these lines explicitly consider the rank 2 skew matrix

$$L = \begin{pmatrix} 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & \zeta^2 & \zeta \\ 1 & -1 & 0 & \zeta & \zeta^2 \\ -1 & -\zeta^2 & -\zeta & 0 & \zeta - \zeta^2 \\ -1 & -\zeta & -\zeta^2 & -\zeta + \zeta^2 & 0 \end{pmatrix} \in \mathrm{Gr}(2, W) \subset \mathbb{P}(\wedge^2(W)),$$

with $\dim(W) = 5$, and a diagonal matrix D with entries ± 1 on the diagonal. Then we check that DLD^t represents a line on X. This gives the 16 lines on X which are permuted by β and γ. Observe that the line represented by L is γ-invariant.

[3] `warwick.ac.uk/fac/sci/maths/people/staff/boehning/m2filesequivariantderived`

We now choose 6 lines whose classes are a basis of $\mathrm{Pic}(X)$ as follows. There are precisely 5 lines $L_1, \dots, L_5$ that intersect the line represented by L. Two of these lines are γ-invariant. Without loss of generality, we can assume these are L_1 and L_2. Finally there is a unique fourth γ-invariant line L_6.

Then $L_1, \dots, L_6$ form a basis of $\mathrm{Pic}(X)$: indeed their intersection matrix can be computed as

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We can now compute the representation of the other lines in this basis by considering their intersections with the lines in the given basis. We find representations

$$B = \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of β and γ, respectively. Moreover, the canonical class K_X is determined by the fact that the intersection number of $-K_X$ with all lines is equal to 1. One finds:

$$-K_X = 2L_1 + 2L_2 - L_3 - L_4 - L_5 + 3L_6,$$

which is also equal to the sum of the four γ-invariant lines L, L_1, L_2, L_6.

Following [BGvBS14, Section 3], we work out the Euler pairing explicitly. The Chern character

$$\begin{aligned} \mathrm{ch}\colon \mathrm{K}_0(X) &\to \mathrm{CH}^*(X)_{\mathbb{Q}} \\ [\mathcal{E}] &\mapsto \mathrm{rk}(E) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2} \end{aligned}$$

is an injective ring homomorphism with values in the sublattice

$$\Lambda := \left\{ x + y_1 l_1 + y_2 l_2 + \cdots + y_6 l_6 + \frac{1}{2} z p \right\} \simeq \mathbb{Z}^8 \subset \mathrm{CH}^*(X)_{\mathbb{Q}},$$

where $(x, y_1, y_2, \dots, y_6, z) \in \mathbb{Z}^8$, p is the class of a point, and $l_i := c_1(\mathcal{O}(L_i))$. We set $v = (x, y, z)$, where

$$y = y_1 l_1 + y_2 l_2 + \cdots + y_6 l_6.$$

Thus Λ is generated by $\mathrm{CH}^0(X) \simeq \mathbb{Z}$, $\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X) \simeq \mathbb{Z}^6$ and $\frac{1}{2}\mathrm{CH}^2(X)$, where $\mathrm{CH}^2(X) \simeq \mathbb{Z}$ is generated by the Chern character of the skyscraper sheaf of a point p, which is just the class of p in the Chow ring. Its image is an index 2 sublattice $\mathrm{ch}(\mathrm{K}_0(X)) \subset \Lambda$: indeed, $\frac{1}{2}p$ is not in $\mathrm{ch}(\mathrm{K}_0(X))$ since for the Euler pairing χ

$$\chi(\mathcal{O}_X, \mathcal{O}_p) = 1$$

and χ takes integral values on $\mathrm{ch}(\mathrm{K}_0(X))$. The class of $\frac{1}{2}p$ generates the quotient $\Lambda/\mathrm{ch}(\mathrm{K}_0(X))$. By Riemann-Roch,

$$\chi(X, \mathcal{E}) = \deg\left(\mathrm{ch}(\mathcal{E}).\mathrm{td}(\mathcal{T}_X)\right)_2 ,$$

where

$$\mathrm{td}(\mathcal{T}_X) = 1 - \frac{1}{2}K_X + \frac{1}{12}(K_X^2 + c_2) = 1 - \frac{1}{2}K_X + p.$$

The subscript 2 in the second to last formula means that one only considers the top-dimensional component. Hence in terms of $v = (x, y, z)$,

$$\chi(X, \mathcal{E}) = x - \frac{1}{2}y.K_X + \frac{1}{2}z\,.$$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are bundles, then

$$\chi(\mathcal{E}_1, \mathcal{E}_2) = \chi(X, \mathcal{E}_1^\vee \otimes \mathcal{E}_2)$$

and

$$\begin{aligned}\mathrm{ch}(\mathcal{E}_1^\vee \otimes \mathcal{E}_2) = \mathrm{ch}(\mathcal{E}_1^\vee).\mathrm{ch}(\mathcal{E}_2) &= (x_1 - y_1 + \frac{1}{2}z_1)(x_2 + y_2 + \frac{1}{2}z_2)\\ &= x_1x_2 + (x_1y_2 - x_2y_1) + \frac{1}{2}(x_1z_2 + x_2z_1 - 2y_1y_2),\end{aligned}$$

whence

$$\chi(\mathcal{E}_1, \mathcal{E}_2) = x_1x_2 - \frac{1}{2}(x_1y_2 - x_2y_1).K_X + \frac{1}{2}(x_1z_2 + x_2z_1 - 2y_1y_2)\,.$$

We work out the Euler pairing χ on the lattice Λ in the above $\mathbb{Z}$-basis:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We now show that equations $(*)$ cannot be solved even in the larger lattice Λ. Namely, consider the subgroup $H \subset G$ generated by β. The invariants of β in Λ are

$$v = (z_1, -2\,z_3, -2\,z_3, -z_2 - z_3, z_2 + 3\,z_3, z_3, -3\,z_3, z_4)$$

for $z_i \in \mathbb{Z}$. Now

$$1 = \chi\big(v, v\big) = z_1^2 + 2\, z_2^2 + 8\, z_2 z_3 + 4\, z_3^2 + z_1 z_4,$$
$$0 = \chi\big(v, vC\big) = z_1^2 - z_2^2 - 4\, z_2 z_3 - 8\, z_3^2 + z_1 z_4.$$

Subtracting the second equation from the first we obtain

$$1 = 3\, z_2^2 + 12\, z_2 z_3 + 12\, z_3^2$$

which has no solution modulo 3.

The same computation can be done for all nontrivial proper subgroups of G. ☐

Implications. Figure 1 shows relations between the different notions for a minimal G-Del Pezzo surface X with $\operatorname{rk}\operatorname{Pic}(X)^G = 1$.

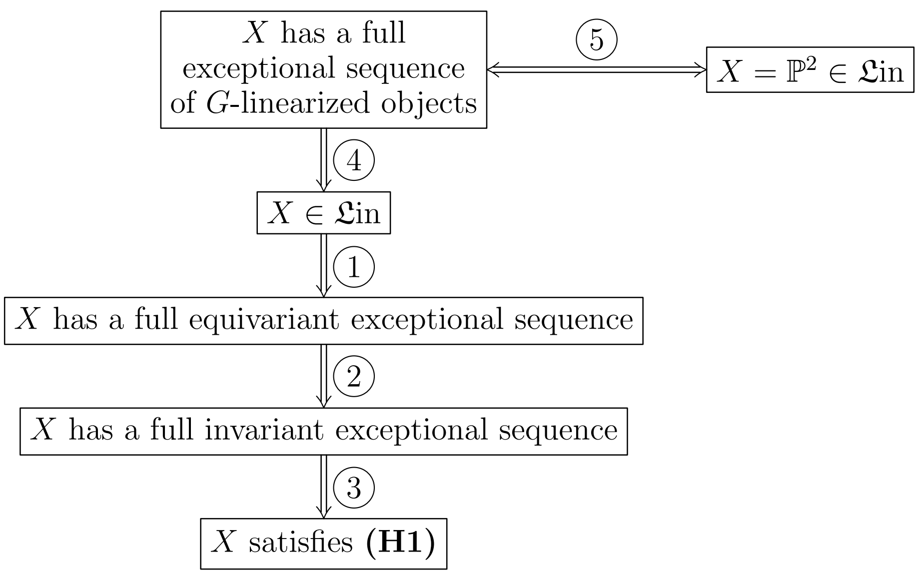

FIGURE 1

- The implications (1)-(4) are strict, see below for references to proofs.
- (3) is proven in Lemma 19, whereas (2), (4) are immediate from the definitions once (5) is proven.
- (1) is not reversible, e.g., for X a DP6.
- (5) follows from Proposition 15 since X then has Picard rank 1, and this also shows that (4) is not reversible.
- (2) is not reversible, by Lemma 20.

- (3) is not reversible, by Theorem 21.

The main result, which requires a longer argument, is the implication (1). We will prove this more generally whenever X is a smooth rational G-surface.

Theorem 22. *A smooth projective rational G-surface that is linearizable has a full G-equivariant exceptional sequence.*

The proof will occupy the remainder of this section. It is based on a detailed analysis of actions, following [DI09] and [Pro15].

Proof. We assume that X is linearizable.

Step 1. We reduce to G-minimal surfaces: Indeed, consider a blowup $\tilde{X} \to X$ in a G-invariant set of points. By Orlov's blowup formula [Orl92], if X admits a full G-equivariant exceptional sequence, then so does $\tilde{X}$. The stabilizer $G_x \subseteq G$ of a point $x \in X$ acts linearly on the tangent bundle of X at x; hence the G_x-action on the sheaves $\mathcal{O}_E(r)$ (where E is the exceptional divisor over x) is linearized.

Step 2. By [DI09, Thm. 3.8], a minimal rational G-surface X either admits a structure of a G-conic bundle over $\mathbb{P}^1$ with $\mathrm{Pic}(X)^G \simeq \mathbb{Z}^2$ or X is isomorphic to a Del Pezzo surface with $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$. We proceed via classification in [DI09, Section 8], depending on the possible values of $d = K_X^2$.

Step 3.
Case $d \leq 0$: X is a rigid G-conic bundle with $8-d$ singular fibres, and in particular, $X \notin \mathfrak{L}\mathrm{in}$.

Case $d = 1$: X is a rigid G-Del Pezzo surface, thus $X \notin \mathfrak{L}\mathrm{in}$, or a G-conic bundle, treated in Lemma 17.

Case $d = 2$: X is a rigid G-Del Pezzo surface, thus $X \notin \mathfrak{L}\mathrm{in}$, or a G-conic bundle. If the conic bundle is not exceptional, it is rigid; exceptional conic bundles with $g = 2$ are treated in Lemma 18.

Case $d = 3$: X is either a minimal G-Del Pezzo surface that is rigid, thus $X \notin \mathfrak{L}\mathrm{in}$; or a minimal G-conic bundle, in which case G contains three commuting involutions two of which have fixed point curves of genus 2, yielding the **(H1)**-obstruction to linearizability, contradicting the assumption.

Case $d = 4$**:** X can be a minimal G-Del Pezzo surface. If $X^G = \emptyset$, X is either rigid or superrigid, hence $X \notin \mathfrak{L}\mathrm{in}$. If $X^G \neq \emptyset$, then X is G-birational to a minimal conic bundle with $d = 3$ and we conclude as in the previous case.

If X is a minimal G-conic bundle, then either X is an exceptional conic bundle with $g = 1$: assuming that X is linearizable, [Pro15, Theorem 8.3] implies that the kernel of

$$\varrho\colon G \to \mathrm{Aut}(\mathrm{Pic}(X))$$

is non-trivial, since otherwise K_X^2 has to be odd. Then [DI09, Classification in §8.1] implies that no elementary transformation is possible and X is not G-birational to any Del Pezzo surface, hence $X \notin \mathfrak{L}\mathrm{in}$.

Secondly, X can also be a G-Del Pezzo surface with two sections with self-intersection -1 intersecting at one point. In this case, X is obtained by regularizing a de Jonquières involution; since such a de Jonquières involution is not conjugate to a projective involution, $X \notin \mathfrak{L}\mathrm{in}$.

Case $d = 5$**:** has been considered above.

Case $d = 6$**:** X always has a full G-invariant exceptional sequence. If $X \in \mathfrak{L}\mathrm{in}$, Lemma 14 applies.

Case $d = 8$**:** If $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then it has the full exceptional sequence, see [Kap88],

$$\mathbf{E} = (\mathcal{O}(-1,-1), \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1)),$$

which is invariant under the full automorphism group

$$\mathrm{Aut}(X) = \mathrm{PGL}_2(\mathbb{C}) \wr C_2.$$

If X is linearizable for a subgroup $G \subset \mathrm{Aut}(X)$, then Lemma 14 applies. In that case, every G-invariant full exceptional sequence is a G-equivariant full exceptional sequence.

When $X = \mathbb{F}_n$ with $n \geq 2$, we apply Proposition 23.

Case $d = 9$**:** $X = \mathbb{P}^2$, and there is nothing to show. □

Proposition 23. *Let X be a G-Hirzebruch surface $\mathbb{F}_n$, $n \geq 2$, that is G-linearizable. Then X admits a full G-equivariant exceptional sequence.*

Proof. If $X = \mathbb{F}_n$, $n \geq 2$, [DI09, Theorem 4.10] shows that any finite subgroup $G \subset \mathrm{Aut}(X)$ is contained in $\mathrm{GL}_2(k)/\mu_n$, which is embedded into $\mathrm{Aut}(X)$ as follows: view $\mathbb{F}_n$ as the quotient $(\mathbb{A}^2\backslash\{0\})^2/\mathbb{G}_m^2$, acting by

$$\begin{aligned}\mathbb{G}_m^2 \times (\mathbb{A}^2\backslash\{0\})^2 &\to (\mathbb{A}^2\backslash\{0\})^2\\ \big((\lambda,\mu),(x_0,x_1,y_0,y_1)\big) &\mapsto (\lambda\mu^{-n}x_0, \lambda x_1, \mu y_0, \mu y_1),\end{aligned}$$

and with projection

$$\begin{aligned}\pi\colon \mathbb{F}_n &\to \mathbb{P}^1\\ (x_0,x_1,y_0,y_1) &\mapsto (y_0 : y_1),\end{aligned}$$

identifying

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

as a $\mathbb{P}^1$-bundle. Letting $A = (a_{ij}) \in \mathrm{GL}_2(k)$ act on the y-coordinates

$$A \cdot (x_0,x_1,y_0,y_1) = (x_0, x_1, a_{11}y_0 + a_{12}y_1, a_{21}y_0 + a_{22}y_1),$$

we obtain an action of $\mathrm{GL}_2(k)$ on $\mathbb{F}_n$; clearly $\mu_n \subset \mathrm{GL}_2(k)$ acts trivially on $\mathbb{F}_n$, and we get an induced action of $\mathrm{GL}_2(k)/\mu_n$. Actually the full automorphism group of $\mathbb{F}_n$ is a semidirect product of $\mathrm{GL}_2(k)/\mu_n$ by a normal subgroup k^{n+1}, thought of as the space of binary forms of degree n with its natural action of $\mathrm{GL}_2(k)/\mu_n$, because $\mathbb{F}_n$ can also be realized as the blowup of the weighted projective space $\mathbb{P}(1,1,n)$ at its singular point.

The group $\mathrm{GL}_2(k)/\mu_n$ is a central product of $k^\times$, embedded diagonally, and $\mathrm{SL}_2(k)$, intersecting in the subgroup generated by $-\mathrm{id}$, for n odd. Then every term in the exceptional sequence (using the relative version of Beilinson's theorem as in [Orl92])

$$(*) \qquad (\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1), \pi^*(\mathcal{O}_{\mathbb{P}^1}), \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)),$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the relative hyperplane bundle on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$, is invariant under $\mathrm{GL}_2(k)/\mu_n$. We conclude by Lemma 14. □

5. Threefolds

There is a wealth of results concerning linearizability of G-actions on rational threefolds, in the context of birational rigidity. In absense of this property, only few examples are known. Of particular interest are threefolds without obvious obstructions, such as nontriviality of the Amitsur invariant or failure of **(H1)**.

In the arithmetic context, rationality over nonclosed fields of smooth geometrically rational Fano threefolds, e.g., those of Picard number one, has been investigated in [HT21a], [HT21b], [BW23], [KP23]:

(1) V_5,
(2) $\mathbb{P}^3$, Q_3, X_{12}, X_{22},
(3) V_4, X_{16}, X_{18}

(we use standard notation for the threefolds from those papers). As is well-known, V_5, a Del Pezzo threefold of degree 5, is always rational. The rationality of forms of varieties in group (2) is controlled by the existence of rational points. In addition to this condition, rationality of varieties in group (3), i.e., forms of complete intersections of two quadrics, Fano threefolds of degree 16, 18, requires the existence, over the ground field, of lines, twisted cubics, or conics, respectively. The papers [KP23], [Kuz22] put this into the framework of derived categories, investigating the semiorthogonal decompositions in this context. In particular, the only cases where the semiorthogonal decomposition does not involve Brauer classes from the base, are V_5 and X_{12}, by [Kuz22, Theorem 1.1 and Theorem 1.3].

We turn to the equivariant setting and linearizability questions. The case of quadrics is already involved [TYZ23, Section 9]:

- existence of fixed points is not necessary for linearizability, when G is nonabelian,
- there are cases, when linearizability is obstructed by the Burnside formalism, but stable linearizability is open,
- there are cases with no visible obstructions, but resistant to all attempts to linearize the action.

As we are interested in situations where no obstructions are visible in the derived category, we focus on V_5 and X_{12}.

Quintic Del Pezzo threefolds. As in Example 7, let W be a faithful 5-dimensional representation of a finite group G, such that $\wedge^2(W)$ contains a faithful 7-dimensional subrepresentation V, so that

$$X = \mathrm{Gr}(2, W) \cap \mathbb{P}(V),$$

is a smooth threefold, with generically free action of G, a quintic Del Pezzo threefold. The restriction $\mathcal{U}_X$ of the universal rank-2 subbundle $\mathcal{U}$ over $\mathrm{Gr}(2, W)$ to X is naturally G-linearized. Orlov [Orl91] showed that there is a full exceptional sequence in $\mathsf{D}^b(X)$ of the form

$$\langle (W \otimes \mathcal{O})/\mathcal{U} \otimes \mathcal{O}(-1), \mathcal{U}, \mathcal{O}, \mathcal{O}(1)\rangle.$$

This is a sequence of G-linearized objects.

There is a distinguished such threefold, with $G = \mathfrak{A}_5$-action, considered in Example 7. It is:

- G-birationally rigid, and thus $X \notin \mathfrak{PLin}$, and
- $X \in \mathfrak{SLin}$.

On the other hand, recall that there is a longstanding conjecture in the context of derived categories relating the rationality of a smooth projective variety (over an algebraically closed field of charactertistic zero) to the existence of a full exceptional sequence in $\mathsf{D}^b(X)$. The above example contradicts the most suggestive analog of this conjecture in the equivariant context. Note that in the arithmetic context, every form of a quintic Del Pezzo threefold is rational, see, e.g., [KP23, Theorem 1.1].

Fano threefolds of genus 7. We follow the discussion in [Muk92, Muk95] and its summary in [Kuz05] and [PS99]. Consider a ten-dimensional complex vector space V with a non-degenerate symmetric bilinear form on it, and denote by Spin_{10} the associated spinor group with 16-dimensional half-spinor representations $\mathrm{S}^{\pm}V$. Consider the Lagrangian Grassmannian of 5-dimensional isotropic subspaces of V: it has two connected components $\mathrm{LGr}_+(V)$ and $\mathrm{LGr}_-(V)$ that can be identified with the closed orbits of the group Spin_{10} in $\mathbb{P}(\mathrm{S}^+V)$ and $\mathbb{P}(\mathrm{S}^-V)$. The representations S^+ and S^- are dual to each other. Choose a pair of subspaces and their orthogonal subspaces (subscripts denote dimensions)

$$A_8 \subset A_9 \subset \mathrm{S}^+V, \quad B_7 \subset B_8 \subset \mathrm{S}^-V.$$

Let

$$\begin{aligned} X :=& \mathrm{LGr}_+(V) \cap \mathbb{P}(A_9) \subset \mathbb{P}(\mathrm{S}^+V), \\ S :=& \mathrm{LGr}_+(V) \cap \mathbb{P}(A_8) \subset \mathbb{P}(\mathrm{S}^+V) \end{aligned}$$

and

$$\begin{aligned} C^\vee :=& \mathrm{LGr}_-(V) \cap \mathbb{P}(B_7) \subset \mathbb{P}(\mathrm{S}^-V), \\ S^\vee :=& \mathrm{LGr}_-(V) \cap \mathbb{P}(B_8) \subset \mathbb{P}(\mathrm{S}^-V). \end{aligned}$$

It is known that X is smooth if and only if $C^\vee$ is smooth, and S is smooth if and only if $S^\vee$ is smooth, which we will now assume. Then $X = X_{12}$ is an index 1 degree 12 genus 7 Fano threefold with a smooth K3 hyperplane section S (a polarized K3 surface of degree 12); all such pairs (X, S) are obtained via the above linear algebra construction by [Muk92, Muk95]. Moreover, $S^\vee$ is also a K3 surface of degree 12 and $C^\vee$ is a canonically embedded curve of genus 7. Note that restricting the universal bundle $\mathcal{U}$ from $\mathrm{Gr}(5, V)$, we obtain rank 5 vector bundles $\mathcal{U}_+, \mathcal{U}_-$ on $\mathrm{LGr}_+(V)$ and $\mathrm{LGr}_-(V)$.

It is known that the Sarkisov link with center a general point $x \in X$ gives a birational map to a quintic Del Pezzo threefold, hence X is rational [KP23, Theorem 5.17 (i)], [PS99].

We now pivot to the equivariant setup, following [Pro12, Example 2.11]. Consider $G = \mathrm{SL}_2(\mathbb{F}_8)$ and let U be a 9-dimensional irreducible representation of G. There is a unique G-invariant quadric $Q \subset \mathbb{P}(U)$, with generically free G-action. Note that, quite generally, the spinor varieties for Spin_{2n-1} and Spin_{2n} are isomorphic, indeed, projectively equivalent as subvarieties of projective space $\mathbb{P}^N$, $N = 2^{n-1} - 1$, in their spinor embeddings. So we can also think of $\mathrm{LGr}_+(V)$ as well as $\mathrm{LGr}_-(V)$ as the spinor variety for Spin_9, parametrizing projective spaces of dimension 3 on a smooth quadric in $\mathbb{P}^8$. Hence G acts on $\mathrm{LGr}_+(V)$ (and $\mathrm{LGr}_-(V)$). The embedding of these into $\mathbb{P}(S^+V)$ and $\mathbb{P}(S^-V)$ is given by the positive generator of the Picard group, eight times of which is the anticanonical bundle. According to [Pro12, Example 2.11], the group G acts in $\mathbb{P}^{15}$ with invariant projective subspaces of dimensions 8 and 6 such that we get an action of G on X and $C^\vee$.

The group G contains the Frobenius group $\mathfrak{F}_8$, which does not act on $\mathbb{P}^3$, so that the G-action on X is not linearizable.

The derived category of X is described in [Kuz05, Theorem 4.4] and [Kuz22, Theorem 5.15]. It has a semiorthogonal decomposition

$$\mathsf{D}^b(X) = \langle \mathcal{U}_+, \mathcal{O}_X, \mathsf{D}^b(C^\vee) \rangle.$$

The proof uses the interpretation of $C^\vee$ as the moduli space of stable rank 2 vector bundles on X with $c_1 = 1, c_2 = 5$ given in [IM04]. Indeed, define

$$\mathcal{A}_X := {}^\perp\langle \mathcal{U}_+, \mathcal{O}_X \rangle.$$

Kuznetsov constructs a fully faithful Fourier-Mukai functor

$$\Phi_{\mathcal{E}} \colon \mathsf{D}^b(C^\vee) \to \mathcal{A}_X \subset \mathsf{D}^b(X)$$

from the universal bundle $\mathcal{E}$ on $X \times C^\vee$, and this is an equivalence of categories.

In our context, we need to check that $\Phi_{\mathcal{E}}$ is a morphism of G-categories, in the sense of, e.g., [BO23, Section 2]. We do this by showing that the G-category structure on $\mathrm{D}^b(C^\vee)$, given by the geometric action of G on $C^\vee$, and the G-category structure on $\mathcal{A}_X$ as the left orthogonal to the exceptional sequence of G-linearized objects $\mathcal{U}_+, \mathcal{O}_X$, coincide. This follows directly from the fact that the Fourier-Mukai kernel bundle $\mathcal{E}$ is a G-linearized vector bundle on $X \times C^\vee$. The easiest way to see this is to use the explicit description of $\mathcal{E}$ in [Kuz05, Constructions in §2 and Corollary 2.5]. Indeed, denoting by $\mathcal{U}_+^X$ and $\mathcal{U}_-^{C^\vee}$ the pullbacks of the tautological subbundles from $\mathrm{LGr}_+(V)$ and $\mathrm{LGr}_-(V)$ to

$$X \times C^\vee \subset \mathrm{LGr}_+(V) \times \mathrm{LGr}_-(V),$$

the bundle $\mathcal{E}$ is cokernel of the morphism

$$\xi \colon \mathcal{U}_-^{C^\vee} \hookrightarrow V \otimes \mathcal{O}_{X\times C^\vee} \simeq V^* \otimes \mathcal{O}_{X\times C^\vee} \to (\mathcal{U}_+^X)^\vee,$$

where the isomorphism in the middle is given by the quadratic form and the other maps are the canonical inclusion and surjection.

In summary, the genus 7 G-Fano threefold X furnishes another example with a nonlinearizable action, where all pieces in a semiorthogonal decomposition of the derived category are "geometric", i.e., equivalent as G-categories to derived categories of G-varieties (of dimension ≤ 1). Thus the pieces of these decompositions fail to detect the nonlinearizability of X.

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