Burnside rings and volume forms with logarithmic poles

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Abstract. — We develop a theory of Burnside rings in the context of birational equivalences of algebraic varieties equipped with logarithmic volume forms. We introduce a residue homomorphism and construct an additive invariant of birational morphisms. We also define a specialization homomorphism.

Résumé. — Nous proposons une théorie d'anneaux de Burnside dans le contexte de la géométrie birationnelle des variétés algébriques munies d'une forme volume à pôles logarithmiques. Nous introduisons un homomorphisme « résidu », construisons un invariant additif des morphismes birationnels. Nous définissons aussi un homomorphisme de spécialisation.

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Contents

1. Introduction

The study of birationality of algebraic varieties is a classical and well studied subject, with many open problems. In some cases, it is interesting to study birational maps preserving additional structure, for example group actions, symplectic forms, or volume forms. Such a study is already implicit in many questions of birational geometry, *eg*, in the notion of crepant resolution of singularities.

In this paper, we consider the case of varieties endowed with volume forms with logarithmic poles and develop a formalism of Burnside rings along the lines of their counterpart introduced by KONTSEVICH & TSCHINKEL (2019) to establish the specialization of rationality, and its equivariant version by KRESCH & TSCHINKEL (2022*b*).

Let k be a field of characteristic zero. For each integer n, we define

$$\mathbf{Burn}_n(k)$$

as the free abelian group on birational equivalence classses of pairs (X, ω) consisting of an integral smooth proper k-variety X of dimension n equipped with an n-form ω with at most logarithmic poles.

The graded abelian group

$$\mathbf{Burn}(k) = \bigoplus_{n\in\mathbf{N}} \mathbf{Burn}_n(k)$$

carries a ring structure, induced by taking products of varieties, decomposed into irreducible components, and equipped with the external product of the volume forms.

In section 4, we define morphisms of abelian groups

$$\partial\colon\ \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k).$$

When X is smooth and the polar divisor of ω has strict normal crossings, the image of the class $[\mathrm{X}, \omega]$ is given by the following formula. Let $(\mathrm{D}_\alpha)_{\alpha\in\mathscr{A}}$ be the family of irreducible components of the polar divisor of ω. For each subset A of $\mathscr{A}$, the intersection $\mathrm{D}_\mathrm{A} = \bigcap_{\alpha\in\mathrm{A}} \mathrm{D}_\alpha$ is a union of integral smooth varieties of codimension $|\mathrm{A}|$; taking iterated residues, we may equip it with a volume form with logarithmic poles ω_A. Then

$$\partial([\mathrm{X}, \omega]) = \sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}} (-1)^{|\mathrm{A}|-1} [\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}] \cdot \mathbf{T}^{|\mathrm{A}|-1},$$

where

$$\mathbf{T} = [\mathbf{P}^1, \mathrm{d}t/t].$$

In particular, the existence of the map ∂ relies on the birational invariance of this expression, see theorem 4.7.

This construction is reminiscent of the boundary map in polar homology (KHESIN & ROSLY, 2003; KHESIN *et al*, 2004; GORCHINSKIY & ROSLY, 2015). However, apart from the obvious difference that we only record birational types of strata, rather than the strata themselves, our formula takes into account strata of all codimensions, rather than those of codimension one.

The map ∂ is additive. Furthermore, we prove in theorem 4.10 that

$$\partial(a \cdot b) = \varepsilon^n \cdot \partial(a) \cdot b + a \cdot \partial(b) - \mathbf{T} \cdot \partial(a) \cdot \partial(b),$$

when $a \in \mathbf{Burn}_m(k)$ and $b \in \mathbf{Burn}_n(k)$. Here ε is the class of the point $\mathrm{Spec}(k)$ equipped with the volume form equal to -1.

In theorem 5.1, we show that

$$\partial \circ \partial = 0.$$

These formulas may look complicated. However, as we explain in §6, they simplify significantly after inverting 2.

Inspired by the constructions of Lin *et al* (2020); Lin & Shinder (2022); Kresch & Tschinkel (2022*a*), we define in §7 a homomorphism

$$\mathbf{c}\colon \mathrm{Bir}(\mathrm{X}, \omega) \to \mathbf{Burn}_{n-1}(k),$$

from the group of birational automorphisms of the pair (X, ω), where X is an n-dimensional integral proper smooth variety equipped with a logarithmic volume form ω. As in the above references, our map $\mathbf{c}$ is defined at the groupoid level of birational maps preserving logarithmic volume forms.

When the birational isomorphism $\varphi\colon (\mathrm{X}, \omega) \dashrightarrow (\mathrm{Y}, \eta)$ is described by a diagram

$$\begin{tikzcd} & \mathrm{W} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{X} \arrow[rr, dashed, "\varphi"] & & \mathrm{Y} \end{tikzcd}$$

of smooth proper integral k-varieties, with birational morphisms p and q, the two logarithmic volume forms $p^*\omega$ and $q^*\eta$ on W are equal, and the element $\mathbf{c}(\varphi) \in \mathbf{Burn}_{n-1}(k)$ is given by

$$\mathbf{c}(\varphi) = \sum_{\mathrm{E} \in \mathrm{Exc}(q)} [\mathrm{E}, p^*\omega_{\mathrm{E}}] - \sum_{\mathrm{D} \in \mathrm{Exc}(p)} [\mathrm{D}, q^*\eta_{\mathrm{D}}]$$

where $\mathrm{Exc}(p)$ is the set of irreducible components of the exceptional divisor of p, and where, for each such component D, the logarithmic volume form $p^*\omega_{\mathrm{D}}$ on D is obtained by taking the residue of $p^*\omega$ along D (and similarly for q).

Finally, consider a discrete valuation ring with residue field k and field of fractions K and let t be a uniformizing element. In this context, we define a specialization map

$$\rho_t\colon \mathbf{Burn}_n(\mathrm{K}) \to \mathbf{Burn}_n(k).$$

The image of the class $[\mathrm{X}, \omega]$ involves the combinatorics of a good model $(\mathscr{X}, \omega)$ over the valuation ring, and a certain subcomplex of the Clemens complex of the special fiber. In the particular case where $\mathscr{X}$ is smooth, the polar divisor of ω is a relative divisor with normal crossings, and denoting by ω_o the restriction of ω to the special fiber $\mathscr{X}_o$, one has

$$\rho_t([\mathrm{X}, \omega]) = [\mathscr{X}_o, \omega_o].$$

Note that the existence of such a specialization map implies, as in theorem 1 of Kontsevich & Tschinkel (2019), or as in (Nicaise & Shinder, 2019), that birational equivalence of varieties with logarithmic volume forms is preserved under "good specializations".

In the geometric case, where the valuation is the local ring of a curve C at point o, the construction of the specialization map can be viewed as a restriction to the special fiber of a normalization of a global residue map ∂ that takes place on a proper model whose special fiber is a divisor with normal crossings. The normalization procedure extracts a subcomplex of the Clemens complex of the special fiber. A similar situation appeared in the study of Tamagawa measures on analytic manifolds (Chambert-Loir & Tschinkel, 2010).

Related constructions emerged from the seminal work of Kontsevich & Soibelman (2006) inspired by mirror symmetry, and its subsequent developments, *eg*, by Mustaţă & Nicaise (2015); Nicaise & Xu (2016); Boucksom & Jonsson (2017); Jonsson & Nicaise (2020).

Our constructions use essentially only formal properties of the residue maps. Consequently, one can envision analogous theories for logarithmic forms of smaller degree, Milnor K-theory, or even for the cycle modules of Rost (1996).

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2. Logarithmic differential forms

2.1. Kähler differentials. — Let k be a field of characteristic zero and let K be a finitely generated extension of k; let n be its transcendence degree. The space of Kähler differentials $\Omega_{\mathrm{K}/k}$ is the K-vector space generated by symbols $\mathrm{d}a$, for $a \in \mathrm{K}$, subject to the relations:

(1) For $a \in k$, one has $\mathrm{d}a = 0$;
(2) For $a, b \in \mathrm{K}$, one has $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = a\mathrm{d}b + b(\mathrm{d}a)$.

For any integer $m \geqslant 0$, we may consider its mth exterior power $\Omega^m_{\mathrm{K}/k}$, which is a K-vector space of dimension $\binom{n}{m}$; in particular, it vanishes if $m > n$, $\Omega^1_{\mathrm{K}/k}$ has dimension n, and $\Omega^n_{\mathrm{K}/k}$ has dimension one. One has $\Omega^0_{\mathrm{K}/k} = k$, canonically.

Elements of $\Omega^n_{\mathrm{K}/k}$, for $n = \operatorname{tr.deg}_k(\mathrm{K})$, are also called *volume forms*.

For $a \in \mathrm{K}^\times$, we also write $\mathrm{dlog}\, a = \mathrm{d}a/a \in \Omega_{\mathrm{K}/k}$.

2.2. Models. — Let m be an integer and let $\omega \in \Omega^m_{\mathrm{K}/k}$. A *model* of K is an integral k-scheme X together with a k-isomorphism $\mathrm{K} \simeq k(\mathrm{X})$; we say that this model is proper, resp. smooth if X is proper, resp. smooth over k. Given such a model, ω induces a meromorphic global section ω_{X} of $\Omega^m_{\mathrm{X}/k}$. The polar ideal of ω_{X} is the subsheaf of $\mathscr{O}_{\mathrm{X}}$ whose local sections are the $a \in \mathscr{O}_{\mathrm{X}}$ such that $a\omega_{\mathrm{X}}$ is induced by a regular m-form. Let D be the zero-locus of this ideal. Its complement U is the largest open subscheme of X such that ω_{X} is induced by a regular m-form on U. If X is smooth, then ω_{X} is locally free, hence the scheme D is an effective divisor (Hartogs's principle); we call it the *polar divisor* of ω on X.

2.3. Logarithmic forms. — By Hironaka's theorem on embedded resolution of singularities, there exist smooth projective models $(\mathrm{X}, \omega_{\mathrm{X}})$ of (K, ω) such that the polar divisor D of ω_{X} has normal crossings.

Following (Deligne, 1970, chap. 2, §3), we then say that ω_{X} has at most logarithmic poles, or that ω has at most logarithmic poles on X, if both ω_{X} and $\mathrm{d}\omega_{\mathrm{X}}$ have at most simple poles along D.

The following lemma implies that this condition is essentially independent of the choice of X such that the polar divisor of ω_{X} has normal crossings.

Lemma 2.4. — *Let $g\colon \mathrm{X}' \to \mathrm{X}$ be a morphism of smooth k-varieties, let* D *be a divisor with normal crossings in* X *and let* D′ *be a divisor with normal crossings in* X′ *such that $\mathrm{D}' = g^{-1}(\mathrm{D})$. Let ω be a regular m-form on $\mathrm{X} - \mathrm{D}$ and let $\omega' = g^*\omega$.*

(1) *If ω has at most logarithmic poles along* D*, then ω' has at most logarithmic poles along* D′.

(2) *The converse holds if g is proper and surjective.*

Proof. — The first assertion is (Deligne, 1970, chap. II, prop. 3.2, (iv)). Let us prove the second one.

Consider the generic point η of X and a point $\eta' \in \mathrm{X}' - \mathrm{D}'$ which is algebraic over $k(\eta)$. The Zariski closure X'_1 of η' is proper and generically finite over X, and $\mathrm{D}'_1 = \mathrm{D}' \cap \mathrm{X}'_1$ is a divisor. There is a proper modification $h\colon \mathrm{X}'_2 \to \mathrm{X}'_1$ such that $\mathrm{D}'_2 = h^{-1}(\mathrm{D}'_1)$ has normal crossings. By the first part, the form $h^*\omega'|_{\mathrm{X}'_1}$ has at most logarithmic poles along D'_2. Replacing g by $g \circ h$, we may assume that g is generically finite.

Since the sheaf of forms with at most logarithmic poles along D is locally free and X is smooth, we can delete from X a subset of codimension at least 2. Thus, we may assume that g is flat, D is smooth and irreducible, and g is étale outside of D. It suffices to argue étale locally at the generic point of D. By the local description of ramified morphisms, there are étale local coordinates $(z_1, \dots, z_n)$ on X such that $\mathrm{D}_{\mathrm{red}} = \mathrm{V}(z_1)$, local coordinates $(z'_1, \dots, z'_n)$ on X′ such that $g^*z_1 = (z'_1)^e$, $g^*z_2 = z'_2$, etc., where e is the ramification index of g along D. Let d be the order of the pole of ω along D; write $\omega = \alpha/z_1^d + \beta \wedge \mathrm{d}z_1/z_1^d$, where α, β are regular forms which do not involve $\mathrm{d}z_1$. Then

$$\omega' = g^*\omega = g^*\alpha/(z'_1)^{de} + e\, g^*\beta \wedge \mathrm{d}z'_1/(z'_1)^{1+(d-1)e}.$$

Assume, by contradiction, that $d \geqslant 2$, so that $de \geqslant 2$ and $1 + (d-1)e \geqslant 2$. Since ω' has at most logarithmic poles along D, we get $g^*\alpha = 0$ and $g^*\beta = 0$. This implies that both α and β are multiples of z_1, contradicting the hypothesis that d was the order of the pole of ω along D. Therefore, $d \leqslant 1$. This concludes the proof. □

2.5. — We say that an m-form $\omega \in \Omega^m_{\mathrm{K}/k}$ is *logarithmic* if for all proper smooth models X of K such that the polar divisor of ω_{X} has normal crossings, the meromorphic differential form ω_{X} has at most logarithmic poles. By resolution of singularities

(HIRONAKA, 1964), two models are dominated by a third one, hence lemma 2.4, implies that it suffices that this condition is satisfied on some proper smooth model for which the polar divisor of ω_{X} has normal crossings.

Analogously, if X is a reduced k-variety, then we say that a meromorphic m-form ω on X is logarithmic "everywhere" if for all proper birational models (X', ω') of (X, ω), the meromorphic m-form ω' on X' has at most logarithmic poles. It suffices that this holds on one such model.

3. Burnside rings for logarithmic forms

3.1. Burnside rings. — Let k be a field of characteristic zero and n an integer such that $n \geqslant 0$. KONTSEVICH & TSCHINKEL (2019) defined the Burnside group $\mathrm{Burn}_n(k)$ as the free abelian group on isomorphism classes of finitely generated extensions of k of transcendence degree n.

Any integral k-variety X of dimension n has a class [X] in $\mathrm{Burn}_n(k)$. This gives rise to alternative useful presentations of $\mathrm{Burn}_n(k)$, for example involving only classes of integral projective smooth varieties.

The group

$$\mathrm{Burn}(k) = \bigoplus_{n \geq 0} \mathrm{Burn}_n(k)$$

carries a natural commutative ring structure, with multiplication defined by taking products of (smooth projective) k-varieties:

$$[\mathrm{X}] \cdot [\mathrm{X}'] = [\mathrm{X} \times \mathrm{X}'].$$

3.2. Definition of a Burnside group for volume forms. — Let k be a field of characteristic zero and let n be an integer $\geqslant 0$. We define $\mathbf{Burn}_n(k)$ to be the free abelian group on isomorphisms classes of pairs (K, ω), where

– K is a finitely generated extension of k of transcendence degree n and
– $\omega \in \Omega^n_{\mathrm{K}/k}$ is a logarithmic volume form.

We write

$$[\mathrm{K}, \omega] \in \mathbf{Burn}_n(k)$$

for the class of a pair (K, ω).

Remark 3.3. — This definition has obvious more geometric formulations. For example, we can take for generators equivalence classes of pairs (X, ω), where

– X is a smooth integral k-scheme of dimension n, and
– ω a regular volume form on X which is logarithmic "everywhere",

modulo the smallest equivalence relation that identifies (X, ω) and (X', ω') if there exists an open immersion $f \colon \mathrm{X}' \to \mathrm{X}$ such that $\omega' = f^*\omega$.

Alternatively, we can assume that X is proper, smooth and integral, the form ω is a logarithmic volume form on X, and consider the smallest equivalence relation that identifies (X, ω) and (X', ω') if there exists a proper birational morphism $f \colon \mathrm{X}' \to \mathrm{X}$ such that $\omega' = f^*\omega$. By the weak factorization theorem of (ABRAMOVICH *et al*, 2002), this equivalence relation is generated by such morphisms f which are blowing-ups along smooth centers in good position with respect to the polar divisor of X.

In both contexts, if X is an n-dimensional k-variety and ω is a meromorphic n-form on X which is logarithmic "everywhere", then we define $[\mathrm{X},\omega]$ to be the sum, over all irreducible components Y of X which have dimension n, of the classes $[\mathrm{Y},\omega|_{\mathrm{Y}}]$.

Example 3.4. — Finitely generated extensions of k of transcendence degree 0 are finite extensions of k. Let K be such an extension. Since k has characteristic zero, one has $\Omega^1_{\mathrm{K}/k}=0$. However, $\Omega^0_{\mathrm{K}/k}$, which is its 0th exterior power, is canonically isomorphic to K. Consequently, $\mathbf{Burn}_0(k)$ is the free abelian group on isomorphism classes of pairs (K,λ), where K is a finite extension of k and $\lambda\in\mathrm{K}$.

We will let $\mathbf{1}=[\mathrm{Spec}(k),1]$ and $\varepsilon=[\mathrm{Spec}(k),-1]$.

Example 3.5. — Let $\mathrm{K}=k(t)$. The differential form $\mathrm{d}t/t$ is a logarithmic volume form; indeed $\mathrm{X}=\mathbf{P}^1_k$ is a model of K and this form has poles of order 1 at 0 and ∞, and no other poles. We write $\mathbf{T}$ for the class of $(k(t),\mathrm{d}t/t)$.

Note that the k-isomorphism of K that maps t to $1/t$ maps $\mathrm{d}t/t$ to its opposite; consequently, we also have $\mathbf{T}=[k(t),-\mathrm{d}t/t]=\varepsilon\cdot\mathbf{T}$.

In the context of birational geometry in presence of logarithmic volume forms, "rational varieties" would have class in $\mathbf{T}^n$, and similarly for stable birationality.

3.6. Multiplicative structure. — We view the direct sum

$$\mathbf{Burn}(k)=\bigoplus_{n\in\mathbf{N}}\mathbf{Burn}_n(k)$$

as a graded abelian group. It is endowed with a multiplication such that

$$[\mathrm{X},\omega]\cdot[\mathrm{X}',\omega']=[\mathrm{X}\times\mathrm{X}',\omega\wedge\omega']$$

when X,X' are proper, smooth and integral and ω, resp. ω' are logarithmic volume forms on X, resp. X′, and Y ranges over the set of irreducible components of $\mathrm{X}\times\mathrm{X}'$.

Let $s\colon\mathrm{X}'\times\mathrm{X}\to\mathrm{X}\times\mathrm{X}'$ be the isomorphism exchanging the two factors. One has

$$s^*(\omega\wedge\omega')=(-1)^{nn'}\omega'\wedge\omega,$$

if $n=\dim(\mathrm{X})$, $n'=\dim(\mathrm{X}')$, ω is a volume form on X and ω' is a volume form on X′. Consequently,

$$a\cdot b=\varepsilon^{nn'}\cdot b\cdot a$$

for $a\in\mathbf{Burn}_n(k)$ and $b\in\mathbf{Burn}_{n'}(k)$. In particular, classes in $\mathbf{Burn}_n(k)$, for even n, are central in $\mathbf{Burn}(k)$.

We remark that the element $\mathbf{T}\in\mathbf{Burn}_1(k)$ is central as well. Let indeed $a\in\mathbf{Burn}_n(k)$. If n is even, then $a\cdot\mathbf{T}=\mathbf{T}\cdot a$. Otherwise, we have $a\cdot\mathbf{T}=\varepsilon\cdot\mathbf{T}\cdot a$, but we have seen in example 3.5 that $\mathbf{T}=\varepsilon\cdot\mathbf{T}$. As a consequence, $a\cdot\mathbf{T}=\mathbf{T}\cdot a$.

However, the ring $\mathbf{Burn}(k)$ is *not* commutative. Indeed, consider curves X, X′ without automorphisms and no nonconstant morphism between them. Then the switch is the only isomorphism from $\mathrm{X}'\times\mathrm{X}$ to $\mathrm{X}\times\mathrm{X}'$. Take nonzero logarithmic

1-forms ω, ω' on X, X' respectively. The classes $[X \times X', \omega \wedge \omega']$ and $[X' \times X, \omega' \wedge \omega]$ are then distinct.

3.7. Functoriality. — Let k' be an extension of k. Then there is a natural ring homomorphism

$$\mathbf{Burn}(k) \to \mathbf{Burn}(k')$$

described as follows. Let (X, ω) be an integral k-variety of dimension n equiped with a logarithmic q-form. Let $X' = X \otimes_k k'$ be its base change to k', and let ω' be the volume form on X' deduced from ω by base change. Then the class of (X, ω) maps to the sum of classes $(Y, \omega'|_Y)$, where Y runs the (finite) set of irreducible components of X'.

If k' is a finite extension of k, we also have a trace map

$$\mathrm{Tr}_{k'/k} \colon \mathbf{Burn}(k') \to \mathbf{Burn}(k)$$

obtained by averaging over a set of representatives of automorphisms of the Galois closure of k' over k modulo those preserving k'.

3.8. Relation with the classical Burnside group. — Forgetting the form ω gives a ring morphism

$$\pi \colon \mathbf{Burn}(k) \to \mathrm{Burn}(k).$$

On the other hand, if K is a finitely generated extension of k of transcendence degree n, we can endow it with the zero n-form. The resulting map

$$\varpi \colon \mathrm{Burn}(k) \to \mathbf{Burn}(k)$$

identifies $\mathrm{Burn}(k)$ with an ideal of $\mathbf{Burn}(k)$. One has $\pi \circ \varpi = \mathrm{id}$.

3.9. Variations on the theme. — The construction of the Burnside ring $\mathbf{Burn}(k)$ admits several natural variants that are relevant in more specific contexts. Some of them will be used in later sections.

3.9.1. *A relative ring.* — Let n be an integer. For any k-scheme S, we define $\mathbf{Burn}_n(S/k)$ as the free abelian group on triples (X, ω, u) where X is an integral smooth n-dimensional k-scheme, $\omega \in \Omega^n_{X/k}$ is a regular volume form which is logarithmic "everywhere", and $u \colon X \to S$ is a morphism, modulo the smallest equivalence relation that identifies (X, ω, u) and (X', ω', u') if there exists an open immersion $f \colon X' \to X$ such that $\omega' = f^*\omega$ and $u' = u \circ f$.

Let $h \colon S \to T$ be a morphism of k-schemes. It induces a morphism of abelian groups

$$h_* \colon \mathbf{Burn}_n(S/k) \to \mathbf{Burn}_n(T/k)$$

such that $h_*([X, \omega, u]) = [X, \omega, h \circ u]$ for any triple (X, ω, u) as above.

3.9.2. *Pluriforms.* — One can replace volume forms with volume r-pluriforms, that is, elements of $(\Omega^n_{K/k})^{\otimes r}$, for some given integer r. The corresponding logarithmic condition requires that the pluriform has poles of order at most r on an adequate model. Note that when r is even, the obtained ring is commmutative.

3.9.3. *Forms up to scalars.* — In the construction, we may wish to identify (K,ω) and (K',ω') if there exists $\lambda\in k^\times$, resp. $\lambda\in\{\pm1\}$, and a k-isomorphism $f\colon \mathrm{K}\to\mathrm{K}'$ such that $f^*\omega'=\lambda\omega$. These variants also give rise to a commutative ring.

3.9.4. *Group actions.* — Let G be a profinite group scheme over k. One can also consider pairs (K,ω), where the field K is endowed with an action of G leaving the form ω invariant. The obtained ring will be denoted by $\mathbf{Burn}^{\mathrm{G}}(k)$.

4. Residues

4.1. Residue of a volume form. — Let X be an equidimensional smooth k-variety of dimension n.

Let D be a smooth divisor on X. We denote by $\Omega^m_{\mathrm{X}/k}(\log\mathrm{D})$ the sheaf of m-forms on X with logarithmic poles along D, locally of the form $\eta\wedge\mathrm{d}\log f+\eta'$, where η and η' are regular and f is a local equation of D. The residue map is the homomorphism of $\mathscr{O}_{\mathrm{X}}$-modules

$$\rho_{\mathrm{D}}\colon \Omega^m_{\mathrm{X}/k}(\log\mathrm{D})\to\Omega^{m-1}_{\mathrm{D}/k},$$

characterized by the relation

$$\rho_{\mathrm{D}}(\eta\wedge\mathrm{d}\log f+\eta')=\eta|_{\mathrm{D}}$$

for every local sections $\eta\in\Omega^{m-1}_{\mathrm{X}/k}$ and $\eta'\in\Omega^m_{\mathrm{X}/k}$, and any local generator f of the ideal of D.

If ω is a logarithmic m-form on X, there is an open subset U of X such that $\mathrm{U}\cap\mathrm{D}\neq\emptyset$ and such that $\omega|_{\mathrm{U}}$ belongs to $\Omega^m_{\mathrm{X}/k}(\log\mathrm{D})$. Its residue $\rho_{\mathrm{D}}(\omega|_{\mathrm{U}})$ is then a meromorphic section of $\Omega^{m-1}_{\mathrm{D}/k}$.

Lemma 4.2. — *Let ω be a logarithmic differential form of degree m on* X. *Then $\rho_{\mathrm{D}}(\omega)$ is a logarithmic $(m-1)$-form on* D.

Proof. — We may assume that the sum of D and of the polar divisor of ω has strict normal crossings. The assertion is then evident in local coordinates. □

4.3. Blowing-ups and normal bundles. — Let Y be a smooth closed subscheme of X. The blow-up $\mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})$ of X along Y is a smooth k-variety. The blowing-up morphism $b_{\mathrm{Y}}\colon \mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})\to\mathrm{X}$ is an isomorphism over the complement of Y. If Y is nowhere dense and nonempty, then $\mathrm{E}_{\mathrm{Y}}=b_{\mathrm{Y}}^{-1}(\mathrm{Y})$ is a smooth divisor in $\mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})$.

In general, $\mathrm{E}_{\mathrm{Y}}=b_{\mathrm{Y}}^{-1}(\mathrm{Y})$ identifies, as an Y-scheme, with the projectivization of the normal bundle $\mathscr{N}_{\mathrm{Y}}(\mathrm{X})$ of Y in X.

Let W be a closed smooth subscheme of X. Assume that W and Y are transversal. Then the Zariski closure of $b_{\mathrm{Y}}^{-1}(\mathrm{W}\smallsetminus(\mathrm{Y}\cap\mathrm{W}))$ is called the strict transform of W in $\mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})$. It identifies with $\mathrm{Bl}_{\mathrm{Y}\cap\mathrm{W}}(\mathrm{W})$.

Let now ω be a logarithmic m-form on X. Then the form $b_{\mathrm{Y}}^*\omega$ on $\mathrm{Bl}_{\mathrm{Y}}(\mathrm{X})$ is logarithmic; assuming that Y is nonempty and nowhere dense, we can consider the residue $\rho_{\mathrm{Y}}(\omega)$ of $b_{\mathrm{Y}}^*\omega$ along E_{Y}. It is a logarithmic $(n-1)$-form on $\mathbf{P}(\mathscr{N}_{\mathrm{Y}}(\mathscr{X}))$.

Definition 4.4. — *Let* X *be an irreducible proper smooth* k*-variety, let* n *be its dimension and let* $\omega \in \Omega^n_{\mathrm{X}/k}$ *be logarithmic volume form whose polar divisor* D *has strict normal crossings. Let* $(\mathrm{D}_\alpha)_{\alpha\in\mathscr{A}}$ *be the family of its irreducible components; for* $\mathrm{A} \subseteq \mathscr{A}$*, we let* $\mathrm{D}_\mathrm{A} = \bigcap_{\alpha\in\mathrm{A}} \mathrm{D}_\alpha$*. We then define an element* $\rho(\mathrm{X},\omega)$ *in* $\mathbf{Burn}_{n-1}(\mathrm{X}/k)$ *by the formula:*

$$\rho(\mathrm{X},\omega) = \sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}} (-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X},\omega).$$

(In this formula and all similar ones below, it is always implicit that the terms where $\mathrm{D}_\mathrm{A} = \emptyset$ are omitted.)

4.5. Iterated residues. — We retain the notation of definition 4.4

Fix a logarithmic volume form ω on X and a nonempty subset A of $\mathscr{A}$ such that $\mathrm{D}_\mathrm{A} \neq \emptyset$. It will be useful to compute inductively the logarithmic volume form $\rho_{\mathrm{D}_\mathrm{A}}(\omega)$ that appears in definition 4.4.

Let $b_\mathrm{A}\colon \tilde{\mathrm{X}} \to \mathrm{X}$ be the blowing-up of X along D_A and let E be its exceptional divisor.

When $\mathrm{A} = \{\alpha\}$ has a single element, D_A is the divisor D_α, the blowing-up morphism b_A is an isomorphism and the exceptional divisor identifies with D_A. Then

$$\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X},\omega) = [\mathrm{D}_\alpha, \rho_{\mathrm{D}_\alpha}(\omega), j_\alpha],$$

where j_α is the immersion of D_α into X.

This construction can be pursued in higher codimension, using iterated residues. Fix a total order on $\mathscr{A}$. There is a unique, strictly increasing sequence $(\alpha_1,\dots,\alpha_m)$ in $\mathscr{A}$ such that $\mathrm{A} = \{\alpha_1,\dots,\alpha_m\}$. Given the chosen order on $\mathscr{A}$, we may apply the iterated residues construction and obtain a logarithmic form of degree $n-m$

$$\rho_{\mathrm{D}_\mathrm{A}}(\omega) = \rho_{\mathrm{D}_{\alpha_1}} \circ \dots \circ \rho_{\mathrm{D}_{\alpha_m}}(\omega).$$

On a nonempty open subset U of X that meets D_A, we may write

$$\omega = \eta \wedge \mathrm{dlog}(f_{\alpha_1}) \wedge \dots \mathrm{dlog}(f_{\alpha_m}),$$

for a regular form η, and then one has $\rho_{\mathrm{D}_\mathrm{A}}(\omega) = \eta|_{\mathrm{U}\cap\mathrm{D}_\mathrm{A}}$.

Denote by b_A the blowing-up of X along D_A and by E_A its exceptional divisor; recall that E_A identifies with the projectivized normal bundle $\mathscr{N}_{\mathrm{D}_\mathrm{A}}(\mathrm{X})$ of D_A in X. Using local equations for the divisors D_α, for $\alpha \in \mathrm{A}$, we trivialize $\mathscr{N}_{\mathrm{D}_\mathrm{A}}(\mathrm{X})$ on a dense open subscheme of D_A; this gives a birational isomorphism of E_A with $\mathrm{D}_\mathrm{A} \times \mathbf{P}^{m-1}$ (with $m = |\mathrm{A}|$), and a local computation gives the formula

$$\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X},\omega) = [\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega)] \cdot \mathbf{T}^{m-1}$$

in $\mathbf{Burn}_{n-1}(\mathrm{D}_\mathrm{A}/k)$.

When $m \geqslant 2$, the definition of $\rho_{\mathrm{D}_\mathrm{A}}$ actually depends on the chosen order of $\mathscr{A}$, but only up to a sign, so that the class $[\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega)]$ is well defined up to multiplication by the class $\varepsilon \in \mathbf{Burn}_0(k)$. On the other hand, it is multiplied by $\mathbf{T}^{m-1}$ and we have observed that $\varepsilon \cdot \mathbf{T} = \mathbf{T}$.

Proposition 4.6. — *Let* (X,ω), D, *and* $(\mathrm{D}_\alpha)_{\alpha\in\mathscr{A}}$ *be as in definition 4.4. Let* Y *be a strict irreducible subvariety of* X*; let* $\mathscr{A}_\mathrm{Y}$ *be the set of all* $\alpha\in\mathscr{A}$ *such that* $\mathrm{Y}\not\subseteq\mathrm{D}_\alpha$*; we assume that* $\sum_{\alpha\in\mathscr{A}_\mathrm{Y}}\mathrm{D}_\alpha$ *meets* Y *transversally.*

Let $g\colon \mathrm{X}'\to\mathrm{X}$ *be the blowing-up of* X *along* Y *and let* $\omega'=g^*\omega$*; it is a logarithmic form, its polar divisor has strict normal crossings, and we have*

$$g_*\rho(\mathrm{X}',\omega')=\rho(\mathrm{X},\omega)$$

in $\mathbf{Burn}(\mathrm{X}/k)$*.*

Proof. — Let $\mathrm{E}=g^{-1}(\mathrm{Y})$ be the exceptional divisor; for each $\alpha\in\mathscr{A}$, let D'_α be the strict transform of D_α. The blow-up X′ is smooth; the divisor $\mathrm{E}+\sum_{\alpha\in\mathscr{A}}\mathrm{D}'_\alpha$ has strict normal crossings and contains the polar divisor of ω'.

Let B be the set of all $\beta\in\mathscr{A}$ such that $\mathrm{Y}\subseteq\mathrm{D}_\beta$, so that D_B is the minimal stratum containing Y.

We now split the discussion into two cases.

(1) *Assume that* $\dim(\mathrm{Y})<\dim(\mathrm{D}_\mathrm{B})$. Since g is ramified along E, its Jacobian vanishes along E. Since ω has poles of order at most one, the form $\omega'=g^*\omega$ is regular at the generic point of E. Consequently, the polar divisor of ω' does not contain E and we have to compare

$$\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}'_\mathrm{A}}(\omega')$$

with

$$\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}_\mathrm{A}}(\omega).$$

Since g is a local isomorphism around the generic points of D_α, for $\alpha\in\mathscr{A}$, we see that the polar divisor of ω' is equal to $\sum_{\alpha\in\mathscr{A}}\mathrm{D}'_\alpha$. For every nonempty subset A of $\mathscr{A}$, one has

$$g_*\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X}',\omega')=\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X},\omega)$$

for every nonempty subset A of $\mathscr{A}$, which implies the desired formula in this case.

(2) *Assume that* $\dim(\mathrm{Y})=\dim(\mathrm{D}_\mathrm{B})$. In this case, Y is an irreducible component of D_B. Since $\mathrm{D}_\emptyset=\mathrm{X}$ and $\mathrm{Y}\neq\mathrm{X}$, we have $\mathrm{B}\neq\emptyset$. We have to compare the expression

$$\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}'_\mathrm{A}}(\omega')+\sum_{\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|}\rho_{\mathrm{E}\cap\mathrm{D}'_\mathrm{A}}(\omega')$$

with

$$\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}_\mathrm{A}}(\omega).$$

The argument takes place in a neighborhood of Y, which allows us to assume that $\mathrm{Y}=\mathrm{D}_\mathrm{B}$.

Let A be a nonempty subset of $\mathscr{A}$. One has $\mathrm{D}'_\mathrm{A}=\emptyset$ whenever $\mathrm{B}\subseteq\mathrm{A}$, and the corresponding terms are absent from the second expression. On the other hand, if $\mathrm{B}\not\subseteq\mathrm{A}$, the morphism g identifies D'_A with the blow-up of D_A along $\mathrm{D}_\mathrm{A}\cap\mathrm{Y}=\mathrm{D}_{\mathrm{A}\cup\mathrm{B}}$. In particular, g induces a birational isomorphism from D'_A to D_A, so that

$g_*\rho_{\mathrm{D}'_{\mathrm{A}}}(\mathrm{X}',\omega') = \rho_{\mathrm{D}_{\mathrm{A}}}(\mathrm{X},\omega)$. Moreover, $\mathrm{E}\cap\mathrm{D}'_{\mathrm{A}}$ is the projectivized normal bundle $\mathbf{P}\mathscr{N}_{\mathrm{D}_{\mathrm{A}\cup\mathrm{B}}}(\mathrm{D}_{\mathrm{A}})$, and

$$g_*\rho_{\mathrm{E}\cap\mathrm{D}'_{\mathrm{A}}}(\mathrm{X}',\omega') = \rho_{\mathrm{D}_{\mathrm{A}\cup\mathrm{B}}}(\mathrm{X},\omega).$$

Similarly, one has

$$g_*\rho_{\mathrm{E}}(\mathrm{X}',\omega') = \rho_{\mathrm{D}_{\mathrm{B}}}(\mathrm{X},\omega).$$

This gives a formula of the form

$$\begin{aligned} g_*\rho(\mathrm{X}',\omega') &= \sum_{\substack{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}\\ \mathrm{B}\not\subseteq\mathrm{A}}} (-1)^{|\mathrm{A}|-1}\rho_{\mathrm{D}_{\mathrm{A}}}(\mathrm{X},\omega) + \sum_{\substack{\mathrm{A}\subseteq\mathscr{A}\\ \mathrm{B}\not\subseteq\mathrm{A}}} (-1)^{|\mathrm{A}|}\rho_{\mathrm{D}_{\mathrm{A}\cup\mathrm{B}}}(\mathrm{X},\omega) \\ &= \sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}} n'_{\mathrm{A}}\rho_{\mathrm{D}_{\mathrm{A}}}(\mathrm{X},\omega), \end{aligned}$$

where

$$n'_{\mathrm{A}} = \begin{cases} (-1)^{|\mathrm{A}|-1} & \text{if } \mathrm{B}\not\subseteq\mathrm{A},\\ \sum_{\substack{\mathrm{C}\subseteq\mathscr{A}\\ \mathrm{B}\not\subseteq\mathrm{C}\\ \mathrm{C}\cup\mathrm{B}=\mathrm{A}}} (-1)^{|\mathrm{C}|} & \text{if } \mathrm{B}\subseteq\mathrm{A}. \end{cases}$$

It suffices to prove that $n'_{\mathrm{A}} = n_{\mathrm{A}}$ for any nonempty subset A of $\mathscr{A}$. This is obvious when $\mathrm{B}\not\subseteq\mathrm{A}$, so let us assume that $\mathrm{B}\subseteq\mathrm{A}$. In the sum that defines n'_{A}, we write $\mathrm{C} = (\mathrm{C}-\mathrm{B})\cup\mathrm{C}'$, where $\mathrm{C}' = \mathrm{C}\cap\mathrm{B}$ is a subset of B; the condition $\mathrm{C}\cup\mathrm{B} = \mathrm{A}$ means $\mathrm{C}-\mathrm{B} = \mathrm{A}-\mathrm{B}$; the condition $\mathrm{B}\not\subseteq\mathrm{C}$ means $\mathrm{C}'\neq\mathrm{B}$. Consequently, we have

$$\begin{aligned} n'_{\mathrm{A}} &= (-1)^{|\mathrm{A}-\mathrm{B}|}\sum_{\substack{\mathrm{C}'\subseteq\mathrm{B}\\ \mathrm{C}'\neq\mathrm{B}}} (-1)^{|\mathrm{C}'|} \\ &= (-1)^{|\mathrm{A}-\mathrm{B}|}\left(\sum_{\mathrm{C}'\subseteq\mathrm{B}} (-1)^{|\mathrm{C}'|} - (-1)^{|\mathrm{B}|}\right) \\ &= (-1)^{|\mathrm{A}-\mathrm{B}|}\left((1-1)^{|\mathrm{B}|} - (-1)^{|\mathrm{B}|}\right) \\ &= (-1)^{|\mathrm{A}|-1}, \end{aligned}$$

since $|\mathrm{B}|\geqslant 1$. This concludes the proof of the proposition. □

Theorem 4.7. — *Let* (X,ω) *be as in definition 4.4. If* X *is proper, then the image of* $\rho(\mathrm{X},\omega)$ *in* $\mathbf{Burn}_{n-1}(k)$ *only depends on the class* $[\mathrm{X},\omega]\in\mathbf{Burn}_n(k)$. *It gives rise to a morphism of abelian groups*

$$\partial_n\colon\ \mathbf{Burn}_n(k)\to\mathbf{Burn}_{n-1}(k).$$

Proof. — By the definition of $\mathbf{Burn}_n(k)$ involving pairs (X,ω) where X is proper, it suffices to consider two pairs (X,ω) and (X',ω') as in definition 4.4 which are related by a proper birational morphism $g\colon \mathrm{X}'\to\mathrm{X}$ such that $g^*\omega = \omega'$. By the weak factorization theorem of Abramovich *et al* (2002), in order to prove the theorem, we may assume that g is a blowing-up of X along a smooth subvariety which is transversal to the polar divisor of ω. In this case, proposition 4.6 asserts that $g_*\rho(\mathrm{X}',\omega') = \rho(\mathrm{X},\omega)$ in $\mathbf{Burn}(\mathrm{X}/k)$. In particular, the images in $\mathbf{Burn}(k)$ of $\rho(\mathrm{X}',\omega')$ and $\rho(\mathrm{X},\omega)$ are equal. □

Example 4.8. — The meromorphic differential form $\mathrm{d}t/t$ on $\mathbf{P}^1_k$ has residues 1 and -1 at 0 and ∞ respectively. By construction, we thus have

$$\partial_1(\mathbf{T}) = [\mathrm{Spec}(k), 1] + [\mathrm{Spec}(k), -1] = \mathbf{1} + \varepsilon.$$

Let n be an integer such that $n \geqslant 2$ and let us compute $\partial_n(\mathbf{T}^n)$. We view $\mathbf{T}^n$ as the class of $\mathbf{P}^n$, with homogeneous coordinates $[1 : x_1 : \ldots : x_n]$, and with the toric differential form

$$\omega_n = (\mathrm{d}x_1/x_1) \wedge \ldots (\mathrm{d}x_n/x_n).$$

Its divisor is the sum of the toric hyperplanes $\mathrm{D}_0, \ldots, \mathrm{D}_n$. Each of these hyperplanes identifies with $\mathbf{P}^{n-1}$, and $\rho_{\mathrm{D}_j}(\omega_n)$ is $(-1)^{n-j}\omega_{n-1}$. Let $\mathscr{A} = \{0, \ldots, n\}$. If $\mathrm{A} = \mathscr{A}$, then $\mathrm{D}_\mathrm{A} = \emptyset$. Otherwise, we see by induction that D_A is isomorphic to $\mathbf{P}^{n-|\mathrm{A}|}$ and $\rho_{\mathrm{D}_\mathrm{A}}(\omega_n)$ identifies with $\pm\omega_{n-|\mathrm{A}|}$, so that

$$[\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega_n)] \cdot \mathbf{T}^{|\mathrm{A}|-1} = [{\mathbf{G}_\mathrm{m}}^{n-1}, \pm\omega_{n-1}] = \mathbf{T}^{n-1},$$

since $n - 1 \geqslant 1$. Then,

$$\begin{aligned}\partial_n(\mathbf{T}^n) &= \sum_{\emptyset\neq\mathrm{A}\subsetneq\mathscr{A}} (-1)^{|\mathrm{A}|-1}[\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega_n)] \cdot \mathbf{T}^{|\mathrm{A}|-1} \\ &= \sum_{\emptyset\neq\mathrm{A}\subsetneq\mathscr{A}} (-1)^{|\mathrm{A}|-1}\mathbf{T}^{n-1}.\end{aligned}$$

Now,

$$\sum_{\emptyset\neq\mathrm{A}\subsetneq\mathscr{A}} (-1)^{|\mathrm{A}|-1} = 1 - (1-1)^{n+1} + (-1)^{n+1} = \begin{cases} 2 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.}\end{cases}$$

We get $\partial_n(\mathbf{T}^n) = 2\mathbf{T}^{n-1}$ if n is odd and $\partial_n(\mathbf{T}^n) = 0$ if n is even. (Remind that $n \geqslant 2$.) Since $\mathbf{T} = \varepsilon \cdot \mathbf{T}$, the following formula unifies the various cases: for $n \geqslant 1$, we have

$$\partial_n(\mathbf{T}^n) = (1 + (-1)^{n-1}\varepsilon) \cdot \mathbf{T}^{n-1}.$$

Proposition 4.9. — *For every class $b \in \mathbf{Burn}_n(k)$, we have*

$$\partial_{n+1}(b \cdot \mathbf{T}) = -\partial_n(b) \cdot \mathbf{T} + b \cdot \partial_1\mathbf{T}.$$

Proof. — We may assume that $b = [\mathrm{X}, \omega]$, where X is a proper integral smooth variety of dimension n, and ω is a logarithmic volume form on X whose polar divisor has strict normal crossings. Let $(\mathrm{D}_\alpha)_{\alpha\in\mathscr{A}}$ be the family of its irreducible components. We view $b \cdot \mathbf{T}$ as the class of $[\mathrm{X} \times \mathbf{P}^1, \omega \wedge \mathrm{d}t/t]$. The polar divisor of $\omega \wedge \mathrm{d}t/t$ is equal to

$$\sum_{\alpha\in\mathscr{A}} \mathrm{D}_\alpha \times \mathbf{P}^1 + \mathrm{X} \times \{0\} + \mathrm{X} \times \{\infty\}.$$

It has strict normal crossings, and its strata are of the form $\mathrm{D}_\mathrm{A} \times \mathbf{P}^1$, for nonempty $\mathrm{A} \subseteq \mathscr{A}$, or $\mathrm{D}_\mathrm{A} \times \{0\}$, or $\mathrm{D}_\mathrm{A} \times \{\infty\}$, for $\mathrm{A} \subseteq \mathscr{A}$. This decomposes $\partial_{n+1}(b \times \mathbf{T})$ as the sum of three terms.

The first one is

$$\sum_{\emptyset \neq \mathrm{A} \subseteq \mathscr{A}} [\mathrm{D}_\mathrm{A} \times \mathbf{P}^1, \rho_{\mathrm{D}_\mathrm{A} \times \mathbf{P}^1}(\omega \wedge \mathrm{d}t/t)] \cdot \mathbf{T}^{|\mathrm{A}|-1}.$$

For any nonempty subset A of $\mathscr{A}$, one has

$$\rho_{\mathrm{D}_\mathrm{A} \times \mathbf{P}^1}(\omega \wedge \mathrm{d}t/t) = \pm \rho_{\mathrm{D}_\mathrm{A}}(\omega) \wedge \mathrm{d}t/t,$$

so that

$$[\mathrm{D}_\mathrm{A} \times \mathbf{P}^1, \rho_{\mathrm{D}_\mathrm{A} \times \mathbf{P}^1}(\omega \wedge \mathrm{d}t/t)] \cdot \mathbf{T}^{|\mathrm{A}|-1} = [\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega)] \cdot \mathbf{T} \cdot \mathbf{T}^{|\mathrm{A}|-1}.$$

Consequently, the first term equals $\partial_n(b) \times \mathbf{T}$.

Write $\mathrm{D}_0 = \mathrm{X} \times \{0\}$ and $\mathrm{D}_\infty = \mathrm{X} \times \{\infty\}$, and identify both divisors to X. For a subset A of $\mathscr{A}$, we have

$$\rho_{\mathrm{D}_{\mathrm{A} \cup \{0\}}}(\omega \wedge \mathrm{d}t/t) = \rho_{\mathrm{D}_\mathrm{A}} \circ \rho_{\mathrm{D}_0}(\omega \wedge \mathrm{d}t/t) = \rho_{\mathrm{D}_\mathrm{A}}(\omega).$$

Consequently, the second term is equal to

$$\sum_{\mathrm{A} \subseteq \mathscr{A}} (-1)^{|\mathrm{A}|} [\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_\mathrm{A}}(\omega)] \cdot \mathbf{T}^{|\mathrm{A}|} = [\mathrm{X}, \omega] - \partial_n(b) \cdot \mathbf{T}.$$

Similarly, the third term is equal to

$$[\mathrm{X}, -\omega] - \partial_n(b) \cdot \mathbf{T}.$$

Summing up these three terms, we get

$$\partial_{n+1}(b \times \mathbf{T}) = -\partial_n(b) \cdot \mathbf{T} + [\mathrm{X}, \omega] + [\mathrm{X}, -\omega].$$

We now recall that $\partial_1(\mathbf{T}) = [\mathrm{Spec}(k), 1] + [\mathrm{Spec}(k), -1]$, so that

$$[\mathrm{X}, \omega] + [\mathrm{X}, -\omega] = [\mathrm{X}, \omega] \cdot \partial_1(\mathbf{T}) = b \cdot \partial_1(\mathbf{T}).$$

This concludes the proof. □

Theorem 4.10. — *Let $a \in \mathbf{Burn}_m(k)$ and $b \in \mathbf{Burn}_n(k)$; we have*

$$\partial_{m+n}(a \cdot b) = \varepsilon^n \cdot \partial_m(a) \cdot b + a \cdot \partial_n(b) - \mathbf{T} \cdot \partial_m(a) \cdot \partial_n(b)$$

in $\mathbf{Burn}_{m+n-1}(k)$.

Proof. — It suffices to treat the case where a and b are classes of proper integral smooth varieties (X, ω), (Y, η), endowed with meromorphic volume forms whose polar divisors have strict normal crossings and no multiplicities. Let $(\mathrm{D}_\alpha)_{\alpha \in \mathscr{A}}$ be the irreducible components of the polar divisor of ω, let $(\mathrm{E}_\beta)_{\beta \in \mathscr{B}}$ be the irreducible components of the polar divisor of η. Then $[\mathrm{X}, \omega] \cdot [\mathrm{Y}, \eta]$ is the class of $[\mathrm{X} \times \mathrm{Y}, \omega \wedge \eta]$; the polar divisor of $\omega \wedge \eta$ is equal to

$$\sum_{\alpha \in \mathscr{A}} \mathrm{D}_\alpha \times \mathrm{Y} + \sum_{\beta \in \mathscr{B}} \mathrm{X} \times \mathrm{E}_\beta.$$

We fix a total order on the disjoint union of $\mathscr{A}$ and $\mathscr{B}$ such that the elements of $\mathscr{A}$ are smaller than those of $\mathscr{B}$. For any subsets A, B of $\mathscr{A}$ and $\mathscr{B}$, observe that we have

$$\rho_{\mathrm{D}_{\mathrm{A}\cup\mathrm{B}}}(\omega\wedge\eta)=\pm\rho_{\mathrm{D}_\mathrm{A}}(\omega)\wedge\rho_{\mathrm{E}_\mathrm{B}}(\eta),$$

where $\rho_{\mathrm{D}_\mathrm{A}}$ has to be understood as the identity when A is empty, and similarly for $\rho_{\mathrm{E}_\mathrm{B}}$. The sign is 1 when $\mathrm{A}=\emptyset$; when $\mathrm{B}=\emptyset$, it is equal to $(-1)^{|\mathrm{A}|n}$; we won't need to use its explicit value in the other cases. Then we can write $\partial([\mathrm{X},\omega]\cdot[\mathrm{Y},\eta])$ as

$$\sum_{\substack{\mathrm{A}\subseteq\mathscr{A}\\ \mathrm{B}\subseteq\mathscr{B}\\ \mathrm{A}\cup\mathrm{B}\neq\emptyset}}(-1)^{|\mathrm{A}|+|\mathrm{B}|-1}[\mathrm{D}_\mathrm{A}\times\mathrm{E}_\mathrm{B},\pm\rho_\mathrm{A}(\omega)\wedge\rho_{\mathrm{E}_\mathrm{B}}(\eta)]\cdot\mathbf{T}^{|\mathrm{A}\cup\mathrm{B}|-1}$$

and we split it into the sum of three terms, according to which $\mathrm{B}=\emptyset$, or $\mathrm{A}=\emptyset$, or none of them is empty. The first two terms are respectively equal to

$$\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}[\mathrm{D}_\mathrm{A}\times\mathrm{Y},(-1)^{n|\mathrm{A}|}\rho_{\mathrm{D}_\mathrm{A}}(\omega)\wedge\eta]\cdot\mathbf{T}^{|\mathrm{A}|-1}=\partial([\mathrm{X},(-1)^n\omega])\cdot[\mathrm{Y},\eta]$$

and

$$\sum_{\emptyset\neq\mathrm{B}\subseteq\mathscr{B}}(-1)^{|\mathrm{B}|-1}[\mathrm{X}\times\mathrm{E}_\mathrm{B},\omega\wedge\rho_{\mathrm{E}_\mathrm{B}}(\eta)]\cdot\mathbf{T}^{|\mathrm{B}|-1}=[\mathrm{X},\omega]\cdot\partial([\mathrm{Y},\eta]),$$

since $\mathbf{T}$ belongs to the center of $\mathbf{Burn}(k)$. As for the third one, we obtain

$$-\sum_{\emptyset\neq\mathrm{B}\subseteq\mathscr{B}}(-1)^{|\mathrm{B}|-1}\sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}}(-1)^{|\mathrm{A}|-1}[\mathrm{D}_\mathrm{A},\rho_{\mathrm{D}_\mathrm{A}}(\omega)]\cdot[\mathrm{E}_\mathrm{B},\rho_{\mathrm{E}_\mathrm{B}}(\eta)]\cdot\mathbf{T}^{|\mathrm{A}|+|\mathrm{B}|-2}$$

which equals

$$-\partial([\mathrm{X},\omega])\cdot\partial([\mathrm{Y},\eta])\cdot\mathbf{T}.$$

Finally, we get

$$\begin{aligned}\partial_{m+n}(a\cdot b)&=\partial_{m+n}([\mathrm{X},\omega]\cdot[\mathrm{Y},\eta])\\&=\partial_m([\mathrm{X},(-1)^n\omega)\cdot[\mathrm{Y},\eta]+[\mathrm{X},\omega]\cdot\partial_n([\mathrm{Y},\eta])\\&\qquad-\mathbf{T}\cdot\partial_m([\mathrm{X},\omega])\cdot\partial_n([\mathrm{Y},\eta])\\&=\varepsilon^n\cdot\partial_m(a)\cdot b+a\cdot\partial_n(b)-\mathbf{T}\cdot\partial_m(a)\cdot\partial_n(b)\end{aligned}$$

as was to be shown. ☐

In particular, using the computation of example 4.8, we obtain the following generalization of proposition 4.9.

Corollary 4.11. — *For any $a\in\mathbf{Burn}_m(k)$ and any integer n, we have*

$$\partial_{m+n}(a\cdot\mathbf{T}^n)=\begin{cases}\partial_m(a)\cdot\mathbf{T}^n & \text{if } n \text{ is even;}\\ -\partial_m(a)\cdot\mathbf{T}^n+a\cdot\partial_n(\mathbf{T}^n) & \text{if } n \text{ is odd.}\end{cases}$$

Remark 4.12. — For the variant of $\mathbf{Burn}(k)$ where we consider forms up to sign, the formula of theorem 4.10 simplifies to

$$\partial_{m+n}(a\cdot b)=\partial_m(a)\cdot b+a\cdot\partial_n(b)-\mathbf{T}\cdot\partial_m(a)\cdot\partial_n(b).$$

5. A complex of Burnside rings

Theorem 5.1. — *For any integer $n \geqslant 2$, we have*

$$\partial_{n-1} \circ \partial_n = 0.$$

In other words, the residue morphisms of Burnside groups give rise to a complex

$$\cdots \to \mathbf{Burn}_n(k) \to \mathbf{Burn}_{n-1}(k) \to \cdots \to \mathbf{Burn}_1(k) \to \mathbf{Burn}_0(k)$$

Proof. — It suffices to prove the following result: Let (X, ω) be an integral proper smooth variety of dimension n equipped with a meromorphic volume form ω whose polar divisor has strict normal crossings and no multiplicities; then $\partial_{n-1}(\partial_n([\mathrm{X}, \omega])) = 0$.

Let $(\mathrm{D}_\alpha)_{\alpha \in \mathscr{A}}$ be the family of irreducible components of the polar divisor of ω in X. By definition, one has

$$\partial_n([\mathrm{X}, \omega]) = \sum_{\emptyset \neq \mathrm{A} \subseteq \mathscr{A}} (-1)^{|\mathrm{A}|-1} \rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X}, \omega).$$

Fix a total order on $\mathscr{A}$. Let $(\alpha_1, \ldots, \alpha_m)$ be a strictly increasing sequence in $\mathscr{A}$ and let $\mathrm{A} = \{\alpha_1, \ldots, \alpha_m\}$. We have seen in §4.5 that $\rho_{\mathrm{D}_\mathrm{A}}(\mathrm{X}, \omega)$ can be defined via iterated residue maps:

$$\rho_{\mathrm{D}_\mathrm{A}}([\mathrm{X}, \omega]) = [\mathrm{D}_\mathrm{A}, \rho_{\mathrm{D}_{\alpha_1}} \circ \cdots \circ \rho_{\mathrm{D}_{\alpha_m}}(\omega)] \cdot \mathbf{T}^{|\mathrm{A}|-1} = [\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}] \cdot \mathbf{T}^{|\mathrm{A}|-1}$$

where we wrote ω_A for the composition $\rho_{\mathrm{D}_{\alpha_1}} \circ \cdots \circ \rho_{\mathrm{D}_{\alpha_m}}(\omega)$. When $|\mathrm{A}|$ is odd, we have

$$\partial(\rho_{\mathrm{D}_\mathrm{A}}([\mathrm{X}, \omega])) = \partial([\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}]) \cdot \mathbf{T}^{|\mathrm{A}|-1},$$

while when $|\mathrm{A}|$ is even, we have

$$\partial(\rho_{\mathrm{D}_\mathrm{A}}([\mathrm{X}, \omega])) = -\partial([\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}]) \cdot \mathbf{T}^{a-1} + [\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}] \cdot \partial(\mathbf{T}^{|\mathrm{A}|-1}).$$

Consequently, we have

$$\partial \circ \partial([\mathrm{X}, \omega]) = \sum_{\emptyset \neq \mathrm{A} \subseteq \mathscr{A}} \partial([\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}]) \cdot \mathbf{T}^{|\mathrm{A}|-1} - \sum_{\substack{\emptyset \neq \mathrm{A} \subseteq \mathscr{A} \\ |\mathrm{A}| \text{ even}}} [\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}] \cdot \partial(\mathbf{T}^{|\mathrm{A}|-1}).$$

The polar divisor of the form ω_A on D_A is equal to $\sum_{\beta \notin \mathrm{A}} \mathrm{D}_\beta \cap \mathrm{D}_\mathrm{A}$, so that, by definition (and computation of ∂ via iterated residues),

$$\partial([\mathrm{D}_\mathrm{A}, \omega_\mathrm{A}]) = \sum_{\emptyset \neq \mathrm{B} \subseteq \complement \mathrm{A}} (-1)^{|\mathrm{B}|-1} [\mathrm{D}_{\mathrm{AB}}, \omega_{\mathrm{A} \cup \mathrm{B}}] \cdot \mathbf{T}^{|\mathrm{B}|-1}.$$

Also, when A is nonempty and of even cardinality, $\partial(\mathbf{T}^{|\mathrm{A}|-1}) = 2\mathbf{T}^{|\mathrm{A}|-2}$. When we put these two formulas into the antepenultimate one and collect the various terms, we obtain

$$\partial \circ \partial([\mathrm{X}, \omega]) = \sum_{\substack{\mathrm{C} \subseteq \mathscr{A} \\ 2 \leqslant |\mathrm{C}|}} n_\mathrm{C} [\mathrm{D}_\mathrm{C}, \omega_\mathrm{C}] \cdot \mathbf{T}^{|\mathrm{C}|-2},$$

where

$$n_{\mathrm{C}} = -\sum_{\substack{\emptyset\neq \mathrm{A},\mathrm{B}\\ \mathrm{A}\cup\mathrm{B}=\mathrm{C},\mathrm{A}\cap\mathrm{B}=\emptyset}} (-1)^{|\mathrm{B}|} - 2\delta_{|\mathrm{C}|\text{ is even}}.$$

In the first sum, the terms $\mathrm{A}=\emptyset$ or $\mathrm{B}=\emptyset$ are omitted, while if we put them in, we obtain

$$\sum_{\substack{\mathrm{A}\cup\mathrm{B}=\mathrm{C}\\ \mathrm{A}\cap\mathrm{B}=\emptyset}} (-1)^{|\mathrm{B}|} = \sum_{b=0}^{|\mathrm{C}|} \binom{|\mathrm{C}|}{b} (-1)^b = (1-1)^{|\mathrm{C}|} = 0$$

since $|\mathrm{C}| \geqslant 1$. Consequently,

$$n_{\mathrm{C}} = 1 + (-1)^{|\mathrm{C}|} - 2\delta_{|\mathrm{C}|\text{ is even}} = 0.$$

This concludes the proof. □

6. Algebraic structure of $\mathbf{Burn}(k)$ after localization at 2

In this section, we study the algebraic structure of the Burnside ring $\mathbf{Burn}(k)$, endowed with its elements ε, $\mathbf{T}$ and the operator ∂.

6.1. — By construction, $\mathbf{Burn}(k) = \bigoplus_{n\geqslant 0} \mathbf{Burn}_n(k)$ is an associative unital $\mathbf{Z}_{\geqslant 0}$-graded ring, $\varepsilon \in \mathbf{Burn}_0(k)$, $\mathbf{T} \in \mathbf{Burn}_1(k)$ and ∂ is a homogeneous additive map of degree -1. They satisfy the following relations, for homogeneous elements $a, b \in \mathbf{Burn}(k)$:

(1) $b \cdot a = \varepsilon^{|a||b|} \cdot a \cdot b$ (§3.6);

(2) $\varepsilon^2 = 1$ (example 3.4);

(3) $\mathbf{T} = \varepsilon \cdot \mathbf{T}$ (example 3.5);

(4) $\partial(\mathbf{T}) = 1 + \varepsilon$ (example 4.8);

(5) $\partial(a \cdot b) = \varepsilon^{|b|} \cdot \partial(a) \cdot b + a \cdot \partial(b) - \mathbf{T} \cdot \partial(a) \cdot \partial(b)$ (theorem 4.10);

(6) $\partial(\partial(a)) = 0$ (theorem 5.1).

By (1), the element ε is central, and by (2), we may view $\mathbf{Burn}(k)$ as an algebra over $\mathbf{Z}[\varepsilon]/(\varepsilon^2-1)$. After inverting 2, the algebra $\mathbf{Burn}(k)$ splits into two components $\mathbf{Burn}^{\varepsilon=1}(k)$ and $\mathbf{Burn}^{\varepsilon=-1}(k)$, one over which $\varepsilon = 1$, and the other over which $\varepsilon = -1$.

In the rest of this section, we implicitly assume that 2 *is inverted, without changing the notation.*

6.2. Sector $\varepsilon = -1$. — Here, we have $\mathbf{T} = -\mathbf{T}$, hence $\mathbf{T} = 0$ since 2 is invertible. As a consequence, after replacing ∂ with $\partial' \colon a \mapsto (-1)^{1+|a|}\partial(a)$, one gets from (5) the usual graded Leibniz rule

$$\partial'(a \cdot b) = \partial'(a) \cdot b + (-1)^{|a|} a \cdot \partial'(b)$$

and therefore $\mathbf{Burn}^{\varepsilon=-1}(k)$ is a classical differential graded (super-)commutative algebra, similar to, *eg*, the de Rham complex.

6.3. Sector $\varepsilon = 1$. — The algebra $\mathbf{Burn}^{\varepsilon=1}$ is now commutative (and not graded commutative). This reflects the intuition in our constructions that they speak about volume forms (as opposed to top-degree differential forms) for which we have commutativity (as reflected by the change of order of integration in multiple integrals).

Lemma 6.4. — *The map* $\mathrm{F} \colon a \mapsto a - \mathbf{T} \cdot \partial(a)$ *is a ring endomorphism of* $\mathbf{Burn}^{\varepsilon=1}(k)$*, and* $\mathrm{F}^2 = \mathrm{id}$*. Moreover, one has* $\mathrm{F} \circ \partial = \partial = -\partial \circ \mathrm{F}$*.*

Proof. — This map is additive. One has $\mathrm{F}(1) = 1 - \mathbf{T} \cdot \partial(1) = 1$. Let us show multiplicativity. Indeed, for $a, b \in \mathbf{Burn}^{\varepsilon=1}(k)$, one has

$$\begin{aligned}
\mathrm{F}(a) \cdot \mathrm{F}(b) &= (a - \mathbf{T} \cdot \partial(a)) \cdot (b - \mathbf{T} \cdot \partial(b)) \\
&= a \cdot b - \mathbf{T} \cdot \partial(a) \cdot b - \mathbf{T} \cdot a \cdot \partial(b) + \mathbf{T}^2 \cdot \partial(a) \cdot \partial(b) \\
&= a \cdot b - \mathbf{T} \cdot (\partial(a) \cdot b + a \cdot \partial(b) - \mathbf{T} \cdot \partial(a) \cdot \partial(b)) \\
&= a \cdot b - \mathbf{T} \cdot \partial(a \cdot b) \qquad \text{(using (5))} \\
&= \mathrm{F}(a \cdot b).
\end{aligned}$$

Since $\partial^2 = 0$, one has

$$\mathrm{F}(\partial(a)) = \partial(a) - \mathbf{T} \cdot \partial(\partial(a)) = \partial(a).$$

On the other hand,

$$\begin{aligned}
\partial(\mathrm{F}(a)) &= \partial(a - \mathbf{T} \cdot \partial(a)) \\
&= \partial(a) - \partial(\mathbf{T} \cdot \partial(a)) \\
&= \partial(a) - \partial(\mathbf{T}) \cdot \partial(a) - \mathbf{T} \cdot \partial(\partial(a)) + \mathbf{T} \cdot \partial(\mathbf{T}) \cdot \partial(\partial(a)) \\
&= -\partial(a)
\end{aligned}$$

using that $\partial(\mathbf{T}) = 2$ and $\partial^2 = 0$.

Consequently, for $a \in \mathbf{Burn}^{\varepsilon=1}(k)$, we have

$$\mathrm{F}^2(a) = \mathrm{F}(a) - \mathbf{T} \cdot \partial(\mathrm{F}(a)) = a - \mathbf{T} \cdot \partial(a) + \mathbf{T} \cdot \partial(a) = a$$

since $\partial \circ \mathrm{F} = -\partial$. □

6.5. — To simplify the notation, write $\mathscr{B} = \mathbf{Burn}^{\varepsilon=1}(k)$. Since $\mathrm{F}^2 = \mathrm{id}$ and 2 is invertible, the algebra $\mathscr{B}$ splits as a direct sum

$$\mathscr{B} = \mathscr{B}_+ \oplus \mathscr{B}_-,$$

such that F acts as id on $\mathscr{B}_+$ and as $-\mathrm{id}$ on $\mathscr{B}_-$. Moreover, $\mathscr{B}_+$ is a subalgebra.

Since the operator ∂ anticommutes with F, it induces maps

$$\partial_\pm \colon \mathscr{B}_+ \to \mathscr{B}_-, \qquad \partial_\mp \colon \mathscr{B}_- \to \mathscr{B}_+.$$

Note that

$$\mathrm{F}(\mathbf{T}) = \mathbf{T} - \mathbf{T}\cdot\partial(\mathbf{T}) = -\mathbf{T},$$

so that $\mathbf{T} \in \mathscr{B}_-$. Consequently, the multiplication by $\mathbf{T}$ map induces two maps

$$t_\pm \colon \mathscr{B}_+ \to \mathscr{B}_-, \qquad t_\mp \colon \mathscr{B}_- \to \mathscr{B}_+.$$

Lemma 6.6. — *The map ∂ vanishes on $\mathscr{B}_+$. Equivalently, $\partial_\pm = 0$.*

The maps $\frac{1}{2}\partial_\mp$ and $t_\pm$ are inverses the one of the other.

Proof. — For $a \in \mathscr{B}_+$, one has $\partial(a) = -\partial(\mathrm{F}(a)) = -\partial(a)$, since $\partial\circ\mathrm{F} = -\partial$, hence $\partial(a) = 0$.

On the other hand, for $a \in \mathscr{B}_+$, one has

$$\partial(\mathbf{T}\cdot a) = 2\cdot a + \mathbf{T}\cdot\partial(a) - 2\mathbf{T}\cdot\partial(a) = 2a - \mathbf{T}\cdot\partial(a) = a + \mathrm{F}(a) = 2a$$

while for $a \in \mathscr{B}_-$, we have

$$\mathbf{T}\cdot\partial(a) = a - \mathrm{F}(a) = 2a.$$

This conclude the proof of the lemma. □

In particular, we see that the cohomology of the differential ∂ vanishes in the sector $\mathbf{Burn}^{\varepsilon=1}(k) = \mathscr{B}$.

6.7. — It follows from the lemma that we have a ring isomorphism

$$\mathscr{B} = \mathscr{B}_+[t](t^2 - \mathbf{T}^2),$$

from which we see that all the algebraic structure of $\mathscr{B}_+$ (namely δ, $\mathbf{T}$, F) can be canonically reconstructed from a unital commutative associative $\mathbf{Z}_{\geqslant 0}$-graded ring $\mathscr{B}_+$ endowed with an element in degree $+2$ (namely, the element $\mathbf{T}^2$).

Remark 6.8. — The situation clarifies even more if we invert the class $\mathbf{T}$. Then we can write $\partial(a) = (a - \mathrm{F}(a))/\mathbf{T}$, and all relations happen to follow from the fact that F is an involution such that $\mathrm{F}(\mathbf{T}) = -1$. Indeed,

$$\partial^2(a) = \frac{\partial(a) - \mathrm{F}(\partial(a))}{\mathbf{T}} = \frac{1}{\mathbf{T}}\left(\frac{a-\partial(a)}{\mathbf{T}} - \mathrm{F}(\frac{a-\partial(a)}{\mathbf{T}}\right) = 0$$

explains that $\partial^2 = 0$. Moreover, for $a, b \in \mathscr{B}$, we have

$$\begin{aligned}\partial(a\cdot b) &= \frac{a\cdot b - \mathrm{F}(a\cdot b)}{\mathbf{T}} = \frac{a\cdot b - \mathrm{F}(a)\cdot\mathrm{F}(b)}{\mathbf{T}} \\ &= \frac{a-\mathrm{F}(a)}{\mathbf{T}}\cdot b + a\cdot\frac{b-\mathrm{F}(b)}{\mathbf{T}} - \mathbf{T}\cdot\frac{a-\mathrm{F}(a)}{\mathbf{T}}\cdot\frac{b-\mathrm{F}(b)}{\mathbf{T}} \\ &= \partial(a)\cdot b + a\cdot\partial(b) - \mathbf{T}\cdot\partial(a)\cdot\partial(b).\end{aligned}$$

7. Birational morphisms preserving volume forms

7.1. — Let $(\mathrm{X}, \omega_{\mathrm{X}})$ be a smooth integral k-variety of dimension n equipped with a logarithmic volume form, and let $f\colon \mathrm{Y} \to \mathrm{X}$ be a proper birational morphism.

Let E be an exceptional divisor in Y, that is, such that $\dim(f(\mathrm{E})) < \dim(\mathrm{E})$. By lemma 2.4, and lemma 4.2, the residue $\rho_{\mathrm{E}}(f^*\omega_{\mathrm{X}})$ along E of the meromorphic form $f^*\omega$ is a logarithmic volume form on E.

We define $\mathbf{c}(f; \mathrm{X}, \omega)$ to be the sum of all such classes $[\mathrm{E}, \rho_{\mathrm{E}}(f^*\omega)]$ in the free abelian group $\mathbf{Burn}_{n-1}(k)$.

Lemma 7.2. — *Let $g\colon \mathrm{Z} \to \mathrm{Y}$ be a proper birational morphism of smooth integral varieties of dimension n. Then $g \circ f$ is a proper birational morphism and one has*

$$\mathbf{c}(g\circ f; \mathrm{X}, \omega) = \mathbf{c}(g; \mathrm{Y}, f^*\omega) + \mathbf{c}(f; \mathrm{X}, \omega)$$

in $\mathbf{Burn}_{n-1}(k)$.

Proof. — An integral divisor F in Z is exceptional for $g \circ f$ if and only if one of the two mutually excluding situations happen:

– The divisor F is exceptional for g;
– Or $g(\mathrm{F})$ is a divisor in Y which is exceptional for f.

Moreover, any divisor E in Y which is exceptional for f appears once and only as a divisor of the form $g(\mathrm{F})$. The contribution of F to $\mathbf{c}(g\circ f; \mathrm{X}, \omega)$ is given by the volume form $\rho_{\mathrm{F}}((g\circ f)^*\omega)$. In the first case, we write $\rho_{\mathrm{F}}((g\circ f)^*\omega) = \rho_{\mathrm{F}}(g^*(f^*\omega))$, so that the contribution of F coincides with its contribution to the term $\mathbf{c}(g; \mathrm{Y}, f^*\omega)$. In the second case, g induces a birational isomorphism from F to $\mathrm{E} = g(\mathrm{F})$; writing $\rho_{\mathrm{F}}((g\circ f)^*\omega) = g^*(\rho_{\mathrm{F}}(f^*\omega))$, we see that the contribution of F coincides with the contribution of E to $\mathbf{c}(f; \mathrm{X}, \omega)$. This concludes the proof. □

7.3. — Let $(\mathrm{X}, \omega_{\mathrm{X}})$ and $(\mathrm{Y}, \omega_{\mathrm{Y}})$ be proper smooth k-varieties equipped with logarithmic volume forms and let

$$\varphi\colon (\mathrm{X}, \omega_{\mathrm{X}}) \dashrightarrow (\mathrm{Y}, \omega_{\mathrm{Y}})$$

be a birational map preserving the volume forms. By definition, this means that there exists a diagram

$$\begin{tikzcd}
& \mathrm{W} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{X} \arrow[rr, dashed, "\varphi"] & & \mathrm{Y}
\end{tikzcd}$$

of integral k-varieties such that that p and q are proper and birational, and such that $p^*\omega = q^*\omega'$ on W. In this situation, we may assume that W is smooth.

Lemma 7.4. — *With this notation, the element*

$$\mathbf{c}(\varphi) = \mathbf{c}(q) - \mathbf{c}(p) \in \mathbf{Burn}_{n-1}(k)$$

only depends on the birational map φ, and not on the choice of the triple (W, p, q).

Proof. — Consider two possible diagrams $\mathrm{X} \xleftarrow{p} \mathrm{V} \xrightarrow{q} \mathrm{Y}$ and $\mathrm{X} \xleftarrow{r} \mathrm{W} \xrightarrow{s} \mathrm{Y}$ describing φ. Considering for example a resolution of singularities U of $\mathrm{V} \times_{\mathrm{X}} \mathrm{W}$, we can fit these two diagrams in a common commutative diagram of the following form:

$$\begin{tikzcd}
& & \mathrm{U} \arrow[dl, "u"'] \arrow[dr, "v"] & & \\
& \mathrm{V} \arrow[dl, "p"'] \arrow[drrr, "q"] & & \mathrm{W} \arrow[dlll, "r"'] \arrow[dr, "s"] & \\
\mathrm{X} \arrow[rrrr, dashed, "\varphi"] & & & & \mathrm{Y}
\end{tikzcd}$$

The equalities $p^*\omega_{\mathrm{X}} = q^*\omega_{\mathrm{Y}}$ and $r^*\omega_{\mathrm{X}} = s^*\omega_{\mathrm{Y}}$ imply that

$$(p \circ u)^*\omega_{\mathrm{X}} = u^*p^*\omega_{\mathrm{X}} = u^*q^*\omega_{\mathrm{Y}} = (q \circ u)^*\omega_{\mathrm{Y}} = (s \circ v)^*\omega_{\mathrm{Y}}.$$

By lemma 7.2, we then have

$$\mathbf{c}(p) - \mathbf{c}(q) = \mathbf{c}(p \circ u) - \mathbf{c}(q \circ u) = \mathbf{c}(r \circ v) - \mathbf{c}(s \circ v) = \mathbf{c}(r) - \mathbf{c}(s).$$

This concludes the proof. □

Theorem 7.5. — *If $\psi\colon (\mathrm{Y}, \omega_{\mathrm{Y}}) \dashrightarrow (\mathrm{Z}, \omega_{\mathrm{Z}})$ is another birational map preserving volume forms, then one has*

$$\mathbf{c}(\psi \circ \varphi) = \mathbf{c}(\psi) + \mathbf{c}(\psi).$$

Proof. — Consider two diagrams $\mathrm{X} \xleftarrow{p} \mathrm{V} \xrightarrow{q} \mathrm{Y}$ and $\mathrm{Y} \xleftarrow{r} \mathrm{W} \xrightarrow{s} \mathrm{Y}$ describing φ and φ. Considering for example a resolution of singularities U of $\mathrm{V} \times_{\mathrm{Y}} \mathrm{W}$, we can fit these two diagrams in a common commutative diagram of the following form:

$$\begin{tikzcd} & & \mathrm{U} \arrow[dl, "u"'] \arrow[dr, "v"] & & \\ & \mathrm{V} \arrow[dl, "p"'] \arrow[dr, "q"] & & \mathrm{W} \arrow[dl, "r"'] \arrow[dr, "s"] & \\ \mathrm{X} \arrow[rr, dashed, "\varphi"] & & \mathrm{Y} \arrow[rr, dashed, "\psi"] & & \mathrm{Z} \end{tikzcd}$$

and the diagram $\mathrm{X} \xleftarrow{p\circ u} \mathrm{U} \xrightarrow{s\circ v}$ describes the birational map $\psi\circ\varphi$. Since $q\circ u = r\circ v$, we then have

$$\begin{aligned} \mathbf{c}(\psi\circ\varphi) &= \mathbf{c}(p\circ u) - \mathbf{c}(s\circ v) \\ &= \mathbf{c}(p\circ u) - \mathbf{c}(q\circ u) + \mathbf{c}(r\circ v) - \mathbf{c}(s\circ v) \\ &= \mathbf{c}(p) - \mathbf{c}(q) + +\mathbf{c}(r) - \mathbf{c}(s) \\ &= \mathbf{c}(\varphi) + \mathbf{c}(\psi), \end{aligned}$$

as was to be shown. □

Corollary 7.6. — *Let* $\mathrm{Bir}(\mathrm{X},\omega)$ *be the set of birational automorphisms of* X *preserving* ω. *The map* $\mathbf{c}$ *induces a homomorphism of abelian groups*

$$\mathrm{Bir}(\mathrm{X},\omega) \to \mathbf{Burn}_{n-1}(k).$$

Its kernel contains the group of automorphisms of X *that preserve* ω.

8. Specialization

Let K be the field of fractions of a discrete valuation ring R with residue field k. Fix a uniformizer $t \in \mathrm{R}$.

In this context, Kontsevich & Tschinkel (2019) have defined two (distinct) *specialization morphisms*

$$\rho_t\colon\ \mathrm{Burn}_n(\mathrm{K}) \to \mathrm{Burn}_n(k),$$

relating the Burnside groups of K and k (see 3.1), one of which is a ring homomorphism. (The latter homomorphism actually depends on the choice of t, see example 6.2 of (Kresch & Tschinkel, 2022*b*).)

The goal of this section is to define a similar homomorphism

$$\rho_t\colon\ \mathbf{Burn}(\mathrm{K}) \to \mathbf{Burn}(k)$$

for varieties with logarithmic volume forms.

8.1. — Let $\mathscr{X}$ be an integral proper scheme over R, of relative dimension n, whose special fiber Δ is a divisor with strict normal crossings.

Let $(\Delta_\alpha)_{\alpha\in\mathscr{A}}$ be the family of irreducible components of the special fiber Δ; for $\alpha\in\mathscr{A}$, let e_α be the multiplicity of Δ_α in Δ. For every nonempty subset A of $\mathscr{A}$, let Δ_{A} be the intersection of all divisors Δ_α, for $\alpha\in\mathscr{A}$ and e_{A} be the greatest common divisor of the e_α, for $\alpha\in\mathrm{A}$; let also $\Delta_{\mathrm{A}}^\circ$ be the complement $\Delta_{\mathrm{A}} - \bigcup_{\alpha\notin\mathrm{A}}\Delta_\alpha$.

The first specialization morphism of (Kontsevich & Tschinkel, 2019) is defined by

$$\rho_t([\mathscr{X}_{\mathrm{K}}]) = \sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}} (-1)^{|\mathrm{A}|-1}[\Delta_{\mathrm{A}}]\mathbf{L}^{|\mathrm{A}|-1}, \tag{8.2}$$

where $\mathbf{L}\in\mathrm{Burn}(k)$ is the class of the affine line.

Although this map is not multiplicative, it proved sufficient for many applications to rationality problems.

To ensure multiplicativity, a more delicate construction was necessary, valued in the Burnside ring

$$\mathrm{Burn}^{\widehat{\mu}}(k)$$

of varieties endowed with an action of the profinite group $\widehat{\mu}$, limit of finite groups of roots of unity.

Fix a nonempty subset A of $\mathscr{A}$. We identify the normal bundle of Δ_{A} in $\mathscr{X}$ as a direct sum of line bundles:

$$\mathscr{N}_{\Delta_{\mathrm{A}}}(\mathscr{X}) \simeq \bigoplus_{\alpha\in\mathrm{A}} \mathscr{N}_{\Delta_\alpha}(\mathscr{X})|_{\Delta_{\mathrm{A}}}.$$

Let us consider its open subscheme $\mathscr{N}^\circ_{\Delta_{\mathrm{A}}}(\mathscr{X})$ obtained by restricting to $\Delta^\circ_{\mathrm{A}}$ and taking out all "coordinate" hyperplanes. This furnishes a morphism

$$\nu_{\mathrm{A}}\colon \mathscr{N}^\circ_{\Delta_{\mathrm{A}}}(\mathscr{X}) \to \bigotimes_{\alpha\in\mathrm{A}} \mathscr{N}_{\Delta_\alpha}(\mathscr{X})^{\otimes e_\alpha}|_{\Delta^\circ_{\mathrm{A}}}.$$

Since the uniformizer t has divisor $-\sum_{\alpha\in\mathscr{A}} e_\alpha\Delta_\alpha$ on $\mathscr{X}$, it trivializes the line bundle on the target of ν_{A}. We set $\Delta'_{\mathrm{A}} = \nu_{\mathrm{A}}^{-1}(t)$. By construction, the projection $\Delta'_{\mathrm{A}}\to\Delta_{\mathrm{A}}$ is a torsor with group $\mu_{e_{\mathrm{A}}}$.

With this notation, the correct, multiplicative, specialization map of (Kontsevich & Tschinkel, 2019) is given by the formula

$$\widehat{\rho}_t(\mathrm{X}) = \sum_{\emptyset\neq\mathrm{A}\subseteq\mathscr{A}} (-1)^{|\mathrm{A}|-1}[\Delta'_{\mathrm{A}}]\mathbf{L}^{|\mathrm{A}|-1}$$

in $\mathrm{Burn}^{\widehat{\mu}}(k)$.

Remark 8.3. — The relation between the two specialization morphisms is as follows. Fix a nonempty subset A of $\mathscr{A}$. The group $\mathbf{G}_{\mathrm{m}}$ acts diagonally on $\mathscr{N}^\circ_{\Delta_{\mathrm{A}}}(\mathscr{X})$ (the factors of index $\alpha\notin\mathrm{A}$ don't act), and this induces an action of the finite group of roots of unity of order e_{A} on Δ'_{A}, hence an action of $\widehat{\mu}$, so that $\widehat{\rho}_t(\mathrm{X})$ naturally lives in the equivariant Burnside ring $\mathrm{Burn}^{\widehat{\mu}}(k)$. Moreover, taking the $\widehat{\mu}$-invariants of Δ'_{A}, we get $\Delta^{\circ}_{\mathrm{A}}$, so that the specialization map ρ_t is the composition of $\widehat{\rho}_t$ with the map

$$\mathrm{Burn}^{\widehat{\mu}}(k) \to \mathrm{Burn}(k)$$

obtained by taking $\widehat{\mu}$-invariants.

Taking invariants does not commute with taking products, in general, so that ρ_t is not multiplicative.

8.4. — Let us explain how to define analogous specialization homomorphisms in our context of Burnside groups with volume forms.

For simplicity, we only consider the case where K has transcendence degree 1 over k, in which case the idea can be explained geometrically as follows. We assume that there exists an smooth integral curve C together with a k-point $o \in \mathrm{C}(k)$ such that $\mathrm{K} = k(\mathrm{C})$ and $\mathrm{R} = \mathscr{O}_{\mathrm{C},o}$. We fix a local parameter $t \in \mathrm{R}$ such that $\mathrm{V}(t) = o$.

Let us consider a pair (X, ω) consisting of an integral proper K-variety X of dimension n and a logarithmic n-form ω on X.

8.5. — Consider a regular flat proper model $\mathscr{X}$ is of X over C, let $\Delta = (\mathscr{X}_o)_{\mathrm{red}}$ be its reduced special fiber, and consider a divisor $\mathscr{D}$ with relative normal crossings on $\mathscr{X}$. We assume that the divisor $\mathscr{D} + \Delta$ has normal crossings. In this situation, Deligne (1970, §3.3.2) says that a meromorphic relative differential m-form on $\mathscr{X}/\mathrm{C}$ is logarithmic with respect to $\mathscr{D} + \Delta$ if it is (locally) the image of a logarithmic m-form $\tilde{\omega}$ in $\Omega^m_{\mathscr{X}/k}$ with poles $\mathscr{D} + \Delta$ under the natural morphism $\Omega^m_{\mathscr{X}/k} \to \Omega^m_{\mathscr{X}/\mathrm{C}}$.

Consider a logarithmic relative n-form ω on $\mathscr{X}/\mathrm{C}$. We consider an associated volume form ω' on $\mathscr{X}$, defined locally by

$$\omega' = \tilde{\omega} \wedge \mathrm{d}t/t,$$

where $\tilde{\omega}$ is any local lift of ω. This form ω' is logarithmic and we can compute its "residue along Δ" as in §4, only taking into account the strata of the polar divisor of ω' which are contained in the special fiber Δ.

There exists a subset $\mathscr{A}_o$ of $\mathscr{A}$ and a subset $\mathscr{B}_o$ of $\mathscr{B}$ such that the polar divisor of ω' is given by

$$\sum_{\alpha \in \mathscr{A}_o} \Delta_\alpha + \sum_{\beta \in \mathscr{B}_o} \mathscr{D}_\beta.$$

We thus set

$$\rho_t(\mathscr{X}, \omega) = \sum_{\substack{\emptyset \neq \mathrm{A} \subseteq \mathscr{A}_o \\ \mathrm{B} \subseteq \mathscr{B}_o}} (-1)^{|\mathrm{A}|+|\mathrm{B}|-1} \rho_{\Delta_{\mathrm{A}} \cap \mathscr{D}_{\mathrm{B}}}(\mathscr{X}, \omega).$$

This is an element of $\mathbf{Burn}_n(\mathscr{X}_o/k)$.

Proposition 8.6. — *Let $\mathscr{Y}$ be an irreducible closed subscheme of $\mathscr{X}$ which is transverse to $\mathscr{D} + \Delta$ and let $g\colon \mathscr{X}' \to \mathscr{X}$ be the blowing-up of $\mathscr{X}$ along $\mathscr{Y}$. The form $g^*\omega$ on $\mathscr{X}'$ is logarithmic and we have*

$$g_*\rho_t(\mathscr{X}', g^*\omega) = \rho_t(\mathscr{X}, \omega)$$

in $\mathbf{Burn}_n(\mathscr{X}_o/k)$.

Proof. — With the notation of §4, the difference

$$\rho(\mathscr{X},\tilde{\omega}) - \rho_t(\mathscr{X},\omega)$$

is exactly the part of $\rho(\mathscr{X},\tilde{\omega})$ which lies over the complement of the special fiber $\mathscr{X}_o$ in $\mathscr{X}$. We have seen in theorem 4.7 that

$$g_*\rho(\mathscr{X}',\tilde{\omega}') = \rho(\mathscr{X},\omega),$$

and a similar formula holds over $\mathscr{X} - \mathscr{X}_o$. This implies the proposition. □

8.7. — Starting from a smooth proper K-variety X and a logarithmic volume form ω on X, we can define a model $\mathscr{X}/\mathrm{C}$, with $\mathscr{D}$ and Δ as above, but the form ω will not necessarily extend to a logarithmic relative form with respect to $\mathscr{D}+\Delta$, nor does the volume form $\tilde{\omega}$ on $\mathscr{X}$. However, this can be achieved by multiplying ω by a suitable power of the uniformizing element.

Let us write the polar divisor of $\tilde{\omega}$ on $\mathscr{X}$ as

$$\operatorname{div}_{\mathscr{X}}(\tilde{\omega}) = \mathrm{D} + \Delta = \sum_{\alpha\in\mathscr{A}} d_\alpha \Delta_\alpha + \sum_{\beta\in\mathscr{B}} d_\beta \mathscr{D}_\beta.$$

With this notation, the condition for $\tilde{\omega}$ to be logarithmic on $\mathscr{X}$ is just that

$$d_\alpha \geqslant -1, \qquad d_\beta \geqslant -1.$$

In particular, while the conditions at the horizontal components follow from their counterparts on the generic fiber, those for the vertical components are not automatic. On the other hand, for any $\kappa \in \mathbf{Z}$, the form $t^\kappa\tilde{\omega}$ is logarithmic if and only if

$$\kappa e_\alpha + d_\alpha \geqslant -1$$

for all $\alpha \in \mathscr{A}$, that is, if and only if $\kappa \geqslant \kappa(\omega)$, where $\kappa(\omega)$ is defined by

$$\kappa(\omega) = \inf_{\alpha\in\mathscr{A}} \frac{1-d_\alpha}{e_\alpha}.$$

Since the rational number $\kappa(\omega)$ is defined in terms of logarithmic forms, it only depends on the class of (X,ω) in $\mathbf{Burn}_n(\mathrm{K})$, and not on the actual model which is chosen to compute it.

8.8. — We assume for the moment that $\kappa(\omega) \in \mathbf{Z}$. This holds in particular if the special fiber $\mathscr{X}_o$ is reduced. Let then $\mathscr{A}_o$ be the subset of $\mathscr{A}$ consisting of all α such that

$$\kappa(\omega)e_\alpha + d_\alpha = -1,$$

and let $\mathscr{B}_o$ be the subset of $\mathscr{B}$ consisting of all β such that $d_\beta = -1$. The polar divisor of $t^\kappa\tilde{\omega}$ is equal to

$$\sum_{\alpha\in\mathscr{A}_o} \Delta_\alpha + \sum_{\beta\in\mathscr{B}_o} \mathscr{D}_\beta,$$

and we set
$$\rho_t(\mathscr{X},\omega) = \rho_t(\mathscr{X}, t^{\kappa(\omega)}\omega)$$
in $\mathbf{Burn}_n(\mathscr{X}_o/k)$.

In the particular case where $\mathscr{D}$ is empty, the strata of the Clemens complex of the special fiber that actually appear in the definition of this class are those defined by KONTSEVICH & SOIBELMAN (2006), more precisely, by the adjustment provided by MUSTAŢĂ & NICAISE (2015).

8.9. — In the general case, the rational number $\kappa(\omega)$ is not an integer. Let us consider the finite ramified extension $\mathrm{K}_d = \mathrm{K}(t^{1/d})$ of K, whose ramification index d is a multiple of the denominator of $\kappa(\omega)$, but which induces an isomorphism on the residue field. Geometrically, this furnishes a morphism $\pi\colon \mathrm{C}_d \to \mathrm{C}$ which is ramified at the point o, together with a lift of o in $\mathrm{C}_d(k)$ (still denoted by o), and a distinguished uniformizing element $t^{1/d}$.

We consider the extension of (X,ω) to K_d and introduce a model $(\mathscr{X}_d,\omega_d)$ as above, over C_d. Now, the corresponding κ-parameter is integral, so that any choice of a uniformizing element $t^{1/d}$ in R_d induces a class $\rho_{t^{1/d}}(\mathscr{X}_d,\omega_d)$ in $\mathbf{Burn}(k)$. In fact, we can assume that the scheme $\mathscr{X}_d$ carries an action of the group scheme μ_d of dth roots of unity induced by its action on $\operatorname{Spec}(\mathrm{R}_d)$, leaving the logarithmic form ω_d invariant. In other words, we obtain a class in the group $\mathbf{Burn}^{\widehat{\mu}}(k)$.

Combinining these classes, we obtain the desired group homomorphism
$$\widehat{\rho}_t\colon \mathbf{Burn}(\mathrm{K}) \to \mathbf{Burn}^{\widehat{\mu}}(k).$$

In fact, as explained in (NICAISE, 2013, §2.3), especially proposition 2.3.2, one can compute the normalisation of $\mathscr{X} \otimes \mathrm{R}_d$ in terms of the given model $\mathscr{X}$. This gives an explicit decomposition of $\widehat{\rho}_t(\mathrm{X},\omega)$ as a sum
$$\sum_{\emptyset\neq \mathrm{A}\subseteq\mathscr{A}_o} (-1)^{|\mathrm{A}|-1}[\mathrm{D}'_\mathrm{A}, \nu_\mathrm{A}^*\omega_\mathrm{A}]\cdot \mathbf{T}^{|\mathrm{A}|-1},$$
where $\nu_\mathrm{A}\colon \mathrm{D}'_\mathrm{A} \to \mathrm{D}_\mathrm{A}$ is the μ_{d_A}-torsor introduced in §8.1 for the definition of the classical specialization map.

Remark 8.10. — In the case of specialization of rationality, it has proved fruitful to consider models with singularities on the special fiber, mild enough so that the special fiber computes the specialization of the birational type of the generic fiber. This is in particular the case for rational double points.

A parallel study can be developped in the context of varieties with logarithmic forms.

Following (KONTSEVICH & TSCHINKEL, 2019) and keeping track of the various logarithmic volume forms on the strata, we have:

Theorem 8.11. — *The morphism $\widehat{\rho}_t$ is a ring homomorphism.*

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