EQUIVARIANT DERIVED EQUIVALENCE AND RATIONAL POINTS ON K3 SURFACES

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ABSTRACT. We study arithmetic properties of derived equivalent K3 surfaces over the field of Laurent power series, using the equivariant geometry of K3 surfaces with cyclic groups actions.

1. INTRODUCTION

Let X and Y be smooth K3 surfaces over a nonclosed field K. Suppose X and Y are derived equivalent over K, i.e., there is an equivalence of bounded derived categories of coherent sheaves

$$\Phi : D^b(X) \to D^b(Y),$$

as triangulated categories, defined over K. Such a derived equivalence respects (see [HT17, Section 1]):

- the Galois action on geometric Picard groups,
- the Brauer groups,
- the *index*, i.e., the gcd of degrees of field extensions K'/K such that $X(K') \neq \emptyset$.

We are interested in understanding which other *arithmetic* properties are preserved under Φ. Specifically, in [HT17] we asked whether or not

$$X(K) \neq \emptyset \quad \Leftrightarrow \quad Y(K) \neq \emptyset.$$

This is known when

- $K = \mathbb{F}_q$ is finite, $\mathrm{char}(K) > 2$, [LO15], [Huy16a, 16.4.3],
- K is real [HT17, Prop. 25],
- $K = \mathbb{C}((t))$ [HT17, Cor. 30], assuming that local monodromy has trace $\neq -2$, in which case both $X(K), Y(K) \neq \emptyset$,
- K is p-adic, under strong assumptions on the reduction and for $p \geq 7$ [HT17, Prop. 36].

We propose to study this in a very special case – isotrivial families of K3 surfaces over the punctured disc. Let $G = C_N$ be a finite cyclic

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group of order N. Fix projective K3 surfaces X and Y over $\mathbb{C}$ with G-actions and consider the associated isotrivial families

$$\mathcal{X}, \mathcal{Y} \to \Delta_1 := \mathrm{Spec}(\mathbb{C}((t))),$$

with generic fibers $\mathcal{X}_t$ and $\mathcal{Y}_t$ over $K = \mathbb{C}((t))$, as defined in Section 3.2.

Theorem 1. *Suppose that $\mathcal{X}_t$ and $\mathcal{Y}_t$ admit a derived equivalence*

$$\Phi : D^b(\mathcal{X}_t) \simeq D^b(\mathcal{Y}_t),$$

over K. If $\mathcal{X}_t(K) \neq \emptyset$ then $\mathcal{Y}_t(K) \neq \emptyset$.

Related questions were considered by [AAHF21] (hyperkähler fourfolds) and twisted K3 surfaces [ADPZ17]; here the existence of rational points is *not* compatible with derived equivalence. The case of torsors for abelian varieties is addressed in [AKW17].

Our approach is based on the analogy between equivariant geometry and descent for nonclosed fields. Section 2 presents foundations for derived equivalence in the presence of group actions, with a view toward equivariant approaches to the Mukai lattice. We link isotrivial families over fields of Laurent series to equivariant geometry in Section 3. Section 4 presents the proof of Theorem 1 through analysis of fixed points; we close with a discussion of connections with the Burnside formalism and open questions.

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2. Generalities

2.1. **Equivariant derived equivalence.** We follow [Plo07] and refer the reader to [KS15] for a more general approach.

Let k be an algebraically closed field of characteristic zero and X a smooth projective variety over k equipped with the action of a finite group G. We consider the bounded derived category $D^b(X, G)$ of G-equivariant complexes of coherent sheaves on X, i.e., objects are pairs $\mathcal{P} = (P, \rho)$ consisting of complexes P of coherent sheaves and G-linearizations ρ compatible with differentials [Plo07]. This is compatible with intrinsic formulations of G-actions on triangulated categories [Ela11, §9], under our assumptions.

Suppose that X and Y are smooth projective varieties with G-actions. Given an element

$$\mathcal{P} = (P, \rho) \in D^b(X \times Y, G \times G)$$

there is an equivariant Fourier-Mukai transform

$$\mathsf{FM}_{\mathcal{P}}(-, G) : D^b(X, G) \to D^b(Y, G),$$

obtained by pulling back via projection to X, tensoring by $\mathcal{P}$, and pushing forward via projection to Y [Plo07, § 1.2]. This operation makes sense [Plo07, Lemma 5] provided $\mathcal{P}$ is equivariant for the diagonal $G_\Delta \subset G \times G$ only, and the equivariant Fourier-Mukai transform is compatible with the ordinary Fourier-Mukai transform associated with P. (In other words, we can forget the G-actions.) Furthermore, if P induces an equivalence of ordinary derived categories then $\mathcal{P}$ induces an equivalence of the equivariant derived categories.

We assume that G acts faithfully on X and Y. Conversely, suppose that $P \in D^b(X \times Y)$ induces an equivalence. When can it be lifted to an equivariant derived equivalence? By [Plo07, Lem. 4], each kernel P inducing an equivalence must be simple, i.e., every automorphism of P as an element of the derived category may be represented as rescaling of a representative complex. In particular, if P is G-invariant as an element of the derived category – $(g, g)^*P$ is quasi-isomorphic to P for each $g \in G$ – then the underlying complex of sheaves is G-invariant. Using the identification $\mathrm{Aut}(P) = \mathbb{G}_m$, for our desired lifting it is necessary that the resulting cocycle $\alpha \in \mathrm{H}^2(G, \mathbb{G}_m)$ [Plo07, Lem. 1] vanish. When G is cyclic, $\mathrm{H}^2(G, \mathbb{G}_m) = 0$ and α vanishes automatically.

If P does lift to an equivariant complex $\mathcal{P} = (P, \rho)$ then this typically is not unique. We can tensor ρ freely with any character of G.

2.2. **Specialization to K3 surfaces.** We retain the notation of Section 2.1 and assume that X and Y are K3 surfaces with Mukai lattices $\widetilde{\mathrm{H}}(X, \mathbb{Z})$ and $\widetilde{\mathrm{H}}(Y, \mathbb{Z})$. Suppose that X and Y are derived equivalent, with the equivalence realized by an isomorphism

$$i : Y \xrightarrow{\sim} M_v(X),$$

where

$$v = (r, D, S) \in \mathrm{H}^0(X, \mathbb{Z}) \oplus \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z})$$

is the Mukai vector of a moduli space of vector bundles. See [Huy06, Prop. 10.10] for more details; in particular, since v induces a derived equivalence, r and s are relatively prime and we may assume $r > 0$. The kernel $P \in D^b(X \times Y)$ inducing the equivalence may be interpreted as a universal sheaf over $X \times M_v(X)$. We have suppressed the polarization from the notation because it is irrelevant for our analysis; under our assumptions, any ample line bundle will yield a fine moduli space parametrizing stable sheaves [Huy06, Prop. 10.20].

Suppose now that X and Y come with faithful actions by a finite group G, where v is G-invariant so that $M_v(X)$ admits a G-action. Here, we are implicitly using a G-invariant polarization so stability is compatible with the G action.

Fix an equivariant isomorphism $i : Y \xrightarrow{\sim} M_v(X)$ as above. This is not sufficient to produce an equivariant derived equivalence between X and Y. The issue is the existence of an *equivariant* universal sheaf $E \to X \times M_v(X)$. Given an arbitrary universal sheaf E, simplicity of the sheaves parametrized by $M_v(X)$ yields

$$g^*E \simeq E \otimes p_2^*L_g, \quad g \in G,$$

where L_g is a line bundle on $M_v(X)$. The data $(L_g)_{g\in G}$ defines an element in $\mathrm{H}^1(G, \mathrm{Pic}(M_v(X)))$. Assuming this vanishes, we can produce an invariant kernel P on $X \times Y$. As we have seen, the obstruction to lifting P to an equivariant complex $\mathcal{P}$ then lies in $\mathrm{H}^2(G, \mathbb{G}_m)$.

Both these obstructions are encoded by

$$\ker\left(\mathrm{Br}(M_v(X), G) \to \mathrm{Br}(M_v(X))\right)$$

in the equivariant Brauer group, computed by a spectral sequence with E_2-terms [HT22, § 2.3]

$$\mathrm{H}^2(G, \mathbb{G}_m) \text{ and } \mathrm{H}^1(G, \mathrm{Pic}(M_v(X))).$$

Ploog's cocycle α lies in the kernel of the natural arrow

$$\mathrm{H}^2(G, \mathbb{G}_m) \to \mathrm{Br}(M_v(X), G)$$

induced by the structure map of $M_v(X)$. This vanishes when $M_v(X)$ admits a fixed point.

Mukai [Muk87] and Orlov [Orl97, Th. 3.3] have shown that K3 surfaces X and Y are derived equivalent if and only if there is an isomorphism of transcendental lattices

$$T(X) \simeq T(Y),$$

as Hodge structures. This does not suffice in the equivariant case:

Proposition 2. *Let X and Y be complex projective K3 surfaces with faithful actions by a finite group G. Then we have a sequence of implications:*

(1) *there is a G-equivariant derived equivalence $D^b(X) \simeq D^b(Y)$;*
(2) *there is an isomorphism of Mukai lattices*

$$\widetilde{\mathrm{H}}^*(X, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}^*(Y, \mathbb{Z})$$

respecting the Hodge structures and the G-actions;
(3) *there is a G-equivariant isomorphism*

$$T(X) \simeq T(Y)$$

of transcendental lattices, compatible with Hodge structures.

Proof. Suppose that X and Y are equivariantly derived equivalent. Then there is an isomorphism $i : Y \simeq M_v(X)$ such that the universal sheaf

$$E \to X \times M_v(X)$$

admits a G-linearization ρ such that $\mathsf{FM}_{(E,\varrho)}$ is an equivalence. The cohomological Fourier-Mukai transform and i induce an isomorphism

$$i^* \circ \mathsf{FM}_E : \widetilde{\mathrm{H}}^*(X, \mathbb{Z}) \to \widetilde{\mathrm{H}}^*(Y, \mathbb{Z})$$

taking v to $(0, 0, 1)$. The homomorphism $i^* \circ \mathsf{FM}_E$ induces the desired isomorphism of transcendental cohomology groups. □

Reversing the first implication in Proposition 2 is not possible precisely when the obstruction $\alpha \in \mathrm{H}^2(G, \mathbb{G}_m)$ is nonzero. Since the obstruction α vanishes in the cyclic case we have:

Corollary 3. *Suppose that X and Y are complex projective K3 surfaces with faithful actions by a cyclic group G. Then there is a G-equivariant derived equivalence between them iff there is an isomorphism of their Mukai lattices respecting the Hodge structures and the G-actions.*

Remark 4. The second implication in Proposition 2 also fails to be an equivalence in general. To extend an isomorphism $T(X) \simeq T(Y)$ to an isomorphism of Mukai lattices, we require a G-equivariant isomorphism of lattices

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$$

compatible (on discriminant groups) with the given isomorphism of transcendental lattices. By definition, $T(X)$ is the orthogonal complement to $\mathrm{Pic}(X)$ in $\mathrm{H}^2(X, \mathbb{Z})$. Example 5 shows such a homomorphism might not exist.

Example 5. Given a polarized K3 surface of degree two (X, f), $f^2 = 2$, the linear series $|f|$ induces a double cover $X \to \mathbb{P}^2$ [SD74, Th. 3.1 and Prop. 8.1], branched over a smooth plane curve of degree six. The covering involution ι acts on $f^\perp \subset \mathrm{H}^2(X, \mathbb{Z})$ by multiplication by -1.

Let X be a K3 surface surface with

$$\mathrm{Pic}(X) = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & 5 \\ f_2 & 5 & 2 \end{array},$$

with involutions ι_1 and ι_2 associated with the double covers $X \to \mathbb{P}^2$ induced by f_1 and f_2. Each involution acts on the primitive cohomology – hence the transcendental cohomology $T(X)$ – by -1. However, we shall show there is no automorphism of the Mukai lattice

$$a : \widetilde{\mathrm{H}}(X, \mathbb{Z}) \to \widetilde{\mathrm{H}}(X, \mathbb{Z})$$

compatible with Hodge structures and conjugating these involutions. In particular (3) does not imply (2) in Proposition 2.

We argue by contradiction; assume such an a existed. We have

$$\iota_1(2f_2 - 5f_1) = -(2f_2 - 5f_1) \quad \iota_2(2f_1 - 5f_2) = -(2f_1 - 5f_2),$$

the unique (up to sign) elements of the Mukai lattice that are algebraic with eigenvalue -1. Thus we must have

$$a(2f_2 - 5f_1) = \pm(2f_1 - 5f_2).$$

The discriminant group $d(\mathrm{Pic}(X)) = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})/\,\mathrm{Pic}(X)$ is

$$\mathbb{Z}/21\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z},$$

with generators $d_1 = \frac{f_1 - f_2}{3}$ and $d_2 = \frac{f_1 + f_2}{7}$. Our distinguished elements give generators

$$\frac{2f_2 - 5f_1}{21} = -d_1 + 3d_2 \quad \frac{2f_1 - 5f_2}{21} = d_1 - 4d_2.$$

Note that these are not equal, even up to sign. We conclude that any automorphism of the algebraic classes

$$\mathrm{Pic}(X) \oplus \mathrm{H}^0(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z}) \subset \widetilde{\mathrm{H}}(X, \mathbb{Z})$$

conjugating ι_1 and ι_2 acts on the discriminant group by an element $\neq \pm 1$. In particular, this applies to

$$a|_{\mathrm{Pic}(X) \oplus \mathrm{H}^0(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})}$$

The only automorphisms of the transcendental cohomology $T(X)$ – assuming X is general with the stipulated Picard group – are multiplications by ± 1. These are the only elements commuting with the action of the Hodge group of a general such X, which is the identity component of the orthogonal group associated with the intersection form. Thus

$$a|_{T(X)} = \pm 1$$

and the same holds true on the discriminant group. This gives a contradiction: Nikulin's theory gives an isomorphism

$$d(T(X)) \simeq d(\mathrm{Pic}(X))$$

and any automorphism of the full cohomology (compatible with the Hodge decomposition) must respect this isomorphism.

Remark 4 is reminiscent of [HS05, Exam. 4.11]: Isomorphisms of transcendental cohomology groups of *twisted* K3 surfaces need not lift to twisted derived equivalences.

We close with examples of intriguing derived equivalences relating K3 surfaces with involution:

Example 6. Recall that the derived category of any smooth projective variety X has an involution

$$\begin{array}{rcl} i_X : D^b(X) & \to & D^b(X) \\ \mathcal{E} & \mapsto & (\mathcal{E}[1])^\vee \end{array},$$

i.e., the composition of "shift-by-one" and "taking duals". When X is a K3 surface, i_X acts on $\widetilde{\mathrm{H}}(X,\mathbb{Z})$ by the identity on H^2 and multiplication by -1 on H^0 and H^4. Note that i_X is *not* an autoequivalence – indeed it fails to be orientation-preserving, a necessary condition for autoequivalences [HMS09, §4].

We seek degree two K3 surfaces (X,f) and (Y,g) (cf. Example 5) with associated involutions

$$\iota : X \to X, \quad \kappa : Y \to Y,$$

such that $(D^b(X), i_X \circ \iota)$ and $(D^b(Y), i_Y \circ \kappa)$ are C_2-equivariantly derived equivalent but (X,f) and (Y,h) are not isomorphic. Analogous to Corollary 3, we would like equivariant isomorphisms of Mukai lattices (with Hodge structures)

$$a : \widetilde{\mathrm{H}}(X,\mathbb{Z}) \simeq \widetilde{\mathrm{H}}(Y,\mathbb{Z})$$

where there is no equivariant isomorphism

$$\mathrm{H}^2(X,\mathbb{Z}) \not\simeq \mathrm{H}^2(Y,\mathbb{Z}).$$

These may be produced using the theory of binary quadratic forms [Bue89]. Consider even, negative definite, rank-two lattices represented by symmetric integer matrices A and B. We say that they are in the same *genus* if they are p-adically equivalent for all primes p; this is equivalent [Nik79b, Cor. 1.13.4] to stable equivalence

$$A \oplus U \simeq B \oplus U, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There are criteria, expressed via class groups, for the existence of non-isomorphic lattices in the same genus; see [Bue89, App. 1] for tables.

We seek examples of such lattices A and B, subject to the condition that A and B do not represent -2. This last assumption ensures that the divisors f and g are ample. For instance, consider even positive definite binary forms of discriminant -47; the reduced forms are:

$$\begin{pmatrix} 2 & 1 \\ 1 & 24 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 12 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & 12 \end{pmatrix}, \begin{pmatrix} 6 & 1 \\ 1 & 8 \end{pmatrix}, \begin{pmatrix} 6 & -1 \\ -1 & 8 \end{pmatrix}.$$

Only the first of these represents 2 so we could take

$$A = -\begin{pmatrix} 4 & 1 \\ 1 & 12 \end{pmatrix}, \ B = -\begin{pmatrix} 6 & 1 \\ 1 & 8 \end{pmatrix}.$$

We construct the desired K3 surfaces using surjectivity of the Torelli map. Choose a K3 surface X with

$$\mathrm{Pic}(X) = \mathbb{Z}f \oplus A$$

with involution ι fixing f and acting on A and $T(X)$ via -1. There exists a second K3 surface Y with

$$\mathrm{Pic}(Y) = \mathbb{Z}g \oplus B$$

and $T(X) \simeq T(Y)$. This admits an involution κ acting on B and $T(Y)$ via -1. There is no isomorphism $\mathrm{Pic}(X) \simeq \mathrm{Pic}(Y)$ compatible with the involutions. However the stable equivalence of A and B induces

$$\widetilde{\mathrm{H}}(X, \mathbb{Z}) \simeq U \oplus \mathrm{H}^2(X, \mathbb{Z}) \simeq U \oplus \mathrm{H}^2(Y, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}(X, \mathbb{Z}),$$

compatible with ι and κ. The involutions act on the U summands via multiplication by -1.

We will explore this further in [HT23].

3. Isotrivial families

3.1. **Construction.** Let X be a projective K3 surface and

$$G = C_N \subseteq \mathrm{Aut}(X)$$

a finite cyclic subgroup of the automorphism group of X. Let $\Delta_2 = \mathrm{Spec}(\mathbb{C}[[\tau]])$ be a formal disc on which G acts via

$$\tau \mapsto \zeta\tau, \quad \zeta = \exp(2\pi i/N).$$

The G-equivariant projection

$$X \times \Delta_2 \to \Delta_2$$

induces an isotrivial family

$$\pi : \mathcal{X} := (X \times \Delta_2)/G \to \Delta_1 := \Delta_2/G.$$

Let $K = \mathbb{C}((t))$ and $L = \mathbb{C}((\tau))$ denote the fields associated with Δ_1 and Δ_2. We regard $\mathcal{X}_t$ as a K3 surface over K; a K-rational point of $\mathcal{X}_t$ is equivalent to a section of π.

Proposition 7. *Suppose that X and Y are complex K3 surfaces with faithful actions of $G = C_N$; assume they are G-equivariantly derived equivalent. Then $\mathcal{X}_t$ and $\mathcal{Y}_t$ are derived equivalent over K.*

Actually, our proof will give more: It suffices to assume that there exists a G-invariant complex P inducing the equivalence between X and Y (see Section 2).

Proof. Realize

$$i : Y \xrightarrow{\sim} M_v(X)$$

for some Mukai vector v for X, fixed under the G-action. This isomorphism may be chosen to be equivariant under the G-action. Letting $\tau = \sqrt[N]{t}$, we basechange to an isomorphism

$$\mathcal{Y}_\tau \simeq M_v(\mathcal{X}_\tau).$$

This descends to an isomorphism

$$\mathcal{Y}_t \simeq M_v(\mathcal{X}_t),$$

where the latter is the coarse moduli space. To complete the proof, we need that $M_v(\mathcal{X}_t) \times \mathcal{X}_t$ admits a universal sheaf. Since the underlying sheaves are simple, this universal sheaf is unique up to tensoring by line bundles on $M_v(\mathcal{X}_t)$ – a trivial line bundle given our assumption that P is G-invariant. Thus the obstruction to descending the data associated with P to a sheaf defined over K lives in the Brauer group of K. The triviality of $\mathrm{Br}\left(\mathbb{C}((t))\right)$ shows this obstruction vanishes. □

3.2. **Rational points and fixed points.**

Proposition 8. *The morphism*

$$\pi : \mathcal{X} \to \Delta_1$$

admits a section if and only if the action of G on X admits a fixed point.

Proof. If π admits a section $\sigma_1 : \Delta_1 \to \mathcal{X}$ then the induced section $\sigma_2 : \Delta_2 \to \mathcal{X} \times_{\Delta_1} \Delta_2$ is G-invariant, whence $\sigma_2(0)$ is fixed.

Suppose X has a fixed point. Then the resulting constant section of $X \times \Delta_2 \to \Delta_2$ is invariant under the action of G and thus descends to a section of $(X \times \Delta_2)/G \to \Delta_2/G$. $\square$

4. Fixed point analysis

Let X be a K3 surface over an algebraically closed field of characteristic zero and $\sigma \in \mathrm{Aut}(X)$ an automorphism of order N. In the following sections, we analyze the structure of the fixed point locus

$$X^\sigma := \{x \in X \mid \sigma(x) = x\},$$

with the goal of identifying σ such that $X^\sigma = \emptyset$.

4.1. **Cyclic automorphisms.** We review basic properties of finite automorphisms due to Nikulin [Nik79a]. Suppose that $G = \langle \sigma \rangle = C_N$ acts on a K3 surface X. We have an exact sequence

$$0 \to C_n \to G \to \mu_m \to 0, \quad nm = N, \tag{4.1}$$

where C_n is the kernel of the representation of G on the symplectic form. Elements in C_n are called *symplectic*; when $C_n = 1$, the action is called *purely nonsymplectic*. We write $N = n \cdot m$, to emphasize the symplectic versus nonsymplectic actions.

Proposition 9. *Let X_1 and X_2 derived equivalent K3 surfaces. Assume that both carry a faithful action of $G = C_N$ and that the derived equivalence is compatible with G. Then the factorizations*

$$N = n_1 m_1 = n_2 m_2,$$

encoding the symplectic elements, are equal, i.e.,

$$n_1 = n_2 \quad \textit{and} \quad m_1 = m_2.$$

Proof. We can read off the symplectic automorphisms from the action on the Mukai lattice, as the symplectic form is distinguished in its complexification. $\square$

4.2. **Fixed point formulas.** Let $G = \langle\sigma\rangle$ be a cyclic group acting on a K3 surface X. Let

$$\sigma^* : \tilde{\mathrm{H}}(X,\mathbb{Z}) \to \tilde{\mathrm{H}}(X,\mathbb{Z})$$

be the induced action on the Mukai lattice, and

$$\chi(\sigma) := \mathrm{Tr}(\sigma^*)$$

the corresponding trace.

The *topological* fixed point formula takes the form:

$$\chi(X^\sigma) = \chi(\sigma), \tag{4.2}$$

Since $\chi(\sigma)$ may be read off from the action on the Mukai lattice, $\chi(X^\sigma)$ is an invariant of G-equivariant derived equivalence.

Lemma 10. *Let $N = n \cdot m$ with $n \geq 2$. Then X^σ is empty or a finite set of isolated points, and*

$$\chi(X^\sigma) = \#X^\sigma.$$

Proof. By [Nik79a], symplectic automorphisms do not contain curves in their fixed locus (a detailed description of possible X^σ is in Section 4.3). ☐

The *complex* Lefschetz fixed point formula involves sums

$$\sum_{\mathfrak{p}} a(\mathfrak{p}) \;+\; \sum_C b(C), \tag{4.3}$$

of contributions from fixed points and fixed curves; here $\zeta = \zeta_N$ (see, [AS68, p. 567]). The corresponding contributions are given by

$$a(\mathfrak{p}) = \frac{1}{(1-\zeta^i)(1-\zeta^j)},$$

for fixed points $\mathfrak{p}$ with weights $\beta_{\mathfrak{p}} = (i,j)$ in the tangent bundle at $\mathfrak{p}$, and

$$b(C) = \frac{1-g(C)}{1-\zeta^{-r(C)}} - \frac{\zeta^{-r(C)}}{(1-\zeta^{-r(C)})^2}C^2,$$

where $g(C)$ is the genus of C, and $r(C)$ is the weight in the normal bundle to C. For $K3$ surfaces we obtain

$$1+\zeta^{-m} = \sum_{i,j} \frac{a_{ij}}{(1-\zeta^i)(1-\zeta^j)} + \sum_{C\subseteq X^\sigma} (1-g(C))\frac{1+\zeta^n}{(1-\zeta^n)^2}, \tag{4.4}$$

where

- a_{ij} is the number of σ-fixed points $\mathfrak{p}$ with weights $\beta_{\mathfrak{p}} = (i,j)$ in the tangent bundle at $\mathfrak{p}$,
$$i + j \equiv n \pmod N, \quad i, j \neq 0,$$
- $C \subseteq X^\sigma$ are (smooth irreducible) curves,

(see [Nik79a] or [ACV20, Lemma 1.1]).

Formula (4.4) immediately implies:

Lemma 11. *Let $N = n \cdot m$ with $m \neq 2$. Then*
$$X^\sigma \neq \emptyset.$$

Proof. Consider equation (4.4). If $m \neq 2$ then the left-hand side is nonzero. It follows that the sums on the right-hand side are nonempty. Since these are indexed by fixed points or curves, we conclude that $X^\sigma \neq \emptyset$. □

Lemma 11 shows that we always have fixed points in the **symplectic case**. In the **purely nonsymplectic case**, where $N = m$, or equivalently, $n = 1$, Lemma 11 guarantees fixed points, except where $m = N = 2$. In this case, the only fixed-point free action is the Enriques involution. Such an involution is characterized by the sublattice of its fixed classes (see, e.g., [Nik87], [AS15, Th. 1.1], [AST11, Th. 3.1]):
$$\mathrm{Pic}(X)^\sigma \simeq U(2) \oplus E_8(2).$$

We turn to the **mixed case** where $m, n > 1$. Lemma 10 guarantees that the existence of σ-fixed points is governed by the trace of σ on $\widetilde{\mathrm{H}}$, i.e., is a derived invariant. This completes the proof of Theorem 1.

4.3. **Role of classification in the proof.** Despite initial expectations, the proof of Theorem 1 does not hinge on classification. At the same time, the comprehensive enumeration in [BH21] does raise interesting questions.

> Can we explicitly describe all types of cyclic automorphisms with $X^\sigma = \emptyset$? Deeper arithmetic problems – extensions to more complicated isotrivial families or the p-adics – would require understanding of all finite groups of automorphisms.

We present indicative examples of actions with $X^\sigma = \emptyset$.

Nikulin [Nik79a] classified symplectic automorphisms of a K3 surface X of order n (in the notation above, $N = n$ and $m = 1$): We necessarily have $n \leq 8$ and $X^\sigma \neq \emptyset$. Moreover, X^σ is a finite set of isolated points, whose structure is given by

- $n = 2 : 8$ fixed points
- $n = 3 : 6$ fixed points
- $n = 4 : 4$ fixed points (and 4 points with order two stabilizer)
- $n = 5 : 4$ fixed points
- $n = 6 : 2$ fixed points (and 4 points with order three stabilizer, and 6 points with order two stabilizer)
- $n = 7 : 3$ fixed points
- $n = 8 : 2$ fixed points (and 2 points with order four stabilizer, 4 points with order two stabilizer).

Mukai [Muk88] gave a classification of all finite groups acting symplectically.

Detailed results are also available for purely nonsymplectic automorphisms of order m. The cases of *prime* order have been considered in [AST11], and various other special cases in, e.g., [ACV20], [AS15], [Dil12], [SyT21], [Tak10]. A complete classification, including an analysis of possible fixed point configurations, is presented in [BH21, Appendix B]: Let σ be a purely nonsymplectic automorphism of a K3 surface X of order m. Then

$$m \in \{2, \dots, 28\} \setminus \{23\},$$

or

$$m \in \{30, 32, 33, 34, 36, 40, 44, 48, 50, 54, 66\}.$$

We return to our general situation where $C_N, N = nm$, acts on a K3 surface, via mth roots of unity on the symplectic form. Lemma 11 allows us to restrict to $m = 2$.

By [Keu16, Lem. 4.8], $m = 2$ implies that $n \neq 8$. For $n = 7$, the number of fixed points of the subgroup $C_7 = \langle \sigma^2 \rangle \subset G$ is three, thus we are guaranteed σ-fixed points. For $n \leq 6$ there exist fixed-point free actions. We record:

- $N = 2 \cdot 2$: Then X^σ is either empty, or it consists of 2, 4, 6, or 8 points [AS15, Prop. 2]; when $X^\sigma = \emptyset$, the σ^*-action on $\mathrm{H}^2(X, \mathbb{Q})$ has eigenvalues 1 and -1 with multiplicities 6 and 8, this characterizes such actions [AS15, Prop. 2]. K3 surfaces with $N = 2 \cdot 2$ have $\operatorname{rk} \operatorname{Pic}(X) \geq 14$ [AS15, Rema. 1.3]. Examples of such actions can be found in [AS15, Exam. 1.2].
- $N = 3 \cdot 2$: Then X^σ is either empty, or it consists of 2, 4, or 6 points [SyT21, Prop. 3.4].
- $N = 4 \cdot 2$: Here the enumeration of cases is more complicated. The classification in [BH21] of symplectic actions on K3 surfaces lists only *maximal* actions: If G acts symplectically on a

K3 surface X, consider its *saturation*, i.e., the largest subgroup $G' \subset \mathrm{Aut}(X)$ such that $\mathrm{H}^2(X,\mathbb{Z})^{G'} = \mathrm{H}^2(X,\mathbb{Z})^G$ – a finite group acting symplectically on X. Thus the enumeration requires checking many subgroups for the presence of an element of the prescribed order.

Consider, for instance, the group with GAP id `(8,1)` from the second column of Table 3 in [BH21], which lists three types. The possibilities for $\chi(\sigma^r)$, for $r = 1, 2, 4$, are

σ	σ^2	σ^4
0	4	8
2	4	8
4	4	8

- $N = 5\cdot 2$: Note that $C_n, n = 5, 6, 7$ does not appear as the saturation of a mixed action with $m = 2$ [BH21, Table 3]. However, there are larger groups admitting cyclic subgroups of order ten acting on the symplectic form via ± 1.

  For example, suppose that G is an extension
  $$1 \to \mathfrak{A}_6 \to G \to \mu_2 \to 1,$$
  where the alternating group is the maximal symplectic subgroup. Assume that G has GAP id `(720,764)`, which admits elements of order ten. (Of course, $\mathfrak{A}_6$ has no such elements!) There are six different occurences of this group in the classification. The one with K3 id (79.2.1.3) has distinguished generator (in the nomenclature of the data sets supporting [BH21]) σ of order ten with $\chi(\sigma) = 0$.

4.4. **Relations to Burnside invariants.** Brandhorst and Hofmann [BH21] explore cases where the data from the fixed-point formulas are insufficient to characterize the automorphism. These are called *ambiguous cases*, at least in the purely nonsymplectic context [BH21, §7].

It would be interesting to consider these from the perspective of the Burnside group: Given the action of a finite cyclic group G on a K3 surface, there is a combinatorial object consisting of subgroups $G_i \subset G$ indexed by strata $Z_i \subset X$ with nontrivial stabilizer G_i, labeled by the induced action on Z_i, and the representation type of the action of G_i on the normal bundle; data of such type are building blocks of equivariant Burnside groups introduced in [KT22b]. The tables in [BH21, Appendix B] list possible configurations of fixed points and curves, for purely nonsymplectic actions. How much of the Burnside data can be extracted from the representation of G on the Mukai lattice?

The paper [KT22a] explores such a connection for actions of finite groups on del Pezzo surfaces.

Another interesting problem is to identify which actions classified in [BH21] are derived equivalent and even to classify finite groups of autoequivalences of K3 surfaces [Huy16b]. For example, Ouchi [Ouc21, §8] has found symplectic autoequivalences of orders 9 and 11 via cubic fourfolds; these cannot be realized as symplectic actions on K3 surfaces.

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