INVOLUTIONS ON K3 SURFACES AND DERIVED EQUIVALENCE

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Abstract. We study involutions on K3 surfaces under conjugation by derived equivalence and more general relations, together with applications to equivariant birational geometry.

1. Introduction

The structure of $\mathrm{Aut}\, D^b(X)$, the group of autoequivalences of the bounded derived category $D^b(X)$ of a K3 surface X, is very rich but well-understood only when the Picard group $\mathrm{Pic}(X)$ has rank one [BB17]. The automorphism group $\mathrm{Aut}(X)$ of X lifts to $\mathrm{Aut}\, D^b(X)$, and one may consider the problem of classification of finite subgroups $G \subset \mathrm{Aut}(X)$ up to conjugation – either by automorphisms, derived equivalence, or even larger groups. This problem is already interesting for cyclic G, and even for involutions, e.g., Enriques or Nikulin involutions. There is an extensive literature classifying these involutions on a given K3 surface X: topological types, moduli spaces of polarized K3 surfaces with involution, and the involutions on a single X up to automorphisms, see, e.g., [AN06], [vGS07], [Oha07], [SV20], [Zha98].

Here we investigate involutions up to derived equivalence, i.e., derived equivalences respecting involutions. Our interest in "derived" phenomena was sparked by a result in [Sos10]— there exist complex conjugate, derived equivalent nonisomorphic K3 surfaces—as well as our investigations of arithmetic problems on K3 surfaces [HT17], [HT22].

One large class of involutions $\sigma : X \to X$ are those whose quotient $Q = X/\sigma$ is rational. Examples include Q a del Pezzo surface and $X \to Q$ a double cover branched along a smooth curve $B \in |-2K_Q|$. We may allow Q to have ADE surface singularities away from B, or B to have ADE curve singularities; then we take X as the minimal resolution of the resulting double cover of Q. These were studied by Alexeev and Nikulin in connection with classification questions concerning singular

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del Pezzo surfaces [AN06]. Our principal result here (see Section 5) is that

- equivariant derived equivalences of such (X, σ) are in fact equivariant isomorphisms (see Corollary 11).

Our study of stable equivalence of lattices with involution leads us to a notion of *skew equivalence*, presented in Section 6. Here, duality interacts with the involution which is reflected in a functional equations for the Fourier-Mukai kernel. Explicit examples, for anti-symplectic actions with quotients equal to $\mathbb{P}^2$, are presented in Section 7.

Next, we focus on *Nikulin* involutions $\iota : X \to X$, i.e., involutions preserving the symplectic form, so that the resolution of singularities Y of the resulting quotient X/ι is a K3 surface. A detailed study of such involutions can be found in [vGS07]. In addition to the polarization class, the Picard group $\mathrm{Pic}(X)$ contains the lattice $\mathrm{E}_8(-2)$; van Geemen and Sarti describe the moduli and the geometry in the case of minimal Picard rank $\mathrm{rk}\,\mathrm{Pic}(X) = 9$. In Section 8, we extend their results to higher ranks, and

- exhibit nontrivial derived equivalences between Nikulin involutions (Proposition 21).

These, in turn, allow us to construct in Section 9 examples of equivariant birational isomorphisms $\phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ with nonvanishing invariant $C_G(\phi)$, introduced in [LSZ20], [LS22] and extended to the equivariant context in [KT22].

The case of *Enriques* involutions $\epsilon : X \to X$, i.e., fixed-point free involutions, so that the resulting quotient X/ϵ is an Enriques surface, has also received considerable attention. There is a parametrization of such involutions in terms of the Mukai lattice $\tilde{\mathrm{H}}(X)$, and an explicit description of conjugacy classes, up to automorphisms $\mathrm{Aut}(X)$, in interesting special cases, e.g., for K3 surfaces of Picard rank 11, Kummer surfaces of product type, general Kummer surfaces, or singular K3 surfaces [Kon92], [Oha07], [Ser05], [SV20]. In Section 10 we observe that

- the existence of an Enriques involution on a K3 surface X implies that every derived equivalent surface is equivariantly isomorphic to X (Propositions 28 and 29);
- while there are *no* nontrivial equivariant derived autoequivalences, we exhibit nontrivial *orientation reversing* (i.e., skew) equivalences, e.g., on singular K3 surfaces.

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2. Lattice results

We recall basic terminology and results concerning lattices: torsion-free finite-rank abelian groups L together with a nondegenerate integral quadratic form $(\cdot,\cdot)$, which we assume to be even. Basic examples are

$$\mathrm{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and positive definite lattices associated with Dynkin diagrams (denoted by the same letter).

We write $\mathrm{L}(2)$, when the form is multiplied by 2. We let

$$d(\mathrm{L}) := \mathrm{L}^*/\mathrm{L}$$

be the discriminant group and

$$q_{\mathrm{L}} : d(\mathrm{L}) \to \mathbb{Q}/2\mathbb{Z}$$

the induced discriminant quadratic form.

Nikulin's form of Witt cancellation:

Proposition 1. [Nik79b, Cor. 1.13.4] *Given an even lattice* L*,* $\mathrm{L}\oplus\mathrm{U}$ *is the unique lattice with its signature and discriminant quadratic form.*

If lattices L_1 and L_2 are stably isomorphic – become isomorphic after adding unimodular lattices of the same signature – then

$$\mathrm{L}_1 \oplus \mathrm{U} \simeq \mathrm{L}_2 \oplus \mathrm{U}.$$

Nikulin stabilization result: Given a lattice L, write $\mathrm{L}_p = \mathrm{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for the induced p-adic bilinear form. The p-primary part of $d(\mathrm{L})$ depends only on L_p and is written $d(\mathrm{L}_p)$. We use q_{L_p} for the induced discriminant quadratic form on $d(\mathrm{L}_p)$. For a finitely generated abelian group A, let $\ell(A)$ denote the minimal number of generators.

Proposition 2. [Nik79b, Thm. 1.14.2] *Let* L *be an even indefinite lattice satisfying*

- $\mathrm{rank}(\mathrm{L}) \geq \ell(d(\mathrm{L}_p)) + 2$ *for all* $p \neq 2$*;*

- *if* $\mathrm{rank}(\mathrm{L}) = \ell(d(\mathrm{L}_2))$ *then* q_{L_2} *contains* $u_+^{(2)}(2)$ *or* $v_+^{(2)}(2)$ *as a summand, i.e., the discriminant quadratic forms of*

$$\mathrm{U}^{(2)}(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \mathrm{V}^{(2)}(2) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Then the genus of L *admits a unique class and* $\mathrm{O}(\mathrm{L}) \to \mathrm{O}(q_{\mathrm{L}})$ *is surjective.*

Remark 3. [Nik79b, Rem. 1.14.5] The 2-adic condition can be achieved whenever the discriminant group $d(\mathrm{L})$ has $(\mathbb{Z}/2\mathbb{Z})^3$ as a summand.

Thus given a lattice L, any automorphism of $(d(\mathrm{L}), q_{\mathrm{L}})$ may be achieved via an automorphism of $\mathrm{L} \oplus \mathrm{U}$. More precisely, given two lattices L_1 and L_2 of the same rank and signature and an isomorphism

$$\varrho : (d(\mathrm{L}_1), q_{\mathrm{L}_1}) \xrightarrow{\sim} (d(\mathrm{L}_2), q_{\mathrm{L}_2})$$

there exists an isomorphism

$$\rho : \mathrm{L}_1 \oplus \mathrm{U} \xrightarrow{\sim} \mathrm{L}_2 \oplus \mathrm{U}$$

inducing ϱ.

Nikulin imbedding result:

Proposition 4. [Nik79b, Cor. 1.12.3,Thm. 1.14.4] *Let* L *be an even lattice of signature* (t_+, t_-) *and discriminant group* $d(\mathrm{L})$. *Then* L *admits a primitive embedding into a unimodular even lattice of signature* (ℓ_+, ℓ_-) *if*

- $\ell_+ - \ell_- \equiv 0 \mod 8$;
- $\ell_+ \geq t_+$ *and* $\ell_- \geq t_-$;
- $\ell_+ + \ell_- - t_+ - t_- > \ell(d(\mathrm{L}))$, *the rank of* $d(\mathrm{L})$.

This embedding is unique up to automorphisms if

- $\ell_+ > t_+$ *and* $\ell_- > t_-$;
- $\ell_+ + \ell_- - t_+ - t_- \geq 2 + \ell(d(\mathrm{L}))$.

In particular, *any* even nondegenerate lattice of signature $(1, 9)$ admits a unique embedding into the K3 lattice $\mathrm{U}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$.

3. Mukai lattices and derived automorphisms

Throughout, we work over the complex numbers $\mathbb{C}$. Let X be a complex K3 surface and

$$\mathrm{Pic}(X) \subset \mathrm{H}^2(X, \mathbb{Z}) \simeq \mathrm{E}_8(-1)^{\oplus 2} \oplus \mathrm{U}^3$$

its Picard lattice, a sublattice of a lattice of signature $(3, 19)$, with respect to the intersection pairing. The Picard lattice determines the automorphisms of X: the natural map

$$\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Pic}(X))/\langle \text{reflections by } (-2)\text{-classes}\rangle,$$

to the quotient of the orthogonal group of the Picard lattice, has finite kernel and cokernel. All possible finite $G \subset \mathrm{Aut}(X)$ have been classified, see [BH21]. Classification of $\mathrm{Aut}(X)$-conjugacy classes of elements or subgroups boils down to lattice theory of $\mathrm{Pic}(X)$; we will revisit it in special cases below.

The *transcendental lattice* of X is the orthogonal complement

$$T(X) := \mathrm{Pic}(X)^{\perp} \subset \mathrm{H}^2(X, \mathbb{Z}).$$

This lattice plays a special role: two K3 surfacs X_1, X_2 are *derived equivalent* if and only if there exists an isomorphism of lattices

$$T(X_1) \xrightarrow{\sim} T(X_2),$$

compatible with Hodge structures [Orl97]. Derived equivalence also means that the lattices $\mathrm{Pic}(X_1)$ and $\mathrm{Pic}(X_2)$ belong to the same genus. Over nonclosed fields, or in equivariant contexts, derived equivalence is a subtle property, see, e.g., [HT17], [HT22].

We recall standard examples of Picard lattices of derived equivalent but not isomorphic K3 surfaces

Remark 5. In Picard rank one: the number of nonisomorphic derived equivalent surfaces is governed by the number of prime divisors of the polarization degree $2d$; see [HLOY04, Cor. 2.7] and Remark 5. The isomorphisms classes correspond to solutions of the congruence

$$x^2 \equiv 1 \pmod{4d} \tag{3.1}$$

modulo ± 1. When $d > 1$ the number of derived equivalent K3 surfaces is $2^{\tau(d)-1}$, where τ is the number of distinct prime factors of d.

In Picard rank two: derived equivalences among lattice-polarized K3 surfaces of square-free discriminant are governed by the genera in the class group of the corresponding real quadratic field [HLOY04, Sect. 3].

Here are instances where derived equivalence is trivial

Proposition 6. [HLOY04, Cor. 2.6, 2.7] *Derived equivalence implies isomorphism in each of the following cases:*

- *if the Picard rank is ≥ 12;*
- *if the surfaces admits an elliptic fibration with a section;*

- *if the Picard rank is ≥ 3 and the discriminant group of the Picard group is cyclic.*

We give a further example in Proposition 21.

Let
$$\tilde{\mathrm{H}}(X) := \mathrm{H}^0(X,\mathbb{Z})(-1) \oplus \mathrm{H}^2(X,\mathbb{Z}) \oplus \mathrm{H}^4(X,\mathbb{Z})(1)$$
be its *Mukai lattice*, a lattice of signature $(4, 20)$, with respect to the Mukai pairing. There is a surjective homomorphism [HMS09, Cor. 3]
$$\mathrm{Aut}\, D^b(X) \to \mathrm{O}^+(\tilde{\mathrm{H}}(X)) \subset \mathrm{O}(\tilde{\mathrm{H}}(X))$$
onto the group of *signed* Hodge isometries, a subgroup of the orthogonal group of the Mukai lattice preserving orientations on the positive 4-planes.

We retain the notation from [HT22, Sect. 2], where we discussed the notion and basic properties of equivariant derived equivalences between K3 surfaces. We recall:

> Let X_1 and X_2 be K3 surfaces equipped with a generically free action of a finite cyclic group G. Then X_1 and X_2 are G-equivariantly derived equivalent if and only if there exists a G-equivariant isomorphism of their Mukai lattices
> $$\tilde{\mathrm{H}}(X_1) \xrightarrow{\sim} \tilde{\mathrm{H}}(X_2)$$
> respecting the Hodge structures.

Note that the G-action is necessarily trivial on
$$\mathrm{H}^0(X,\mathbb{Z})(-1) \oplus \mathrm{H}^4(X,\mathbb{Z})(1).$$

Even in the event of an isomorphism $X_1 \simeq X_2$, equivariant derived equivalences are interesting: indeed, there are actions of finite groups G that are not conjugate in $\mathrm{Aut}(X)$ but are conjugate via $\mathrm{Aut}\, D^b(X)$ as the action of the latter group is visibly larger.

Let G be a finite group and X_1 and X_2 K3 surfaces with G-actions. For simplicity, assume that G acts on $T(X_i)$ via $\pm\mathrm{I}$. (This is the case if the transcendental cohomology is simple.)

Given a G-equivariant isomorphism $T(X_1) \simeq T(X_2)$, can we lift to a G-equivariant isomorphism of Mukai lattices
$$\tilde{\mathrm{H}}(X_1,\mathbb{Z}) \simeq \tilde{\mathrm{H}}(X_2,\mathbb{Z}),$$
where G acts trivially on the hyperbolic summand
$$\mathrm{U} = \mathrm{H}^0 \oplus \mathrm{H}^4?$$

Clearly the answer is NO. Suppose that $G = C_2 = \langle\epsilon\rangle$ and the $\epsilon = -1$ eigenspaces are stably isomorphic but not isomorphic. Adding U does nothing to achieve the desired stabilization.

In other words, U is "too small". We need to add summands where G acts nontrivially to achieve stabilization across all the various isotypic components.

4. Cohomological Fourier-Mukai transforms

Let X_1 and X_2 be smooth projective complex K3 surfaces. A fundamental result of Orlov [Orl97] shows that any equivalence

$$\Phi : D^b(X_1) \to D^b(X_2)$$

arises from a kernel $\mathcal{K} \in D^b(X_1 \times X_2)$ through a Fourier-Mukai transform

$$\begin{array}{rcl} \Phi_{\mathcal{K}} : D^b(X_1) & \to & D^b(X_2) \\ \mathcal{E} & \mapsto & \pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}). \end{array}$$

All the indicated functors are taken in their derived senses. Given such a kernel, there is also a Fourier-Mukai transform in the opposite direction

$$\begin{array}{rcl} \Psi_{\mathcal{K}} : D^b(X_2) & \to & D^b(X_1) \\ \mathcal{E} & \mapsto & \pi_{1*}(\pi_2^*\mathcal{E} \otimes \mathcal{K}). \end{array}$$

Mukai has computed the kernel of the inverse

$$\Phi_{\mathcal{K}}^{-1} = \Psi_{\mathcal{K}^\vee[2]}$$

i.e., a twist of the dual to our original kernel. See [Muk87, 4.10], [BBHR97, § 4.3], and [Huy06, p. 133] for details. The computation relies on Grothendieck-Serre Duality, so the appearance of the dualizing complex is natural. This machinery [Huy06, § 3.4] also allows us to analyze how Fourier-Mukai transforms interact with taking duals:

$$\begin{aligned} \Phi_{\mathcal{K}}(\mathcal{E}^\vee) &= \pi_{2*}(\mathcal{K} \otimes \pi_1^*(\mathcal{E}^\vee)) \\ &= ((\pi_{2*}(\mathcal{K}^\vee \otimes \pi_1^*\mathcal{E}))^\vee)[-2] \\ &= ((\Phi_{\mathcal{K}^\vee}\mathcal{E})[2])^\vee \\ &= (\Phi_{\mathcal{K}^\vee[2]}\mathcal{E})^\vee \end{aligned}$$

Suppose that X_1 and X_2 are equivalent through an isomorphism

$$X_2 = M_v(X_1),$$

i.e., the moduli space of simple sheaves $\mathcal{E}_p, p \in X_2$, on X_1 with Mukai vector

$$v(\mathcal{E}_p) = (r, D, s) \in \tilde{\mathrm{H}}(X, \mathbb{Z}).$$

Here r is the rank of $\mathcal{E}_p$, $D = c_1(\mathcal{E}_p)$, and $s = \chi(\mathcal{E}_p) - r$. We assume there exists another Hodge class $v' \in \tilde{\mathrm{H}}(X_1, \mathbb{Z})$ such that $\langle v, v' \rangle = 1$; in particular, v is primitive. Let $\mathcal{E} \to X_1 \times X_2$ denote a universal sheaf; by simplicity of the sheaves, $\mathcal{E}$ is unique up to tensoring by a line bundle from X_2. We may use $\mathcal{E}$ as a kernel inducing a derived equivalence between X_1 and X_2 [Huy06, 10.25]. Our formulas for inverses are compatible with tensoring the kernel by line bundles from one of the factors.

In searching for Fourier-Mukai kernels, cohomological Fourier-Mukai transforms play a crucial role. Let $\omega_i \in \mathrm{H}^4(X_i, \mathbb{Z})$ denote the point class and set [Muk87, §1], [Huy06, p. 128]

$$Z_{\mathcal{K}} := \pi_1^*(1 + \omega_1)\,\mathrm{ch}(\mathcal{K})\pi_2^*(1 + \omega_2) \in \mathrm{H}^*(X_1 \times X_2, \mathbb{Z}),$$

where the middle term is the Chern character. Then $Z_{\mathcal{K}}$ induces an integral isomorphism of Hodge structures

$$\phi_{\mathcal{K}} : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{\mathrm{H}}(X_2, \mathbb{Z})$$

compatible with Mukai pairings; this is called the *cohomological Fourier-Mukai transform.* For $\mathcal{E} \in D^b(X_1)$, we have the identity

$$\phi_{\mathcal{K}}(v(\mathcal{E})) = v(\Phi_{\mathcal{K}}(\mathcal{E})).$$

We use $\psi_{\mathcal{K}}$ to denote the cohomological transform of $\Psi_{\mathcal{K}}$.

Most cohomological Fourier-Mukai transforms are induced by kernels

Proposition 7. [Orl97, HMS09] *Given an orientation-preserving integral Hodge isometry*

$$\phi : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \to \tilde{\mathrm{H}}(X_2, \mathbb{Z})$$

there exists a derived equivalence

$$\Phi_{\mathcal{K}} : D^b(X_1) \to D^b(X_2)$$

such that ϕ is the cohomological Fourier-Mukai transform of $\Phi_{\mathcal{K}}$.

Suppose that (X_1, f_1) is a polarized K3 surfaces of degree $2r_0 s$ where r_0 and s are relatively prime positive integers. Let d_0 be an integer prime to r_0 and fix the isotropic Mukai vector

$$v_0 = (r_0, d_0 f_1, d_0^2 s) \in \tilde{\mathrm{H}}(X_1, \mathbb{Z}).$$

Since r_0 and $d_0^2 s$ are relatively prime, there exists a Mukai vector $v' = (m, 0, n)$ such that $\langle v_0, v' \rangle = 1$. Let $X_2 = M_v(X_1)$, also a K3 surface, and choose a universal sheaf $\mathcal{E} \to X_1 \times X_2$. Our goal is to describe the induced isomorphism

$$\phi_{\mathcal{E}} : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \tilde{\to} \tilde{\mathrm{H}}(X_2, \mathbb{Z}).$$

Following [HL10, Ch. 8] and [Yos99, §2], the polarization on X_2 is given by

$$\det(\pi_{2*}(\mathcal{E} \otimes \mathcal{O}_H(s(r_0 - 2d_0))))^\vee, \quad H \in |f_1|,$$

a primitive ample divisor f_2 on X_2. More generally, we have an isomorphism of Hodge structures

$$\mathrm{H}^2(X_2, \mathbb{Z}) = (v_0^\vee)^\perp / \mathbb{Z} v_0^\vee,$$

where the perpendicular subspace is taken with respect to the Mukai pairing.

Proposition 8. [Yos99] *Choose integers d_1 and ℓ such that $sd_0d_1 - r_0\ell = 1$ and take $\mathcal{K} = \mathcal{E} \otimes \pi_2^* L$ for some line bundle L on X_2. With respect to the bases*

$$(1, 0, 0), (0, f_i, 0), (0, 0, 1) \in \tilde{\mathrm{H}}(X_i, \mathbb{Z})$$

the matrix of the cohomological Fourier-Mukai transform takes the form

$$\phi_{\mathcal{K}} := \begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \ell & 2d_0 d_1 s - 1 & d_1 \\ \ell^2 r_0 & 2d_1 s \ell r_0 & d_1^2 s \end{pmatrix}.$$

The inverse is obtained reversing the sign of the middle basis vector and interchanging the role of d_0 and d_1:

$$\begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \ell & 2d_0 d_1 s - 1 & d_1 \\ \ell^2 r_0 & 2d_1 s \ell r_0 & d_1^2 s \end{pmatrix} \begin{pmatrix} d_1^2 s & -2d_1 s r_0 & r_0 \\ -d_1 \ell & 2d_0 d_1 s - 1 & -d_0 \\ \ell^2 r_0 & -2d_0 s \ell r_0 & d_0^2 s \end{pmatrix} = I.$$

The formula

$$\phi_{\mathcal{K}} \psi_{\mathcal{K}^\vee} = I$$

is the cohomological realization of the identity

$$\Phi_{\mathcal{K}} \Psi_{\mathcal{K}^\vee[2]} = I.$$

The third column of $\phi_{\mathcal{K}}^{-1}$ is the Mukai vector $v_0^\vee$, as

$$\Phi_{\mathcal{K}}^{-1}(\mathcal{O}_p) = \mathcal{E}_p^\vee, \quad p = [\mathcal{E}_p] \in X_2 = M_{v_0}(X_1).$$

Example 9. Suppose that (X_1, f_1) is a degree 12 K3 surface. Consider the isotropic Mukai vector $v = (2, f_1, 3)$ so that

$$X_2 := M_v(X_1)$$

is also a K3 surface derived equivalent to X_1. Taking

$$r_0 = 2, s = 3, d_0 = 1, d_1 = \ell = 1$$

we obtain

$$\begin{aligned}(1,0,0) &\mapsto (3, f_2, 2)\\ (0, f_1, 0) &\mapsto (12, 5f_2, 12)\\ (0,0,1) &\mapsto (2, f_2, 3)\end{aligned}$$

with matrix

$$\varphi := \begin{pmatrix} 3 & 12 & 2 \\ 1 & 5 & 1 \\ 2 & 12 & 3 \end{pmatrix}. \tag{4.1}$$

The determinant is 1 with one eigenvector $(1, 0, -1)$ with eigenvalue 1; thus this is orientation preserving. Note that

$$(2, -f_1, 3) \mapsto (0, 0, 1)$$

whence

$$X_1 = M_{(2,f_2,3)}(X_2), \quad X_2 = M_{(2,-f_1,3)}(X_1).$$

The fact that $(1, 0, -1)$ has eigenvalue 1 gives

$$X_1^{[2]} \overset{\sim}{\dashrightarrow} X_2^{[2]}.$$

5. Generalities concerning involutions on K3 surfaces

Let $i : X \to X$ be an involution on a complex projective K3 surface, which acts faithfully on $\mathrm{H}^2(X, \mathbb{Z})$ by the Torelli Theorem. It is *symplectic* (resp. *anti-symplectic*) if

$$i^*\omega = \omega \quad (\text{resp. } -\omega),$$

where ω is a holomorphic two-form. Nikulin [Nik79a] showed that any symplectic involution fixes eight isolated points and that all such involutions are topologically conjugate; these are the *Nikulin involutions* studied in Section 8. An involution without fixed points was classically known to be an *Enriques involution* arising from a double cover $X \to S$ of an Enriques surface.

The case of anti-symplectic involutions with fixed points is more complicated. Nikulin enumerated 74 cases beyond the Enriques case; see [AN06, BH21, AE22, Ale22] for details of the various cases.

Given an anti-symplectic involution $i : X \to X$ on a K3 surface, we recall the Nikulin invariants (r, a, δ) [AE22, §2]: Let r denote the rank of the lattice

$$S = \mathrm{H}^2(X, \mathbb{Z})^{i=1},$$

which is indefinite if $r > 1$. We are using the fact that transcendental classes of X are anti-invariant under i, as the quotient X/i admits no holomorphic two-form. We write

$$T = \mathrm{H}^2(X, \mathbb{Z})^{i=-1} = S^{\perp}$$

for the complementary lattice with signature $(2, 20-r)$, which is indefinite if $r < 20$. The discriminant group $d(S) \simeq d(T)$ is a 2-elementary group; its rank is denoted by a. This group comes with a quadratic form

$$q_S : d(S) \to \mathbb{Q}/2\mathbb{Z}.$$

The *coparity* δ equals 0 if $q_S(x) \in \mathbb{Z}$ for each $x \in d(S)$ and equals 1 otherwise.

We relate this to geometric invariants. For an anti-symplectic involution, there are no isolated fixed points so the fixed locus $R = X^i$ is of pure dimension one or empty. Suppose there are $k + 1$ irreducible components, with genera summing to g. Then we have cf. [AE22, p.5]

$$g = 11 - (r + a)/2 \quad k = (r - a)/2,$$

excluding the Enriques case $(r, k, \delta) = (10, 10, 0)$.

Nikulin classifies even indefinite 2-elementary lattices L. They are determined uniquely by (r, a, δ) and $\mathrm{O(L)} \to \mathrm{Aut}(d(\mathrm{L}))$ is surjective. In the definite case, *a priori* there are multiple classes in each genus but this is not relevant for our applications. Indeed, the possibilities include

- $r = a = 1$: X is a double cover of $\mathbb{P}^2$ branched along a sextic plane curve.
- The case where T is definite $(r = 20, a = 2, g = 0, k = 9)$, we have $d(T) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ thus is equal to

  $$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

  Even in this case, automorphisms of the discriminant group are realized by automorphisms of the lattice.

Theorem 10 (Alexeev-Nikulin)**.** *For each admissible set of invariants (r, a, δ), there is a unique orthogonal pair of lattices (S, T) embedded in the K3 lattice Λ, up to automorphisms of Λ. There are 75 such cases.*

Corollary 11. *Any equivariant derived equivalence of K3 surfaces with anti-symplectic involutions induces an equivariant isomorphism between the underlying K3 surfaces.*

Proof. Suppose that (X_1, i_1) and (X_2, i_2) are derived equivalent, compatibly with their anti-symplectic involutions.

Indeed, derived equivalence shows that the invariant (resp. anti-invariant) sublattices of the Picard group are stably equivalent (resp. equivalent):

$$\mathrm{Pic}(X_1)^{i_1=1} \oplus U \simeq \mathrm{Pic}(X_2)^{i_2=1} \oplus U, \quad \mathrm{Pic}(X_1)^{i_1=-1} \simeq \mathrm{Pic}(X_2)^{i_2=-1}.$$

Since the possibilities for the invariant sublattices are characterized by their 2-adic invariants, we have

$$\mathrm{Pic}(X_1)^{i_1=1} \simeq \mathrm{Pic}(X_2)^{i_2=1}.$$

We have already observed that all the possible isomorphisms between their discriminants

$$\left(d(\mathrm{Pic}(X_1)^{i_1=1}), q_1\right) \simeq \left(d(\mathrm{Pic}(X_2)^{i_2=1}), q_2\right)$$

are realized by isomorphisms of the lattices. In particular, there exists a choice compatible with the isomorphism

$$\mathrm{H}^2(X_1, \mathbb{Z})^{i_1=-1} \xrightarrow{\sim} \mathrm{H}^2(X_2, \mathbb{Z})^{i_2=-1}$$

induced by the derived equivalence. Thus we obtain isomorphisms on middle cohomology, compatible with the involutions. The Torelli Theorem gives an isomorphism $X_1 \xrightarrow{\sim} X_2$ respecting the involutions. $\square$

Corollary 12. *Let (X_1, σ_1) and (X_2, σ_2) denote K3 surfaces with involutions that are C_2-equivariantly derived equivalent. If X_1/σ_1 is rational then X_2/σ_2 is rational as well.*

Indeed, the rationality of the quotient forces the involution to be anti-symplectic.

Example 13. Having an anti-symmetric involution is *not* generally a derived property. For example, consider Picard lattices

$$A_1 = \begin{pmatrix} 2 & 13 \\ 13 & 12 \end{pmatrix} \quad A_2 = \begin{pmatrix} 8 & 15 \\ 15 & 10 \end{pmatrix}.$$

These forms are stably equivalent but not isomorphic. As in Remark 5 – see [HT17, Sec. 2.3] for details – choose derived equivalent K3 surfaces X_1 and X_2 with $\mathrm{Pic}(X_1) = A_1$ and $\mathrm{Pic}(X_2) = A_2$. Note that A_2 does not represent two and admits no involution acting via ± 1 on $d(A_2)$; thus X_2 does not admit an involution.

This should be compared with Proposition 28: Having an Enriques involution is a derived invariant.

6. Orientation reversing conjugation

We continue to assume that i is an anti-symplectic involution on a K3 surfaces X. As we have seen,

$$T(X) \subset \mathrm{H}^2(X, \mathbb{Z})^{i=-1}$$

with complement $\mathrm{Pic}(X)^{i=-1}$, which is negative definite by the Hodge index theorem.

Recall that Orlov's Theorem [Orl97, §3] asserts that for K3 surfaces (without group action) isomorphisms of transcendental cohomology lift to derived equivalences. Given K3 surfaces (X_1, i_1) and (X_2, i_2) with anti-symplectic involutions of the same type in the sense of Alexeev-Nikulin, the existence of an isomorphism

$$T(X_1) \xrightarrow{\sim} T(X_2)$$

seldom induces an equivariant derived equivalence; a notable exception is the case where the anti-invariant Picard group has rank zero or one. We only have that

$$\mathrm{Pic}(X_1)^{i_1=-1}, \quad \mathrm{Pic}(X_2)^{\epsilon_2=-1}$$

are stably equivalent – compatibly with the isomorphism on the discriminant groups of the transcendental lattices – but not necessarily isomorphic.

In light of this, we propose an orientation reversing conjugation of actions, with a view toward realizing isomorphisms of transcendental cohomology.

Assume that $\mathrm{Pic}(X_1)^{i_1=-1}$ and $\mathrm{Pic}(X_2)^{i_2=-1}$ are not isomorphic, so there is no C_2-equivariant derived equivalence

$$D^b(X_1) \xrightarrow{\sim} D^b(X_2)$$

taking i_1 to i_2, by Corollary 11. However, let

$$\mathrm{dual}_k : D^b(X_k) \xrightarrow{\sim} D^b(X_k), \quad k = 1, 2,$$

denote the involution

$$\mathcal{E}_* \mapsto \mathcal{E}_*^\vee.$$

Note that shift and duality commute with each other and with any automorphism of the K3 surface. The action of dual_k on the Mukai lattice $\tilde{\mathrm{H}}(X_k, \mathbb{Z})$ is trivial in degrees 0 and 4 and multiplication by -1 in degree two. Recall that shift acts via -1 in all degrees, so composition with dual_k is trivial in degree 2 and multiplication by -1 in degrees 0 and 4.

We propose a general definition and then explain how it is related to our analysis of quadratic forms with involution:

Definition 14. Let (X_1, i_1) and (X_2, i_2) be smooth projective varieties with involution, of dimension n with trivial canonical class. They are *skew equivalent* if there is a kernel $\mathcal{K}$ on $X_1 \times X_2$, inducing an equivalence between X_1 and X_2, such that

$$(i_1^*, i_2^*)\mathcal{K} = \mathcal{K}^\vee[n]. \tag{6.1}$$

Note that this dualization coincides with the relative dualizing complex for both projections π_1 and π_2.

Suppose again that X_1 and X_2 are K3 surfaces and $\mathcal{K} = \mathcal{E}[1]$ for a universal sheaf

$$\mathcal{E} \to X_1 \times X_2$$

associated with an isomorphism $X_2 = M_v(X_1)$. Then relation (6.1) (with $n = 2$) translates into

$$i_1^* \mathcal{E}_{i_2(x_2)} \simeq (\mathcal{E}_{x_2})^\vee. \tag{6.2}$$

Proposition 15. *Let (X_1, i_1) and (X_2, i_2) be K3 surfaces with involutions. Then the following are equivalent*

- *(X_1, i_1) and (X_2, i_2) are skew derived equivalent;*
- *there exists an orientation-preserving equivalence of Mukai lattices*
$$\phi : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}(X_2, \mathbb{Z}),$$
satisfying

$$\phi(i_1^*(v^\vee)) = (i_2^* \phi(v))^\vee. \tag{6.3}$$

As duality and pull back commute with each other, the order of these operations in (6.3) is immaterial. Furthermore, if ϕ satisfies this relation then so does $-\phi$.

Proof. The forward implication is clear. Indeed, the cohomological Fourier-Mukai transform $\phi_{\mathcal{K}}$ of a skew equivalence satisfies

$$(i_1, i_2)^* \phi_{\mathcal{K}} = \phi_{\mathcal{K}^\vee}$$

but $\phi_{\mathcal{K}^\vee}$ differs from $\phi_{\mathcal{K}}$ by the involution acting via $+1$ on H^0 and H^4 and -1 on H^2. Thus

$$\phi_{\mathcal{K}} : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}(X_2, \mathbb{Z})$$

is an isomorphism equivariant under the prescribed "skew" involutions.

For the reverse implication, we consider the cohomological Fourier-Mukai transform

$$\phi : \tilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}(X_2, \mathbb{Z}).$$

Proposition 7 yields a kernel $\mathcal{K}$ such that $\phi = \phi_{\mathcal{K}}$. We make this more explicit.

Set

$$v = (\phi^{-1}(0, 0, 1))^\vee$$

with view toward relating $\mathcal{K}$ to a universal bundle

$$\mathcal{E} \to X_1 \times X_2,$$

where X_2 is a moduli space of bundles on X_1. Write $v = (r, f_1, s)$; if $r < 0$, replace ϕ by $-\phi$. This leaves (6.3) unchanged and corresponds to replacing $\Phi_{\mathcal{K}}$ by $\Phi_{\mathcal{K}}$ pre-composed (or post-composed) by a shift on X_1 (or X_2).

Thus we may assume $v_0 = (r_0, f_1, s)$ with $r_0 > 0$ and consider a universal bundle

$$\mathcal{E} \to X_1 \times M_{v_0}(X_1) \simeq X_1 \times X_2.$$

We therefore have (see Proposition 8 for the formula and our basis conventions)

$$\phi_{\mathcal{E}} = \begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \ell & 2d_0 d_1 s - 1 & d_1 \\ \ell^2 r_0 & 2d_1 s \ell r_0 & d_1^2 s \end{pmatrix}$$

and

$$\phi_{\mathcal{K}} = \begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \hat{\ell} & 2d_0 \hat{d}_1 s - 1 & \hat{d}_1 \\ \hat{\ell}^2 r_0 & 2\hat{d}_1 s \hat{\ell} r_0 & \hat{d}_1^2 s \end{pmatrix}.$$

Here we have

$$s d_0 d_1 - r_0 \ell = s d_0 \hat{d}_1 - r_0 \hat{\ell} = 1,$$

whence

$$\hat{\ell} = \ell + N s d_0 \quad \hat{d}_1 = d_1 + N r_0.$$

Consider the autoequivalence on X_2 obtained by tensoring with the invertible sheaf $\mathcal{O}_{X_2}(Nf_2)$. This has cohomological Fourier-Mukai transform with matrix

$$t^N := \begin{pmatrix} 1 & 0 & 0 \\ N & 1 & 0 \\ N^2 r_0 s & 2Nr_0 s & 1 \end{pmatrix}.$$

Note however that

$$t^N \phi_{\mathcal{E}} = \phi_{\mathcal{K}}$$

and we can renormalize $\mathcal{E}$ so that it has cohomological Fourier-Mukai transform ϕ. Specifically, there exists an isomorphism

$$X_2 \xrightarrow{\sim} M_{v_0}(X_1)$$

such that the pullback of the universal sheaf $\mathcal{E}$ to $X_1 \times X_2$ induces ϕ. We analyze how i_1 and i_2 act on $\mathcal{E}$, keeping in mind the functional relation. We have that

$$\mathcal{E}, \quad (i_1, i_2)^* \mathcal{E}^\vee$$

are both universal bundles on $X_1 \times X_2$ with the same numerical invariants. The uniqueness of such bundles gives an isomorphism

$$\xi : \mathcal{E} \xrightarrow{\sim} (i_1, i_2)^* \mathcal{E}^\vee$$

over $X_1 \times X_2$, unique up to a scalar. Note there are two distinguished normalizations of this scalar, for which the composition

$$(i_1, i_2)^* \xi^\vee \circ \xi : \mathcal{E} \to \mathcal{E}$$

is the identity. For purposes of establishing derived equivalences, the choice of normalization is immaterial. □

Corollary 16. *Under the assumptions above, the functors* $\mathrm{dual}_1 \circ i_1$ *and* $\mathrm{dual}_2 \circ i_2$ *are* C_2*-equivariantly derived equivalent.*

Remark 17. As we recalled in Section 3, derived equivalences respect orientations on the Mukai lattice [HMS09]. The orientation reversing conjugation violates the orientation condition, in a prescribed way. Duality is the archetypal orientation-reversing Hodge isogeny.

In Sections 7 and 10 we give examples of such equivalences.

7. Rational quotients and skew equivalence

Our first task is to give examples of skew equivalences. Proposition 15 and the discussion preceding it reduce this to exhibiting lattice-polarized K3 surfaces with involution (X_1, i_1) and (X_2, i_2), such that the anti-invariant Picard groups are stably equivalent but inequivalent.

Specifically, we assume X_1 and X_2 are degree two K3 surfaces with

$$\mathrm{Pic}(X_j) = \mathbb{Z}h_j \oplus A_j(-1), \quad h_j^2 = 2,$$

where the involutions fix the h_j and reverse signs on A_j's. If A_1 and A_2 are stably-equivalent, inequivalent positive definite lattices then (X_1, i_1) and (X_2, i_2) are skew equivalent.

In contrast to ordinary equivalences (see 11) we do have antisymplectic actions with nontrivial *skew* equivalences. The resulting quotients are rational surfaces, indeed, $\mathbb{P}^2$.

Example 18 (Explicit matrices)**.** The matrices, in the basis p_j, q_j, for $j = 1, 2$, are given by

$$A_1 := \begin{pmatrix} 4 & 1 \\ 1 & 12 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 6 & 1 \\ 1 & 8 \end{pmatrix}.$$

We extract a stable isomorphism

$$A_1 \oplus \mathrm{U} \simeq A_2 \oplus \mathrm{U}, \quad \mathrm{U} = \langle u_1, v_1 \rangle, \text{ with matrix } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

First, we give an isomorphism

$$A_1 \oplus \langle e_1 \rangle \simeq A_2 \oplus \langle e_2 \rangle, \quad e_1^2 = -2.$$

We put

$$p_1 \mapsto p_2 + e_2,$$

and claim that the orthogonal complements to these are equivalent indefinite lattices. Indeed,

$$p_1^\perp = \langle p_1 - 4q_1, e_1 \rangle = \begin{pmatrix} 188 & 0 \\ 0 & -2 \end{pmatrix},$$

$$\begin{aligned} (p_2 + e_2)^\perp &= \langle p_2 - 6q_2, 2q_2 + e_2 \rangle = \begin{pmatrix} 282 & -94 \\ -94 & 30 \end{pmatrix} \\ &= \langle p_2 + 3e_2, 2q_2 + e_2 \rangle = \begin{pmatrix} -12 & -4 \\ -4 & 30 \end{pmatrix} \end{aligned}$$

These are equivalent via Gaussian cycles of reduced forms

$$\begin{array}{ccccccccc} & 0 & & 18 & & 8 & & 4 & \\ 188 & & -2 & & 26 & & -12 & & 30 \end{array}$$

where the indicated basis elements are

$$p_1 - 4q_1, \quad e_1, \quad p_1 - 4q_1 - 9e_1, \quad p_1 - 4q_1 - 10e_1, \quad 2(p_1 - 4q_1) - 19e_1.$$

The composed isomorphism is

$$\begin{aligned} p_1 - 4q_1 - 10e_1 &\mapsto p_2 + 3e_2, \\ 2(p_1 - 4q_1) - 19e_1 &\mapsto 2q_2 + e_2 \\ p_1 &\mapsto p_2 + e_2 \\ e_1 &\mapsto (2q_2 + e_2) - 2(p_2 + 3e_2) = 2(q_2 - p_2) - 5e_2 \\ q_1 &\mapsto 5(p_2 - q_2) + 12e_2. \end{aligned}$$

We extend the isomorphism above where $e_i = u_i - v_i$

$$\begin{aligned} u_1 + v_1 &\mapsto u_2 + v_2 \\ u_1 - v_1 &\mapsto 2(q_2 - p_2) - 5(u_2 - v_2) \\ p_1 &\mapsto p_2 + (u_2 - v_2) \\ q_1 &\mapsto 5(p_2 - q_2) + 12(u_2 - v_2) \end{aligned}$$

whence we have

$$\begin{aligned} u_1 &\mapsto (q_2 - p_2) - 2u_2 + 3v_2 \\ v_1 &\mapsto (p_2 - q_2) + 3u_2 - 2v_2. \end{aligned}$$

8. Nikulin involutions

General properties. An involution ι on a K3 surface X over $\mathbb{C}$ preserving the symplectic form is called a *Nikulin* involution. We recall basic facts concerning such involutions, following [vGS07]:

- ι has 8 isolated fixed points;
- the (resolution of singularities) $Y \to X/\iota$ is a K3 surface fitting into a diagram

$$\begin{array}{ccc} X & \stackrel{\beta}{\leftarrow} & \widetilde{X} \\ \downarrow & & \downarrow \pi \\ X/\iota & \leftarrow & Y \end{array}$$

  where β blows up the fixed points and the vertical arrows have degree two;

- the action of ι on $\mathrm{H}^2(X,\mathbb{Z})$ is uniquely determined, and there is a decomposition
$$\mathrm{H}^2(X,\mathbb{Z}) = (\mathrm{U}^{\oplus 3})_1 \oplus (\mathrm{E}_8(-1)\oplus \mathrm{E}_8(-1))_P,$$
where the first term is invariant and the second is a permutation module for ι;
- the invariant and the anti-invariant parts of H^2 take the form:
$$\mathrm{H}^2(X,\mathbb{Z})^{\iota=1} \simeq \mathrm{U}^3 \oplus \mathrm{E}_8(-2), \quad \mathrm{H}^2(X,\mathbb{Z})^{\iota=-1} = \mathrm{E}_8(-2)$$

Let $E_1,\ldots,E_8$ denote the exceptional divisors of β and $N_1,\ldots,N_8$ the corresponding (-2)-curves on Y. The union $\cup N_i$ is the branch locus of π so there is a divisor
$$\hat{N} = (N_1+\cdots+N_8)/2$$
saturating $\langle N_1,\ldots,N_8\rangle \subset \mathrm{Pic}(Y)$; the minimal primitive sublattice containing these divisors is called the *Nikulin* lattice, and is denoted by N. We have [vGS07, Prop. 1.8]
$$\begin{array}{rcl} \pi_* : \mathrm{H}^2(\widetilde{X},\mathbb{Z}) & \to & \mathrm{H}^2(Y,\mathbb{Z}) \\ \mathrm{U}^3\oplus \mathrm{E}_8(-1)\oplus \mathrm{E}_8(-1)\oplus \langle -1\rangle^8 & \to & \mathrm{U}(2)^3 \oplus \mathrm{N}\oplus \mathrm{E}_8(-1) \\ (u,x,y,z) & \mapsto & (u,z,x+y) \end{array}$$
and
$$\begin{array}{rcl} \pi^* : \mathrm{H}^2(Y,\mathbb{Z}) & \to & \mathrm{H}^2(\widetilde{X},\mathbb{Z}) \\ \mathrm{U}(2)^3 \oplus \mathrm{N}\oplus \mathrm{E}_8(-1) & \to & \mathrm{U}^3\oplus \mathrm{E}_8(-1)\oplus \mathrm{E}_8(-1)\oplus \langle -1\rangle^8 \\ (u,n,x) & \mapsto & (2u,x,x,2\tilde{n}) \end{array}$$
where if $n=\sum n_i N_i$ then $\tilde{n} = \sum n_i E_i$. Thus we obtain a distinguished saturated sublattice
$$\mathrm{E}_8(-2) \subset \mathrm{Pic}(X)$$
that coincides with the $\iota=-1$ piece.

Proposition 19. *Fix a lattice* L *containing* $\mathrm{E}_8(-2)$ *as a primitive sublattice; assume* L *arises as the Picard lattice of a projective K3 surface. Then there exists a K3 surface X with Nikulin involution ι such that*
$$\mathrm{L} = \mathrm{Pic}(X) \supset \mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

Proof. Let A denote the orthogonal complement of $\mathrm{E}_8(-2)$ is L. There is a unique involution ι on L with
$$\mathrm{L}^{\iota=1} = \mathrm{A}, \quad \mathrm{L}^{\iota=-1} = \mathrm{E}_8(-2).$$

Now ι acts trivially on $d(\mathrm{L})$ – keep in mind $d(\mathrm{E}_8(-2))$ is a two-elementary group – so we may naturally extend ι to the full K3 lattice. (It acts trivially on $\mathrm{L}^{\perp}$.) These lattice-polarized K3 surfaces form our family

Nikulin [Nik79a, §4] explains how to get involutions for generic K3 surfaces with lattice polarization L. Choose such a surface X such that $\mathrm{Pic}(X) = \mathrm{L}$ – a very general member of the family has this property. Clearly X is projective – it admits divisors with positive self-intersection. We claim there is an ample divisor $H \in \mathrm{A}$. Indeed, the ample cone of X is characterized as the chamber of the cone of positive divisors by the group generated by reflections associated with indecomposable (-2)-classes E of positive degree [LP81]. Each (-2)-class E is perpendicular to a unique ray in

$$\mathrm{A} \otimes \mathbb{R} \cap \{ \text{ cone of positive divisors } \}$$

generated by an element $a_E \in \mathrm{A}$. Note that A cannot be contained in $E^{\perp}$ as $\mathrm{E}_8(-2)$ has no (-2)-classes. We conclude that A meets each chamber in the decomposition of the positive cone – it cannot be separated from the ample cone by any of the $\mathrm{E}^{\perp}$.

Once we have the ample cone, we can extract the automorphism group of X via the Torelli Theorem: It consists of the Hodge isometries taking the ample cone to itself. In particular, any Hodge isometry fixing H is an automorphism. Thus ι is an automorphism of X. □

Proposition 20. *Let* L *be an even hyperbolic lattice containing* $\mathrm{E}_8(-2)$ *as a saturated sublattice. Assume that* $d(\mathrm{L})$ *has rank at most* 11*. Then* L *is unique in its genus and the homomorphism*

$$\mathrm{O}(\mathrm{L}) \to \mathrm{O}(q_{\mathrm{L}})$$

is surjective.

The condition on the rank of $d(L)$ is satisfied for Picard lattices of K3 surfaces X. We have

$$\mathrm{Pic}(X) \subset \mathrm{U}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$$

which has rank 22; $d(\mathrm{Pic}(X)) \simeq d(T(X))$ so both groups are generated by ≤ 11 elements.

Proof. We apply Proposition 2. For odd primes p, the conditions are easily checked as the rank r of L exceeds the rank of the p-primary part $d(\mathrm{L})$. If $r \geq 12$ then the discriminant group is generated by ≤ 10 elements and we are done. Thus we focus on the $p = 2$ case with $r = 9, 10$, or 11.

Let A denote the orthogonal complement to $\mathrm{E}_8(-2)$ in L. The overlattice

$$\mathrm{L} \supset \mathrm{A} \oplus \mathrm{E}_8(-2)$$

corresponds to an isotropic subgroup

$$H \subset d(\mathrm{A}) \oplus d(\mathrm{L})$$

with respect to $q_{\mathrm{A}} \oplus q_{\mathrm{L}}$. Projection maps H injectively into each summand – we may interpret these projections as kernels of the natural maps

$$d(\mathrm{A}) \to d(\mathrm{L}), \quad d(\mathrm{E}_8(-2)) \to d(\mathrm{L}).$$

Thus H is a 2-elementary group, of rank at most three. It follows that $d(\mathrm{L})$ contains at least five copies of $\mathbb{Z}/2\mathbb{Z}$. Remark 3 shows this validates the hypothesis of Proposition 2. $\square$

The assumption on the *rank* of the discriminant groups can be replaced by bounds on its *order* [CS99, Cor. 22, p. 395] – at least for purposes of showing there is one class in each genus.

Rank nine examples. We focus on examples with Picard rank nine, following [vGS07, Prop. 2.2] which lists the possible lattices. Suppose that $\mathrm{Pic}(X)^{\iota=1} = \mathbb{Z}f$ with $f^2 = 2d$, which is necessarily ample as there are no (-2)-classes in

$$\mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

We have the lattice

$$\Lambda := (2d) \oplus \mathrm{E}_8(-2),$$

for all d. For even d we have the index-two overlattice $\widetilde{\Lambda} \supset \Lambda$, generated by

$$\frac{f+e}{2},$$

where f is a generator of $(2d)$ and $e \in \mathrm{E}_8(-2)$ is a primitive element with

$$(e,e) = \begin{cases} -4 & \text{if } d = 4m+2 \\ -8 & \text{if } d = 4m. \end{cases}$$

We are using the fact that the lattice E_8 has primitive vectors of lengths 2 and 4. Using the shorthand

$$q(v) = q_{\mathrm{E}_8(-2)}(v) \pmod{2\mathbb{Z}}$$

elements $0 \neq v \in e_8(-2) := d(\mathrm{E}_8(-2))$ are of two types

- 120 elements v with $q(v) = 1$ ($A_1 + E_7$ type);
- 135 elements v with $q(v) = 0$ (D_8 type).

Note that $\widetilde{\Lambda}$ is the unique such overlattice such that $\mathrm{E}_8(-2)$ remains saturated.

Proposition 21. *Let (X_1, f_1) and (X_2, f_2) be polarized K3 surfaces of degree $2d$, derived equivalent via specialization of the construction in Remark 5. If X_1 admits a Nikulin involution fixing f_1 then*

- *X_2 admits a Nikulin involution fixing f_2;*
- *there is an isomorphism*

$$\varphi : X_1 \xrightarrow{\sim} X_2.$$

Proof. The derived equivalence induces an isomorphism of lattices with Hodge structure

$$\mathrm{H}^2(X_1, \mathbb{Z}) \supset f_1^{\perp} \simeq f_2^{\perp} \subset \mathrm{H}^2(X_2, \mathbb{Z}),$$

which means that $f_2^{\perp} \cap \mathrm{Pic}(X_2)$ contains a sublattice isomorphic to $E_8(-2)$. Thus there exists a Hodge involution

$$\iota_2^* : \mathrm{H}^2(X_2, \mathbb{Z}) \to \mathrm{H}^2(X_2, \mathbb{Z})$$

with anti-invariant summand equal to this copy of $\mathrm{E}_8(-2)$. The Torelli Theorem – see [vGS07, Prop. 2.3] – shows that X_2 admits an involution $\iota_2 : X_2 \to X_2$.

Isomorphisms of K3 surfaces specialize in families [MM64, ch. I]. This reduces us to proving the result when the X_k have Picard rank nine, putting us in the case of Proposition 20. The Counting Formula of [HLOY04, §2] – using the conclusions of Proposition 20 – implies that all Fourier-Mukai partners of X_1 are isomorphic to X_1. □

Remark 22. We are *not* asserting that $\varphi^* f_2 = f_1$! Suppose that X_1 and X_2 have Picard rank nine, the minimal possible rank. Then

$$\varphi^* f_2 \equiv \alpha f_1 \pmod{\mathrm{E}_8(-2)}$$

where $\alpha \pmod{4d}$ is the corresponding solution to congruence (3.1).

Thus we obtain nontrivial derived equivalence among Nikulin surfaces even in rank nine!

Rank ten examples. Turning to rank ten, we offer a generalization of [vGS07, Prop. 2.3]:

Proposition 23. *Fix a rank two indefinite even lattice* A *and an even extension*

$$\mathrm{L} \supset \mathrm{A} \oplus \mathrm{E}_8(-2)$$

invariant under ι; here ι fixes A *and acts by multiplication by* -1 *on* $\mathrm{E}_8(-2)$. *Then there exists a K3 surface X with Nikulin involution ι such that*

$$\mathrm{A} = \mathrm{Pic}(X)^{\iota=1} \subset \mathrm{Pic}(X) = \mathrm{L} \supset \mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

Proof. The lattice L embeds uniquely into the K3 lattice by Proposition 4. Proposition 19 gives the desired K3 surface with involution. □

We observed in Proposition 20 that the lattice L are unique in their genus and admit automorphisms realizing the full group $\mathrm{O}(d(\mathrm{L}))$. Repeating the reasoning for Proposition 21 we find:

Proposition 24. *A K3 surface X with involution ι_1, produced following Proposition 23 applied to* A_1, *will have a second involution ι_2 associated with* A_2. *Moreover (X, ι_1) and (X, ι_2) are not equivariantly derived equivalent.*

We elaborate on the overlattices L arising in the assumptions of Proposition 23. What lattices may arise from a given A? Each L arises from a 2-elementary

$$H \subset d(\mathrm{A}) \oplus e_8(-2)$$

isotropic with respect to $q_\mathrm{A} \oplus q_{\mathrm{E}_8(-2)}$.

We consider the orbits of

$$H \simeq (\mathbb{Z}/2\mathbb{Z})^2 \subset e_8(-2)$$

under automorphisms of the lattice. These reflect possible quadratic forms on $(\mathbb{Z}/2\mathbb{Z})^2$. We enumerate the possibilities, relying on description of maximal subgroups of the simple group of $\mathrm{O}_8^+(2)$ (automorphisms of $e_8(-2)$) [CCN+85, p. 85] and subgroups of $W(\mathrm{E}_8)$ (a closely related group) associated with reflections [DPR13, Table 5]. For the reader's reference, we list the root systems associated with the subgroups in parentheses:

(1) isotropic subspaces, where $q|H$ is trivial – 1575 elements ($\mathrm{D}_4 + \mathrm{D}_4$ type);
(2) rank one subspaces, where $q|H$ has a kernel, e.g., $q(x, y) = x^2$ – $3780 = 28 \times 135$ elements ($\mathrm{A}_1 + \mathrm{A}_1 + \mathrm{D}_6$ type);
(3) "minus lines" full rank non-split subspaces, e.g., $q(x, y) = x^2 + xy + y^2$ – $1120 = 28 \cdot 120/3$ elements ($\mathrm{A}_2 + \mathrm{E}_6$ type);
(4) full rank split subspaces, e.g., $q(x, y) = xy$ – 4320 elements.

As a check, the Grassmannian $\mathrm{Gr}(2, 8)$ has Betti numbers

$$1 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 4 \quad 3 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1$$

and thus, by the Weil conjectures, 10795 points of $\mathbb{F}_2$. Note that

$$10795 = 1575 + 3780 + 1120 + 4320.$$

What about arbitrary rank? Let A_1 and A_2 be indefinite lattices of rank $r \geq 2$ in the same genus. Consider overlattices

$$\mathrm{L}_1 \supset \mathrm{A}_1 \oplus \mathrm{E}_8(-2), \quad \mathrm{L}_2 \supset \mathrm{A}_1 \oplus \mathrm{E}_8(-2)$$

associated with subspaces $H \subset e_8(-2)$ in the same orbit, so we have $d(\mathrm{L}_1) \simeq d(\mathrm{L}_2)$. It follows that $\mathrm{L}_1 \simeq \mathrm{L}_2$ provided the $d(\mathrm{L}_i)$ have rank at most 11 (see Proposition 20); this holds for Picard lattices of $K3$ surfaces. Assuming L_1 and L_2 arise as Picard lattices of K3 surfaces, we obtain results as in Propositions 21 and 24.

We conclude with one last observation:

Proposition 25. *The existence of a Nikulin structure for one member of a derived equivalence class induces Nikulin structures on all K3 surfaces in the equivalence class.*

Suppose X_1 and X_2 are derived equivalent and X_1 admits a Nikulin involution. Proposition 20 implies

$$\mathrm{Pic}(X_1) \simeq \mathrm{Pic}(X_2)$$

and we obtain a copy of $\mathrm{E}_8(-2) \subset \mathrm{Pic}(X_2)$. Proposition 19 guarantees X_2 admits a Nikulin involution as well.

9. Geometric application

In this section, we present a geometric application of the study of Nikulin involutions, up to derived equivalence.

Let (X_1, f_1) and (X_2, f_2) denote derived equivalent K3 surfaces of degree 12, admitting Nikulin involutions $\iota_j : X_j \to X_j$ with $\iota_j^* f_j = f_j$ for $j = 1, 2$. We assume Picard groups

$$\mathrm{Pic}(X_j) = \mathbb{Z}f_j \oplus \mathrm{E}_8(-2).$$

Note that the derived equivalence induces natural identifications between the $\mathrm{E}_8(-2)$ summands of $\mathrm{Pic}(X_1)$ and $\mathrm{Pic}(X_2)$. In particular, we obtain bijections between the fixed-point loci

$$X_1^{\iota_1} = X_2^{\iota_2}.$$

Let $Z_j \subset X_j$ denote triples of fixed points compatible with these bijections. Assuming the X_j are generic, i.e. defined by quadratic equations in $\mathbb{P}^7$, the fixed points are not collinear.

Projection from the Z_j gives surfaces

$$\mathrm{Bl}_{Z_j}(X_j) \to Y_j \subset \mathbb{P}^4$$

where the blowup normalizes the image of the projection. These constructions are compatible with the involutions on each side.

We claim that the construction of [HL18] gives a Cremona transform

$$\phi : \mathbb{P}^4 \overset{\sim}{\dashrightarrow} \mathbb{P}^4$$

such that

- the indeterminacy of ϕ is Y_1;
- the indeterminacy of ϕ^{-1} is Y_2;
- ϕ is compatible with the involutions ι_1 and ι_2 induced in the $\mathbb{P}^4$'s.

Indeed, the construction induces an isogeny of $\mathrm{H}^2(X_1, \mathbb{Z})$ and $\mathrm{H}^2(X_2, \mathbb{Z})$ induced by ϕ, restricting to an isomorphism of the primitive cohomology

$$f_1^\perp \overset{\sim}{\to} f_2^\perp.$$

The construction entails designating projection loci $Z'_j \in X_j^{[3]}$ compatible with the associated

$$X_1^{[3]} \overset{\sim}{\dashrightarrow} X_2^{[3]},$$

our stipulation that the Z_j consist of suitable fixed points gives compatible projection loci.

Suppose that $\phi : \mathbb{P}^n \overset{\sim}{\dashrightarrow} \mathbb{P}^n$ is birational and equivariant for the action of a finite group G. In this case, [KT22, Thm. 1] introduces a well-defined invariant

(9.1)

$$C_G(\phi) := \sum_{\substack{E \in \mathrm{Ex}_G(\phi^{-1}) \\ \mathrm{gen.stab}(E)=\{1\}}} [E \circlearrowleft G] - \sum_{\substack{D \in \mathrm{Ex}_G(\phi) \\ \mathrm{gen.stab}(D)=\{1\}}} [D \circlearrowleft G] \in \mathbb{Z}[\mathrm{Bir}_{G,n-1}],$$

taking values in the free abelian group on G-birational isomorphism classes of algebraic varieties of dimension $n-1$. In this case, the terms are the projectivized normal bundles of Y_1 and Y_2, taken with opposite signs. It is worth mentioning that the underlying K3 surfaces X_1 and X_2 are isomorphic by Proposition 21, and the group actions are conjugate under derived equivalences but not under automorphisms. The difference of classes of exceptional loci in (9.1) is *nonzero* due to Proposition 26 below. This gives an instance where the refinement of the invariant $c(\phi)$ in [LSZ20], [LS22] using group actions yields new information.

Proposition 26 (cf. Thm. 2, [LS10])**.** *Let X_1 and X_2 be smooth projective G-varieties that are not uniruled. Then any G-equivariant stable birational equivalence*

$$X_1 \times \mathbb{P}^r \stackrel{\sim}{\dashrightarrow} X_2 \times \mathbb{P}^s,$$

with trivial G-action on the second factors, arises from a G-equivariant birational equivalence

$$X_1 \stackrel{\sim}{\dashrightarrow} X_2.$$

Proof. Our assumption – that X_1 and X_2 are not uniruled – means that

$$X_1 \times \mathbb{P}^r \to X_1, \quad X_2 \times \mathbb{P}^s \to X_2$$

are maximal rationally-connected (MRC) fibrations. Since $X_1 \times \mathbb{P}^r \stackrel{\sim}{\dashrightarrow} X_2 \times \mathbb{P}^s$, the functoriality of MRC fibrations [Kol96, IV.5.5] gives a natural birational map

$$X_1 \stackrel{\sim}{\dashrightarrow} X_2.$$

When the varieties admit G-actions, the induced birational map is compatible with these actions. □

10. Enriques involutions

Let S be an Enriques surface over $\mathbb{C}$. Its universal cover is a K3 surface X with covering involution $\epsilon : X \to X$, a fixed-point-free automorphism of order two, called an *Enriques involution*.

The classification of Enriques surfaces S up to derived equivalence boils down to the classification of pairs (X, ϵ) up to C_2-equivariant derived equivalence [BM01, §6] (and [BM17] more generally). Derived equivalent Enriques surfaces are isomorphic [BM01, Prop. 6.1].

A number of authors have classified Enriques involutions on a given K3 surface X, modulo its automorphisms $\mathrm{Aut}(X)$:

- Kondo gave the first examples with finite $\mathrm{Aut}(S)$ [Kon92].
- Ohashi showed that there finitely many $\mathrm{Aut}(X)$-orbits of such involutions. In the Kummer case, the possible quotients are classified by nontrivial elements of the discriminant group of the Néron-Severi group $\mathrm{NS}(X)$. There are 15 on general Kummer surfaces of product type, 31 in a general Jacobian Kummer surface, but the number is generally unbounded [Oha07], [Oha09].
- Shimada and Veniani consider *singular* (i.e. rank 20) K3 surfaces; one of their results is a parametrization of $\mathrm{Aut}(X)$-orbits on the set of Enriques involutions; the number of such orbits

depends only on the genus of the transcendental lattice $T(X)$ [SV20, Thm. 3.19].

These results are based on lattice theory: two Enriques involutions on a K3 surface X are conjugate via $\mathrm{Aut}(X)$ if an only if the corresponding Enriques quotients are isomorphic [Oha07, Prop. 2.1].

Let

$$\mathrm{M} := \mathrm{U} \oplus \mathrm{E}_8(-1)$$

be the unique even unimodular hyperbolic lattice of rank 10; we have

$$\mathrm{Pic}(S)/\mathrm{torsion} \simeq \mathrm{M}$$

and

$$\mathrm{Pic}(X) \supseteq \mathrm{M}(2)$$

as a primitive sublattice. This coincides with the invariant sublattice

$$\mathrm{Pic}(X)^{\epsilon=1} \subset \mathrm{Pic}(X)$$

under the involution ϵ. Let N denote the orthogonal complement to M in $\mathrm{H}^2(X, \mathbb{Z})$, which coincides with $\mathrm{H}^2(X, \mathbb{Z})^{\epsilon=-1}$; note that $T(X) \subset \mathrm{N}$. We have

$$\mathrm{N} \simeq \mathrm{U} \oplus \mathrm{U}(2) \oplus \mathrm{E}_8(-2)$$

which has signature $(2, 10)$. Thus

$$\mathrm{Pic}(X)^{\epsilon=-1} = T(X)^{\perp} \subset \mathrm{N}$$

has negative definite intersection form. The following result gives a criterion for the existence of Enriques involutions [Keu16, Thm. 1], [Oha07, Thm. 2.2], [SV20, Thm. 3.1.1]:

Proposition 27. *Let X be a K3 surface. Enriques involutions on X correspond to the following data: Primitive embeddings*

$$T(X) \subset \mathrm{N} \subset \mathrm{H}^2(X, \mathbb{Z})$$

such that the orthogonal complement to $T(X)$ in N *does not contain* (-2)*-classes.*

In particular, let X be a K3 surface with an Enriques involution. Then:

- $\mathrm{rk}\,\mathrm{Pic}(X) \geq 10$,
- if $\mathrm{rk}\,\mathrm{Pic}(X) = 10$ then there is a unique such involution,
- if $\mathrm{rk}\,\mathrm{Pic}(X) = 11$ then $\mathrm{Pic}(X)$ is isomorphic to [Oha07, Prop. 3.5]
  - $\mathrm{U}(2) \oplus \mathrm{E}_8 \oplus \langle -2n \rangle$, $n \geq 2$, or
  - $\mathrm{U} \oplus \mathrm{E}_8(2) \oplus \langle -4n \rangle$, $n \geq 1$.

Proposition 28. *Let X and Y be derived equivalent K3 surfaces. Assume that X admits an Enriques involution. Then X is isomorphic to Y. In particular, the existence of an Enriques involution is a derived invariant.*

Proof. In Picard rank ≥ 12, derived equivalence implies isomorphism. If X and Y and derived equivalent of rank 10 and X admits an Enriques involution, then $T(X) \simeq T(Y)$ and $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ are *stably isomorphic*. In Picard ranks 10 and 11, it suffices to show that the lattice $\mathrm{M}(2)$ is unique in its genus and all automorphisms of the discriminant group $(d(\mathrm{M}(2)), q_{\mathrm{M}(2)}))$ lift to automorphisms of $\mathrm{M}(2)$. This is implied by [Nik79b, Thm. 1.14.2]. Indeed, [SV20, Lem. 3.1.7] shows that $\mathrm{Pic}(X)$ satisfies these two conditions whenever X admits an Enriques involution. □

Corollary 11 implies (cf. [BM01, §6]):

Proposition 29. *Any C_2-equivariant derived autoequivalence*

$$(X, \epsilon_1) \sim (X, \epsilon_2)$$

arises from an automorphism of X.

We observe a corollary of Proposition 2: Let (X_1, ϵ_1) and (X_2, ϵ_2) denote K3 surfaces with Enriques involutions. They are orientation reversing (i.e. skew) conjugate if

- $\tau : T(X_1) \xrightarrow{\sim} T(X_2)$ as lattices, with compatible Hodge structures;
- $\mathrm{Pic}(X_1)^{\epsilon_1=-1}$ and $\mathrm{Pic}(X_2)^{\epsilon_2=-1}$ have the same discriminant quadratic form.

We explore this in more detail in the case of singular (rank 20) K3 surfaces. The existence of involutions on singular K3 surfaces is governed by:

Proposition 30. [Ser05] *Let X be a singular K3 surface with transcendental lattice $T(X)$ of discriminant d. There is no Enriques involution on X if and only if $d \equiv 3 \pmod 8$ or*

$$T(X) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \ \textit{or} \ \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$$

The "most algebraic example", i.e. the smallest discriminant admitting an Enriques involution, has

$$T(X) \simeq \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

In this situation there are two possibilities. We write the maximal sublattices

$$\mathrm{N} \subset \mathrm{Pic}(X)$$

such that the involution ϵ acts via -1.

We follow the notation [SV20, Table 3.1]. We consider lattices

$$N_{10,7}^{144}(2), \quad N_{10,7}^{242}(2)$$

where

$$N_{10,7}^{242}(-1) \simeq \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \oplus \mathrm{E}_8$$

with E_8 positive definite and

$$N_{10,7}^{144}(2)(-1) \simeq \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

According to `magma`, these two lattices are inequivalent but are in the same spinor genus thus are stably equivalent.

These involutions are not derived equivalent. Indeed, passing to Mukai lattices adds a hyperbolic summand U on which the involution acts trivially. However, in the case at hand we are stabilizing the (-1)-eigenspace. Thus these involutions are "skew equivalent" in the sense of Section 6.

11. Postscript on involutions in higher dimensions

There are many papers addressing the structure of involutions of higher-dimensional irreducible holomorphic symplectic varieties.

- Symplectic involutions of varieties of $K3^{[n]}$-type and their fixed loci are classified in [KMO22].

- For varieties of Kummer type – arising from an abelian surface A – involutions associated with ± 1 on A are analyzed in [HT13, Th. 4.4] and [KMO22, Th. 1.3].
- Anti-symplectic involutions on varieties of $K3^{[n]}$-type of degree two are studied in [FMOS22].
- Higher-dimensional analogs of Enriques involutions are studied in [OS11].
- Involutions on cubic fourfolds – both symplectic (see [LZ22] and [HT10]) and anti-symplectic – are studied in [Mar22]. The corresponding actions on lattices are described explicitly.
- Involutions on O'Grady type examples are considered in [MM22].

It is natural to consider whether derived equivalences of involutions on K3 surfaces X_1 and X_2 may be understood via equivalences of the induced involutions on punctual Hilbert schemes and other moduli spaces.

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