INVOLUTIONS ON K3 SURFACES AND DERIVED EQUIVALENCE

BRENDAN HASSETT AND YURI TSCHINKEL

Abstract. We study involutions on K3 surfaces under conjugation by derived equivalence and more general relations, together with applications to equivariant birational geometry.

1. Introduction

The structure of $\operatorname{Aut} D^b(X)$, the group of autoequivalences of the bounded derived category $D^b(X)$ of a K3 surface X, is very rich but well-understood only when the Picard group $\operatorname{Pic}(X)$ has rank one [BB17]. The automorphism group $\operatorname{Aut}(X)$ of X lifts to $\operatorname{Aut} D^b(X)$, and one may consider the problem of classification of finite subgroups $G \subset \operatorname{Aut}(X)$ up to conjugation – either by automorphisms, derived equivalence, or even larger groups. This problem is already interesting for cyclic G, and even for involutions, e.g., Enriques or Nikulin involutions. There is an extensive literature classifying these involutions on a given K3 surface X: topological types, moduli spaces of polarized K3 surfaces with involution, and the involutions on a single X up to automorphisms, see, e.g., [AN06], [vGS07], [Oha07], [SV20], [Zha98].

Here we investigate involutions up to derived equivalence, i.e., derived equivalences respecting involutions. Our interest in "derived" phenomena was sparked by a result in [Sos10]—there exist complex conjugate, derived equivalent nonisomorphic K3 surfaces—as well as our investigations of arithmetic problems on K3 surfaces [HT17], [HT23].

One large class of involutions $\sigma : X \to X$ are those whose quotient $Q = X/\sigma$ is rational. Examples include Q a del Pezzo surface and $X \to Q$ a double cover branched along a smooth curve $B \in |-2K_Q|$. We may allow Q to have ADE surface singularities away from B, or B to have ADE curve singularities; then we take X as the minimal resolution of the resulting double cover of Q. These were studied by Alexeev and Nikulin in connection with classification questions concerning singular

Date: August 1, 2024.

del Pezzo surfaces [AN06]. Our principal result here (see Section 4) is that

- equivariant derived equivalences of such (X, σ) are in fact equivariant isomorphisms (see Corollary 4.2).

Our study of stable equivalence of lattices with involution leads us to a notion of *skew equivalence*, presented in Section 7. Here, duality interacts with the involution which is reflected in a functional equations for the Fourier-Mukai kernel. Explicit examples, for anti-symplectic actions with quotients equal to $\mathbb{P}^2$, are presented in Section 8.

Next, we focus on *Nikulin* involutions $\iota : X \to X$, i.e., involutions preserving the symplectic form, so that the resolution of singularities Y of the resulting quotient X/ι is a K3 surface. A detailed study of such involutions can be found in [vGS07]. In addition to the polarization class, the Picard group $\mathrm{Pic}(X)$ contains the lattice $\mathrm{E}_8(-2)$; van Geemen and Sarti describe the moduli and the geometry in the case of minimal Picard rank $\mathrm{rk}\,\mathrm{Pic}(X) = 9$. In Section 9, we extend their results to higher ranks, and

- exhibit nontrivial derived equivalences between Nikulin involutions (Proposition 9.3).

These, in turn, allow us to construct in Section 10 examples of equivariant birational isomorphisms $\phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ with nonvanishing invariant $C_G(\phi)$, introduced in [LSZ23], [LS24] and extended to the equivariant context in [KT22].

The case of *Enriques* involutions $\epsilon : X \to X$, i.e., fixed-point free involutions, so that the resulting quotient X/ϵ is an Enriques surface, has also received considerable attention. There is a parametrization of such involutions in terms of the Mukai lattice $\widetilde{\mathrm{H}}(X)$, and an explicit description of conjugacy classes, up to automorphisms $\mathrm{Aut}(X)$, in interesting special cases, e.g., for K3 surfaces of Picard rank 11, Kummer surfaces of product type, general Kummer surfaces, or singular K3 surfaces [Kon92], [Oha07], [Ser05], [SV20]. In Section 11 we observe that

- the existence of an Enriques involution on a K3 surface X implies that every derived equivalent surface is equivariantly isomorphic to X (Propositions 11.2 and 11.3);
- while there are *no* nontrivial equivariant derived autoequivalences, we exhibit nontrivial *orientation reversing* (i.e., skew) equivalences, e.g., on singular K3 surfaces.

Acknowledgments: The first author was partially supported by Simons Foundation Award 546235 and NSF grants 1701659 and 1929284, the second author by NSF grant 2000099. We are grateful to Nicolas Addington, Arend Bayer, Igor Dolgachev, Sarah Frei, Lisa Marquand, and Barry Mazur for helpful suggestions.

2. Lattice results

We recall basic terminology and results concerning lattices: torsion-free finite-rank abelian groups L together with a nondegenerate integral quadratic form $(\cdot,\cdot)$, which we assume to be even. Basic examples are

$$\mathrm{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and positive definite lattices associated with Dynkin diagrams (denoted by the same letter).

We write $\mathrm{L}(2)$, when the form is multiplied by 2. We let

$$d(\mathrm{L}) := \mathrm{L}^*/\mathrm{L}$$

be the discriminant group and

$$q_{\mathrm{L}} : d(\mathrm{L}) \to \mathbb{Q}/2\mathbb{Z}$$

the induced discriminant quadratic form.

Nikulin's form of Witt cancellation:

Proposition 2.1. [Nik79b, Cor. 1.13.4] *Given an even lattice* L, $\mathrm{L}\oplus\mathrm{U}$ *is the unique lattice with its signature and discriminant quadratic form.*

If lattices L_1 and L_2 are stably isomorphic – become isomorphic after adding unimodular lattices of the same signature – then

$$\mathrm{L}_1 \oplus \mathrm{U} \simeq \mathrm{L}_2 \oplus \mathrm{U}.$$

Nikulin stabilization result: Given a lattice L, write $\mathrm{L}\otimes\mathbb{Z}_p$ for the induced p-adic quadratic form. The genus of L is the collection of all lattices equivalent to L over $\mathbb{Z}_p$ for each prime p and over $\mathbb{R}$. Stably equivalent lattices are in the same genus. The p-primary part of $d(\mathrm{L})$ depends only on $\mathrm{L}\otimes\mathbb{Z}_p$ and is written $d(\mathrm{L}\otimes\mathbb{Z}_p)$. We use $q_{\mathrm{L}\otimes\mathbb{Z}_p}$ for the induced discriminant quadratic form on $d(\mathrm{L}\otimes\mathbb{Z}_p)$, with values in the p-primary part of $\mathbb{Q}/\mathbb{Z}$ for odd p; when L is even and $p=2$ it takes values in the 2-primary part of $\mathbb{Q}/2\mathbb{Z}$. For a finitely generated abelian group A, let $\ell(A)$ denote the minimal number of generators.

Proposition 2.2. [Nik79b, Thm. 1.14.2] *Let* L *be an even indefinite lattice satisfying*

- $\mathrm{rank}(\mathrm{L}) \geq \ell(d(\mathrm{L} \otimes \mathbb{Z}_p)) + 2$ *for all* $p \neq 2$*;*
- *if* $\mathrm{rank}(\mathrm{L}) = \ell(d(\mathrm{L} \otimes \mathbb{Z}_2))$ *then* $q_{\mathrm{L}\otimes\mathbb{Z}_2}$ *contains* $u_+^{(2)}(2)$ *or* $v_+^{(2)}(2)$ *as a summand, i.e., the discriminant quadratic forms of*

$$\mathrm{U}^{(2)}(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \mathrm{V}^{(2)}(2) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Then the genus of L *admits a unique class and* $\mathrm{O(L)} \to \mathrm{O}(q_{\mathrm{L}})$ *is surjective.*

Remark 2.3. [Nik79b, Rem. 1.14.5] The 2-adic condition can be achieved whenever the discriminant group $d(\mathrm{L})$ has $(\mathbb{Z}/2\mathbb{Z})^3$ as a summand.

Thus given a lattice L, any automorphism of $(d(\mathrm{L}), q_{\mathrm{L}})$ may be achieved via an automorphism of $\mathrm{L} \oplus \mathrm{U}$. More precisely, given two lattices L_1 and L_2 of the same rank and signature and an isomorphism

$$\varrho : (d(\mathrm{L}_1), q_{\mathrm{L}_1}) \xrightarrow{\sim} (d(\mathrm{L}_2), q_{\mathrm{L}_2})$$

there exists an isomorphism

$$\rho : \mathrm{L}_1 \oplus \mathrm{U} \xrightarrow{\sim} \mathrm{L}_2 \oplus \mathrm{U}$$

inducing ϱ.

Nikulin imbedding result:

Proposition 2.4. [Nik79b, Cor. 1.12.3,Thm. 1.14.4] *Let* L *be an even lattice of signature* (t_+, t_-) *and discriminant group* $d(\mathrm{L})$*. Then* L *admits a primitive embedding into a unimodular even lattice of signature* (ℓ_+, ℓ_-) *if*

- $\ell_+ - \ell_- \equiv 0 \mod 8$*;*
- $\ell_+ \geq t_+$ *and* $\ell_- \geq t_-$*;*
- $\ell_+ + \ell_- - t_+ - t_- > \ell(d(\mathrm{L}))$*, the rank of* $d(\mathrm{L})$*.*

This embedding is unique up to automorphisms if

- $\ell_+ > t_+$ *and* $\ell_- > t_-$*;*
- $\ell_+ + \ell_- - t_+ - t_- \geq 2 + \ell(d(\mathrm{L}))$*.*

In particular, *any* even nondegenerate lattice of signature $(1, 9)$ admits a unique embedding into the K3 lattice $\mathrm{U}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$.

3. Mukai lattices and derived automorphisms

Throughout, we work over the complex numbers $\mathbb{C}$. Let X be a projective K3 surface and

$$\mathrm{Pic}(X) \subset \mathrm{H}^2(X, \mathbb{Z}) \simeq \mathrm{E}_8(-1)^{\oplus 2} \oplus \mathrm{U}^3$$

its Picard lattice, a sublattice of a lattice of signature $(3, 19)$, with respect to the intersection pairing. The Picard lattice governs the automorphisms of X. The composition

$$\varpi : \mathrm{Aut}(X) \to \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z})) \to \mathrm{O}(\mathrm{Pic}(X))$$

has finite cyclic kernel [Nik79a, Kon92]. The image can be computed explicitly, at least up to finite subgroups, in terms of $\mathrm{Pic}(X)$ [LP81, §2]. Consider the subgroup generated by reflections in (-2)-classes, i.e., indecomposable effective divisors of self-intersection -2; it acts naturally on the positive cone in $\mathrm{Pic}(X)_{\mathbb{R}}$. Then the image of ϖ is a finite-index subgroup of those elements leaving invariant a fundamental domain for this action, i.e. the ample cone. All possible finite $G \subset \mathrm{Aut}(X)$ have been classified, see [BH23]. Classification of $\mathrm{Aut}(X)$-conjugacy classes of elements or subgroups boils down to lattice theory of $\mathrm{Pic}(X)$; we will revisit it in special cases below.

The *transcendental lattice* of X is the orthogonal complement

$$T(X) := \mathrm{Pic}(X)^{\perp} \subset \mathrm{H}^2(X, \mathbb{Z}).$$

This lattice plays a special role: two K3 surfaces X_1, X_2 are *derived equivalent* if and only if there exists an isomorphism of lattices

$$T(X_1) \xrightarrow{\sim} T(X_2),$$

compatible with Hodge structures [Orl97]. Derived equivalence also means that the lattices $\mathrm{Pic}(X_1)$ and $\mathrm{Pic}(X_2)$ belong to the same genus. Over nonclosed fields, or in equivariant contexts, derived equivalence is a subtle property, see, e.g., [HT17], [HT23].

We recall standard examples of Picard lattices of derived equivalent but not isomorphic K3 surfaces

Remark 3.1. In Picard rank one: the number of nonisomorphic derived equivalent surfaces is governed by the number of prime divisors of the polarization degree $2d$; see [HLOY04, Cor. 2.7] and Proposition 2.2. The isomorphisms classes correspond to solutions of the congruence

$$x^2 \equiv 1 \pmod{4d} \tag{3.1}$$

modulo ± 1. When $d > 1$ the number of derived equivalent K3 surfaces is $2^{\tau(d)-1}$, where τ is the number of distinct prime factors of d.

In Picard rank two: derived equivalences among lattice-polarized K3 surfaces of square-free discriminant are governed by the genera in the class group of the corresponding real quadratic field [HLOY04, Sect. 3].

Here are instances where derived equivalence is trivial

Proposition 3.2. [HLOY04, Cor. 2.6, 2.7] *Derived equivalence implies isomorphism in each of the following cases:*

- *if the Picard rank is* ≥ 12*;*
- *if the surface admits an elliptic fibration with a section;*
- *if the Picard rank is* ≥ 3 *and the discriminant group of the Picard group is cyclic.*

We give a further example in Proposition 9.3.

Let

$$\widetilde{\mathrm{H}}(X) := \mathrm{H}^0(X, \mathbb{Z})(-1) \oplus \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^4(X, \mathbb{Z})(1)$$

be its *Mukai lattice*, a lattice of signature $(4, 20)$, with respect to the Mukai pairing. There is a surjective homomorphism [HMS09, Cor. 3]

$$\mathrm{Aut}\, D^b(X) \to \mathrm{O}^+(\widetilde{\mathrm{H}}(X)) \subset \mathrm{O}(\widetilde{\mathrm{H}}(X))$$

onto the group of *signed* Hodge isometries, a subgroup of the orthogonal group of the Mukai lattice preserving orientations on the positive 4-planes.

We retain the notation from [HT23, Cor. 3], where we discussed the notion and basic properties of equivariant derived equivalences between K3 surfaces. We recall:

> Let X_1 and X_2 be K3 surfaces equipped with a generically free action of a finite cyclic group G. Then X_1 and X_2 are G-equivariantly derived equivalent if and only if there exists a G-equivariant isomorphism of their Mukai lattices
>
> $$\widetilde{\mathrm{H}}(X_1) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2)$$
>
> respecting the Hodge structures.

Note that the G-action is necessarily trivial on

$$\mathrm{H}^0(X, \mathbb{Z})(-1) \oplus \mathrm{H}^4(X, \mathbb{Z})(1).$$

Even in the event of an isomorphism $X_1 \simeq X_2$, equivariant derived equivalences are interesting: indeed, there are actions of finite groups G that are not conjugate in $\mathrm{Aut}(X)$ but are conjugate via $\mathrm{Aut}\, D^b(X)$

as the action of the latter group is visibly larger. See Proposition 9.6 for examples.

Let G be a finite group and X_1 and X_2 K3 surfaces with G-actions. For simplicity, assume that G acts on $T(X_i)$ via $\pm\mathrm{I}$. (This is the case if the transcendental cohomology is simple.) Given a G-equivariant isomorphism $T(X_1) \simeq T(X_2)$, can we lift to a G-equivariant isomorphism of Mukai lattices

$$\widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}(X_2, \mathbb{Z}),$$

where G acts trivially on the hyperbolic summand

$$\mathrm{U} = \mathrm{H}^0 \oplus \mathrm{H}^4?$$

Clearly the answer is NO. Suppose that $G = C_2 = \langle \epsilon \rangle$ and the $\epsilon = -1$ eigenspaces are stably isomorphic but not isomorphic. Adding U does nothing to achieve the desired stabilization. In other words, U is "too small". We need to add summands where G acts *nontrivially* to achieve stabilization across all the various isotypic components. See Proposition 4.5 for more on this question.

4. Generalities concerning involutions on K3 surfaces

Let $i : X \to X$ be an involution on a complex projective K3 surface, which acts faithfully on $\mathrm{H}^2(X, \mathbb{Z})$ by the Torelli Theorem. It is *symplectic* (resp. *anti-symplectic*) if

$$i^*\omega = \omega \quad (\text{resp.} \ -\omega),$$

where ω is a holomorphic two-form. Nikulin [Nik79a] showed that any symplectic involution fixes eight isolated points and that all such involutions are topologically conjugate; these are the *Nikulin involutions* studied in Section 9. An involution without fixed points was classically known to be an *Enriques involution* arising from a double cover $X \to S$ of an Enriques surface.

The case of anti-symplectic involutions with fixed points is more complicated. Nikulin enumerated 74 cases beyond the Enriques case; see [AN06, BH23, AE22, Ale22] for details of the various cases.

Given an anti-symplectic involution $i : X \to X$ on a K3 surface, we recall the Nikulin invariants (r, a, δ) [AE22, §2]: Let r denote the rank of the lattice

$$S = \mathrm{H}^2(X, \mathbb{Z})^{i=1},$$

which is indefinite if $r > 1$. We are using the fact that transcendental classes of X are anti-invariant under i, as the quotient X/i admits no holomorphic two-form. We write

$$T = \mathrm{H}^2(X, \mathbb{Z})^{i=-1} = S^{\perp}$$

for the complementary lattice with signature $(2, 20-r)$, which is indefinite if $r < 20$. The discriminant group $d(S) \simeq d(T)$ is a 2-elementary group; its rank is denoted by a. This group comes with a quadratic form

$$q_S : d(S) \to \mathbb{Q}/2\mathbb{Z}.$$

The *coparity* δ equals 0 if $q_S(x) \in \mathbb{Z}$ for each $x \in d(S)$ and equals 1 otherwise.

We relate this to geometric invariants. For an anti-symplectic involution, there are no isolated fixed points so the fixed locus $R = X^i$ is of pure dimension one or empty. Suppose there are $k+1$ irreducible components, with genera summing to g. Then we have cf. [AE22, p.5]

$$g = 11 - (r+a)/2 \quad k = (r-a)/2,$$

excluding the Enriques case $(r, k, \delta) = (10, 10, 0)$.

Nikulin classifies even indefinite 2-elementary lattices L. They are determined uniquely by (r, a, δ) and $\mathrm{O(L)} \to \mathrm{Aut}(d(\mathrm{L}))$ is surjective. In the definite case, *a priori* there are multiple classes in each genus but this is not relevant for our applications. Indeed, the possibilities include

- $r = a = 1$: X is a double cover of $\mathbb{P}^2$ branched along a sextic plane curve.
- The case where T is definite ($r = 20, a = 2, g = 0, k = 9$), we have $d(T) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ thus is equal to

  $$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

  Even in this case, automorphisms of the discriminant group are realized by automorphisms of the lattice.

Theorem 4.1 (Alexeev-Nikulin)**.** *For each admissible set of invariants (r, a, δ), there is a unique orthogonal pair of lattices (S, T) embedded in the K3 lattice Λ, up to automorphisms of Λ. There are 75 such cases.*

Corollary 4.2. *Any equivariant derived equivalence of K3 surfaces with anti-symplectic involutions induces an equivariant isomorphism between the underlying K3 surfaces.*

Proof. Suppose that (X_1, i_1) and (X_2, i_2) are derived equivalent, compatibly with their anti-symplectic involutions.

Indeed, derived equivalence shows that the invariant (resp. anti-invariant) sublattices of the Picard group are stably equivalent (resp. equivalent):

$$\mathrm{Pic}(X_1)^{i_1=1} \oplus \mathrm{U} \simeq \mathrm{Pic}(X_2)^{i_2=1} \oplus \mathrm{U}, \quad \mathrm{Pic}(X_1)^{i_1=-1} \simeq \mathrm{Pic}(X_2)^{i_2=-1}.$$

Since the possibilities for the invariant sublattices are characterized by their 2-adic invariants, we have

$$\mathrm{Pic}(X_1)^{i_1=1} \simeq \mathrm{Pic}(X_2)^{i_2=1}.$$

We have already observed that all the possible isomorphisms between their discriminants

$$\left(d(\mathrm{Pic}(X_1)^{i_1=1}), q_1\right) \simeq \left(d(\mathrm{Pic}(X_2)^{i_2=1}), q_2\right)$$

are realized by isomorphisms of the lattices. In particular, there exists a choice compatible with the isomorphism

$$\mathrm{H}^2(X_1, \mathbb{Z})^{i_1=-1} \xrightarrow{\sim} \mathrm{H}^2(X_2, \mathbb{Z})^{i_2=-1}$$

induced by the derived equivalence. Thus we obtain isomorphisms on middle cohomology, compatible with the involutions. The Torelli Theorem gives an isomorphism $X_1 \xrightarrow{\sim} X_2$ respecting the involutions. □

Corollary 4.3. *Let (X_1, σ_1) and (X_2, σ_2) denote K3 surfaces with involutions that are C_2-equivariantly derived equivalent. If X_1/σ_1 is rational then X_2/σ_2 is rational as well.*

Indeed, the rationality of the quotient forces the involution to be anti-symplectic.

Example 4.4. Having an anti-symmetric involution is *not* generally a derived property. For example, consider Picard lattices

$$A_1 = \begin{pmatrix} 2 & 13 \\ 13 & 12 \end{pmatrix} \quad A_2 = \begin{pmatrix} 8 & 15 \\ 15 & 10 \end{pmatrix}.$$

These forms are stably equivalent but not isomorphic. As in Remark 3.1 – see [HT17, Sec. 2.3] for details – choose derived equivalent K3 surfaces X_1 and X_2 with $\mathrm{Pic}(X_1) = A_1$ and $\mathrm{Pic}(X_2) = A_2$. Note that A_2 does not represent two and admits no involution acting via ± 1 on $d(A_2)$; thus X_2 does not admit an involution.

This should be compared with Proposition 11.2: Having an Enriques involution is a derived invariant.

We collect some lattice-theoretic observations that will serve as a foundation for Section 7:

Proposition 4.5. *Let* (X_1, i_1) *and* (X_2, i_2) *be K3 surfaces with involutions, both symplectic or anti-symplectic. Extend the involutions to actions on the Mukai lattices*

$$\widetilde{i_j} : \widetilde{\mathrm{H}}(X_j, \mathbb{Z}), \quad j = 1, 2,$$

where

$$\widetilde{i_j}|\mathrm{H}^k = \begin{cases} i_j^* & \text{if } k = 2, \\ -\mathrm{I} & \text{if } k = 0, 4. \end{cases}$$

An equivalence of such actions, on Hodge structures of weight two, corresponds to a triple

(1) *an isomorphism of Hodge structures*

$$t : T(X_1) \to T(X_2),$$

(2) *an isomorphism of lattices*

$$\pi^{+1} : \mathrm{Pic}(X_1)^{i_1=1} \to \mathrm{Pic}(X_2)^{i_2=1},$$

(3) *a stable equivalence of lattices*

$$\pi^{-1} : \mathrm{Pic}(X_1)^{i_1=-1} \oplus \mathrm{U} \to \mathrm{Pic}(X_2)^{i_2=-1} \oplus \mathrm{U},$$

satisfying the following conditions

- *the isomorphisms induced by* $\pi^{\pm 1}$ *on discriminant groups agree on the images*

$$\mathrm{Pic}(X_j) \to d(\mathrm{Pic}(X_j)^{i_j=1}) \oplus d(\mathrm{Pic}(X_j)^{i_j=-1}),$$

  which are 2*-elementary groups;*
- *the resulting isomorphism*

$$\mathrm{Pic}(X_1) \to \mathrm{Pic}(X_2)$$

  is compatible with t *on discriminant groups.*

This is proven through two applications of Nikulin's lattice extension theory, first to the Picard group and then to the full cohomology lattice.

Fixing $T(X_j)$ and $\mathrm{Pic}(X_j)^{i_j=1}$, the possible equivalences are indexed by isomorphisms π^{-1} restricting to the identity on the distinguished 2-elementary subgroups. Applying Nikulin stabilization, the equivalent Mukai lattices, with these data, are indexed by the stable isomorphism classes of the anti-invariant Picard groups, where the stable isomorphism restricts to the identity on the distinguished 2-elementary subgroup of the their discriminant groups.

Corollary 4.6. *Suppose the anti-invariant Picard lattice P is unique in its genus. Then the possible Mukai lattices $(\widetilde{\mathrm{H}}, \widetilde{\imath})$ with P are indexed by automorphisms of $d(P)$ restricting to the identity on the two-elementary subgroup.*

This is an equivariant version of the counting results of [HLOY04].

5. Cohomological Fourier-Mukai transforms

Let X_1 and X_2 be smooth projective complex K3 surfaces. A fundamental result of Orlov [Orl97] shows that any equivalence

$$\Phi : D^b(X_1) \to D^b(X_2)$$

arises from a kernel $\mathcal{K} \in D^b(X_1 \times X_2)$ through a Fourier-Mukai transform

$$\begin{array}{rcl} \Phi_{\mathcal{K}} : D^b(X_1) & \to & D^b(X_2) \\ \mathcal{E} & \mapsto & \pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}). \end{array}$$

All the indicated functors are taken in their derived senses. Given such a kernel, there is also a Fourier-Mukai transform in the opposite direction

$$\begin{array}{rcl} \Psi_{\mathcal{K}} : D^b(X_2) & \to & D^b(X_1) \\ \mathcal{E} & \mapsto & \pi_{1*}(\pi_2^*\mathcal{E} \otimes \mathcal{K}). \end{array}$$

Mukai has computed the kernel of the inverse

$$\Phi_{\mathcal{K}}^{-1} = \Psi_{\mathcal{K}^\vee[2]}$$

i.e., a shift of the dual to our original kernel. See [Muk87, 4.10], [BBHR97, § 4.3], and [Huy06, p. 133] for details. The computation relies on Grothendieck-Serre Duality, so the appearance of the dualizing complex is natural. This machinery [Huy06, § 3.4] also allows us to analyze how Fourier-Mukai transforms interact with taking duals:

$$\begin{aligned} \Phi_{\mathcal{K}}(\mathcal{E}^\vee) &= \pi_{2*}(\mathcal{K} \otimes \pi_1^*(\mathcal{E}^\vee)) \\ &= ((\pi_{2*}(\mathcal{K}^\vee \otimes \pi_1^*\mathcal{E}))^\vee)[-2] \\ &= ((\Phi_{\mathcal{K}^\vee}\mathcal{E})[2])^\vee \\ &= (\Phi_{\mathcal{K}^\vee[2]}\mathcal{E})^\vee \end{aligned}$$

Suppose that X_1 and X_2 are equivalent through an isomorphism

$$X_2 = M_H(X_1, v_1),$$

i.e., the moduli space of sheaves $\mathcal{E}_p, p \in X_2$, on X_1 with Mukai vector

$$v_1 = v(\mathcal{E}_p) = (r, D, s) \in \widetilde{\mathrm{H}}(X, \mathbb{Z}),$$

Gieseker-stable with respect to some polarization H on X_1. Here r is the rank of $\mathcal{E}_p$, $D = c_1(\mathcal{E}_p)$, and $s = \chi(\mathcal{E}_p) - r$. We assume there exists another Hodge class $v' \in \widetilde{\mathrm{H}}(X_1, \mathbb{Z})$ such that $\langle v, v' \rangle = 1$; in particular, v is primitive. (For information on how to realize derived equivalences via such moduli spaces, see [Muk87, §4,5] and [Huy08, p. 385], the discussion following Proposition 4.1.) Let $\mathcal{E} \to X_1 \times X_2$ denote a universal sheaf; by simplicity of the sheaves, $\mathcal{E}$ is unique up to tensoring by a line bundle from X_2. We may use $\mathcal{E}$ as a kernel inducing a derived equivalence between X_1 and X_2 [Huy06, 10.25]. Our formulas for inverses are compatible with tensoring the kernel by line bundles from one of the factors.

In searching for Fourier-Mukai kernels, cohomological Fourier-Mukai transforms play a crucial role. Let $\omega_i \in \mathrm{H}^4(X_i, \mathbb{Z})$ denote the point class and set [Muk87, §1], [Huy06, p. 128]

$$Z_{\mathcal{K}} := \pi_1^*(1 + \omega_1)\,\mathrm{ch}(\mathcal{K})\pi_2^*(1 + \omega_2) \in \mathrm{H}^*(X_1 \times X_2, \mathbb{Z}),$$

where the middle term is the Chern character. Then $Z_{\mathcal{K}}$ induces an integral isomorphism of Hodge structures

$$\phi_{\mathcal{K}} : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2, \mathbb{Z})$$

compatible with Mukai pairings; this is called the *cohomological Fourier-Mukai transform*. For $\mathcal{E} \in D^b(X_1)$, we have the identity

$$\phi_{\mathcal{K}}(v(\mathcal{E})) = v(\Phi_{\mathcal{K}}(\mathcal{E})).$$

We use $\psi_{\mathcal{K}}$ to denote the cohomological transform of $\Psi_{\mathcal{K}}$.

Most cohomological Fourier-Mukai transforms are induced by kernels

Proposition 5.1. [Orl97, HMS09] *Given an orientation-preserving integral Hodge isometry*

$$\phi : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \to \widetilde{\mathrm{H}}(X_2, \mathbb{Z})$$

there exists a derived equivalence

$$\Phi_{\mathcal{K}} : D^b(X_1) \to D^b(X_2)$$

such that ϕ is the cohomological Fourier-Mukai transform of $\Phi_{\mathcal{K}}$.

Suppose that (X_1, f_1) is a polarized K3 surface of degree $2r_0 s$, where r_0 and s are relatively prime positive integers. Let d_0 be an integer prime to r_0 and fix the isotropic Mukai vector

$$v_0 = (r_0, d_0 f_1, d_0^2 s) \in \widetilde{\mathrm{H}}(X_1, \mathbb{Z}).$$

Since r_0 and $d_0^2 s$ are relatively prime, there exists a Mukai vector $v' = (m, 0, n)$ such that $\langle v_0, v' \rangle = 1$. Let $X_2 = M_{f_1}(X_1, v_0)$ be the moduli space of torsion-free sheaves with Mukai vector v_0, Gieseker-stable with respect to f_1 – also a K3 surface. Choose a universal sheaf $\mathcal{E} \to X_1 \times X_2$. Our goal is to describe the induced isomorphism

$$\phi_{\mathcal{E}} : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2, \mathbb{Z}).$$

Following [HL10, Ch. 8] and [Yos99, §2], the polarization on X_2 is given by

$$\det(\pi_{2*}(\mathcal{E} \otimes \mathcal{O}_H(s(r_0 - 2d_0))))^\vee, \quad H \in |f_1|,$$

a primitive ample divisor f_2 on X_2. More generally, we have an isomorphism of Hodge structures

$$\mathrm{H}^2(X_2, \mathbb{Z}) = (v_0^\vee)^\perp / \mathbb{Z} v_0^\vee,$$

where the perpendicular subspace is taken with respect to the Mukai pairing.

Proposition 5.2. [Yos99] *Let (X_1, f_1) and (X_2, f_2) be K3 surfaces of Picard rank one with $X_2 \simeq M_{f_1}(X_1, v_0)$ as above. Choose integers d_1 and ℓ such that $sd_0d_1 - r_0\ell = 1$ and take $\mathcal{K} = \mathcal{E} \otimes \pi_2^* L$ for some line bundle L on X_2. With respect to the bases*

$$(1,0,0), (0, f_j, 0), (0,0,1) \in \widetilde{\mathrm{H}}(X_j, \mathbb{Z}) \cap \widetilde{\mathrm{H}}^{1,1}(X_j)$$

the matrix of the cohomological Fourier-Mukai transform takes the form

$$\phi_{\mathcal{K}} := \begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \ell & 2d_0 d_1 s - 1 & d_1 \\ \ell^2 r_0 & 2d_1 s \ell r_0 & d_1^2 s \end{pmatrix}. \tag{5.1}$$

The inverse is obtained reversing the sign of the middle basis vector and interchanging the role of d_0 and d_1:

$$\begin{pmatrix} d_0^2 s & 2d_0 s r_0 & r_0 \\ d_0 \ell & 2d_0 d_1 s - 1 & d_1 \\ \ell^2 r_0 & 2d_1 s \ell r_0 & d_1^2 s \end{pmatrix} \begin{pmatrix} d_1^2 s & -2d_1 s r_0 & r_0 \\ -d_1 \ell & 2d_0 d_1 s - 1 & -d_0 \\ \ell^2 r_0 & -2d_0 s \ell r_0 & d_0^2 s \end{pmatrix} = \mathrm{I}.$$

The formula

$$\phi_{\mathcal{K}} \psi_{\mathcal{K}^\vee} = \mathrm{I}$$

is the cohomological realization of the identity

$$\Phi_{\mathcal{K}} \Psi_{\mathcal{K}^\vee[2]} = \mathrm{I}.$$

The third column of $\phi_{\mathcal{K}}^{-1}$ is the Mukai vector $v_0^\vee$, as

$$\Phi_{\mathcal{K}}^{-1}(\mathcal{O}_p) = \mathcal{E}_p^\vee, \quad p = [\mathcal{E}_p] \in X_2 = M_{f_1}(X_1, v).$$

Remark 5.3. The assumption in Proposition 5.2 on the rank of the Picard groups is not too restrictive, as Proposition 5.1 allows us to specialize from the rank-one case. Derived equivalences satisfying (5.1) exist provided the primitive cohomology groups are isomorphic

$$\mathrm{H}^2(X_1,\mathbb{Z}) \supset f_1^\perp \simeq f_2^\perp \subset \mathrm{H}^2(X_2,\mathbb{Z})$$

as integral Hodge structures. However, these are not given as kernels associated with explicit moduli spaces of sheaves Gieseker-stable with respect to some polarization.

Example 5.4. Suppose that (X_1, f_1) is a degree 12 K3 surface. Consider the isotropic Mukai vector $v = (2, f_1, 3)$ so that

$$X_2 := M_{f_1}(X_1, v)$$

is also a K3 surface derived equivalent to X_1. Taking

$$r_0 = 2, \quad s = 3, \quad d_0 = 1, \quad d_1 = \ell = 1,$$

we obtain

$$\begin{aligned} (1,0,0) &\mapsto (3, f_2, 2) \\ (0, f_1, 0) &\mapsto (12, 5f_2, 12) \\ (0,0,1) &\mapsto (2, f_2, 3) \end{aligned}$$

with matrix

$$\varphi := \begin{pmatrix} 3 & 12 & 2 \\ 1 & 5 & 1 \\ 2 & 12 & 3 \end{pmatrix}. \tag{5.2}$$

The determinant is 1 with one eigenvector $(1, 0, -1)$ with eigenvalue 1; thus this is orientation preserving. Note that

$$(2, -f_1, 3) \mapsto (0, 0, 1)$$

whence

$$X_1 = M_{f_2}(X_2, (2, f_2, 3)), \quad X_2 = M_{f_2}(X_1, (2, -f_1, 3)).$$

The fact that $(1, 0, -1)$ has eigenvalue 1 gives

$$X_1^{[2]} \overset{\sim}{\dashrightarrow} X_2^{[2]}.$$

6. Locally-free kernels and wall-crossing

For applications to skew equivalence, discussed in Section 7, we require derived equivalences between K3 surfaces X_1 and X_2 induced by locally-free kernels

$$\mathcal{E} \to X_1 \times X_2.$$

Many equivalences do not arise in this way.

Example 6.1. Suppose that $X_1 = X_2 = X$ and consider the equivalence arising by interpreting X as the moduli space of ideal sheaves $I_x, x \in X$. These sheaves are not locally-free.

We refer the reader to [HL10, §1.2] for the definitions and background on μ-stable sheaves and relations to Gieseker stability. We use the implications [HL18, Lemma 1.2.13]

$$\mu_H\text{-stable} \Rightarrow \text{stable wrt } H \Rightarrow \text{semistable wrt } H \Rightarrow \mu_H\text{-semistable}.$$

Now μ-stable sheaves on K3 surfaces are typically locally-free: Let E be a simple sheaf on a K3 surface X, with $v(E)$ isotropic, such that μ_H-stable for some polarization H. Then E is locally-free, with the exception of ideal sheaves I_x [Muk87, 3.10], [HL18, 6.1.9]. The problem is that moduli spaces $M_H^{\mu s}(X, v)$ of such sheaves are not always compact, when there are strictly μ_H-semistable sheaves.

We recall criteria guaranteeing that μ stability and semistability coincide. Let $v = (r, D, s)$ be a primitive isotropic Mukai vector of rank $r > 0$ for a K3 surface X. Assume that D is primitive and H is a polarization avoiding "walls", i.e., hyperplanes expressible in the form $\xi^\perp$ for suitable $0 \neq \xi \in D^\perp$. Then we have, by [HL10, 4.C.3],

$$M_H^{\mu s}(X, v) = M_H^{\mu ss}(X, v).$$

The ξ that arise may be characterized in terms of r [HL10, 4.C.2]:

Example 6.2. Suppose that $r = 2$ and E is strictly μ_H-semistable.

One possibility is extensions

$$0 \to \mathcal{O}_X(L) \to E \to \mathcal{O}_X(D - L) \to 0,$$

where L is a divisor with

$$H \cdot L = H \cdot (D - L), \quad \dim \operatorname{Ext}^1(\mathcal{O}_X(D - L), \mathcal{O}_X(L)) = 2.$$

Here $\xi = 2L - D \in H^\perp$ satisfies $\xi^2 = -8$. Writing $\hat{s} = v(L)$ we have:

	v	$\hat{s}$
v	0	0
$\hat{s}$	0	-2

Thus $\hat{s}$ gives rise to a (-2)-class in the Picard group of $M_{H'}(X,v)$ for H' a polarization outside the walls; typically this is the class of a rational curve isomorphic to $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(D-L),\mathcal{O}_X(L)))$, contracted in $M_H^{\mu ss}(X,v)$ with complement $M_H^{\mu s}(X,v)$.

The other possibility is extensions

$$0 \to \mathcal{O}_X(L) \to E \to I_x(D-L) \to 0, \tag{6.1}$$

where $x \in X$ and L is a divisor

$$H \cdot L = H \cdot (D-L), \quad \dim \mathrm{Hom}(L,E) = 1.$$

Here $\xi = 2L - D \in H^{\perp}$ satisfies $\xi^2 = -4$, writing $\hat{s} = v(L)$ we have:

$$\begin{array}{c|cc} & v & \hat{s} \\ \hline v & 0 & -1 \\ \hat{s} & -1 & -2 \end{array}$$

Here $M_H^{\mu s}(X,v) = \emptyset$ reflecting the fact that the extension (6.1) may be trivial or nontrivial. The resulting coarse moduli space is isomorphic to X; this is called a "totally semistable" wall.

This dichotomy in the wall types is typical and explained in [Bri08, §12] (for two-dimensional moduli spaces) and [BM14, Th. 5.7] (in general); we are grateful to Bayer for pointing out this framework. The possible walls are all associated with spherical classes $\hat{s}$ with $v(\hat{s})^2 = -2$ of two types:

- *contracting walls:*
$$\begin{array}{c|cc} & v & \hat{s} \\ \hline v & 0 & 0 \\ \hat{s} & 0 & -2 \end{array}$$
where $M_H^{\mu ss}(X,v)$ has a contractible (-2)-class in its Picard group;
- *totally semistable walls:*
$$\begin{array}{c|cc} & v & \hat{s} \\ \hline v & 0 & -r \\ \hat{s} & -r & -2 \end{array}, \quad r \geq 1,$$
where $M_H^{\mu s}(X,v)$ is empty but the coarse moduli space is left unchanged.

Bridgeland [Bri08] elucidates the typical behavior; we refer the reader to [BM14, §6] for details of the derived equivalences associated with wall crossing arising as compositions of spherical twists associated the (-2)-classes.

Example 6.3. Mukai [Muk87, 3.8] offers examples of the second type. Let F be a rigid vector bundle with $v(F) = \hat{s}$, r its rank, and E the kernel of evaluation at a skyscraper sheaf at $x \in X$:

$$0 \to E \to F^{\oplus r} \to \mathbb{C}(x) \to 0.$$

Note that E has local cohomology at x and thus cannot be locally-free.

Suppose that the polarization varies over the ample cone. As we cross walls of either type, the moduli spaces associated with adjacent chambers are naturally isomorphic. Even for a contracting wall, the minimal resolution of the nodal moduli space is naturally isomorphic to moduli spaces associated with each side. Since the ample cone is simply-connected, for all ample H_1 and H_2 we obtain natural isomorphisms

$$\beta_{H_2,H_1} : M^{\mu ss}_{H_1}(X, v) \xrightarrow{\sim} M^{\mu ss}_{H_2}(X, v); \tag{6.2}$$

see the discussion following [BM14, Th. 1.1]. This is an instance of the general phenomenon that wall-crossing induces birational maps among moduli spaces of vector bundles on surfaces [HL10, 4.C.7]. However, the universal sheaves over these moduli spaces – and the derived equivalences they induce – do vary from chamber to chamber (see[BM14, Th. 1.1(b)]). In particular, explicit formulas as in Proposition 5.2 are not available for higher rank K3 surfaces.

An application of wall-crossing, and a template for our results in Section 7, is the following result of Huybrechts [Huy08, Prop. 4.1]: Let X_1 and X_2 be derived equivalent K3 surfaces. Then there exists a moduli space of μ_H-stable locally-free sheaves with universal family

$$\mathcal{E} \to M^{\mu s}_H(X_2, v) \times X_2$$

and an isomorphism $X_1 \simeq M^{\mu s}_H(X_2, v)$.

7. Orientation reversing conjugation

We continue to assume that i is an anti-symplectic involution on a K3 surface X. As we have seen,

$$T(X) \subset \mathrm{H}^2(X, \mathbb{Z})^{i=-1},$$

with complement $\mathrm{Pic}(X)^{i=-1}$, which is negative definite by the Hodge index theorem.

Recall that Orlov's Theorem [Orl97, §3] asserts that for K3 surfaces (without group action) isomorphisms of transcendental cohomology lift to derived equivalences. Given K3 surfaces (X_1, i_1) and (X_2, i_2) with anti-symplectic involutions of the same type in the sense of Alexeev-Nikulin, the existence of an isomorphism

$$T(X_1) \xrightarrow{\sim} T(X_2)$$

seldom induces an equivariant derived equivalence; a notable exception is the case where the anti-invariant Picard group has rank zero or one. We only have that

$$\mathrm{Pic}(X_1)^{i_1=-1}, \quad \mathrm{Pic}(X_2)^{i_2=-1}$$

are stably equivalent – compatibly with the isomorphism on the discriminant groups of the transcendental lattices – but not necessarily isomorphic.

In light of this, we propose an orientation reversing conjugation of actions, with a view toward realizing isomorphisms of transcendental cohomology.

Assume that $\mathrm{Pic}(X_1)^{i_1=-1}$ and $\mathrm{Pic}(X_2)^{i_2=-1}$ are not isomorphic, so there is no C_2-equivariant derived equivalence

$$D^b(X_1) \xrightarrow{\sim} D^b(X_2)$$

taking i_1 to i_2, by Corollary 4.2. However, let

$$\mathrm{dual}_j : D^b(X_j) \xrightarrow{\sim} D^b(X_j), \quad j = 1, 2,$$

denote the involution

$$\mathcal{E}_* \mapsto \mathcal{E}_*^\vee.$$

Note that shift and duality commute with each other and with any automorphism of the K3 surface. The action of dual_j on the Mukai lattice $\widetilde{\mathrm{H}}(X_j, \mathbb{Z})$ is trivial in degrees 0 and 4 and multiplication by -1 in degree two. Recall that shift acts via -1 in all degrees, so composition with dual_j is trivial in degree 2 and multiplication by -1 in degrees 0 and 4.

We propose a general definition and then explain how it is related to our analysis of quadratic forms with involution:

Definition 7.1. Let (X_1, i_1) and (X_2, i_2) be smooth projective varieties with involution, of dimension n with trivial canonical class. They are *skew equivalent* if there is a kernel $\mathcal{K}$ on $X_1 \times X_2$, inducing an equivalence between X_1 and X_2, and a quasi-isomorphism

$$(i_1^*, i_2^*)\mathcal{K} \xrightarrow{\sim} \mathcal{K}^\vee[n]. \tag{7.1}$$

Note that dualization coincides with the relative dualizing complex for both projections π_1 and π_2. The quasi-isomorphism (7.1) is involutive

$$\mathcal{K} \mapsto \mathcal{K}^\vee[n] \mapsto (\mathcal{K}^\vee[n])^\vee[n] \simeq \mathcal{K},$$

i.e., (i_1, i_2) takes $\mathcal{K}$ to the kernel inducing the inverse of $\Phi_\mathcal{K}$. Since $\mathcal{K}$ is simple and the base field is algebraically closed, the quasi-isomorphism may be normalized so this composition is the identity.

Our first property follows straight from the definition:

Proposition 7.2. *Suppose that (X_1, i_1) and (X_2, i_2) are as specified in Definition 7.1 and $\mathcal{K}$ induces a skew equivalence between them. Consider line bundles L_1 and L_2 on X_1 and X_2 that are anti-invariant under i_1 and i_2*

$$i_j^* L_j \simeq L_j^\vee.$$

Then $\mathcal{K} \otimes (L_1 \boxtimes L_2)$ also induces a skew equivalence.

Our next property makes explicit the behavior under duality:

Proposition 7.3. *Let (X_1, i_1) and (X_2, i_2) be K3 surfaces equipped with involutions. Suppose that $\mathcal{K}$ is a kernel inducing an equivalence between X_1 and X_2, with induced Fourier-Mukai transforms*

$$\Phi_\mathcal{K} : D^b(X_1) \to D^b(X_2), \quad \Psi_\mathcal{K} : D^b(X_1) \to D^b(X_2).$$

Then the following are equivalent:

- $\Phi_\mathcal{K}\,\mathrm{dual}_1\, i_1^* = \mathrm{dual}_2\, i_2^* \Phi_\mathcal{K}$*;*
- $i_1^* = \Psi_\mathcal{K} i_2^* \Phi_\mathcal{K}$*;*
- $\mathcal{K}$ *induces a skew equivalence between* (X_1, i_1) *and* (X_2, i_2)*.*

Proof. Recall the interpretations of duality and inverses of Fourier-Mukai equivalences in Section 5. Let T_j denote the shift on X_j. Applying duality gives to the first expression gives

$$T_2^{-2}\,\mathrm{dual}_2\, \Phi_{\mathcal{K}^\vee} i_1^* = \mathrm{dual}_2\, i_2^* \Phi_\mathcal{K}$$

whence

$$T_2^2 \Phi_{\mathcal{K}^\vee} i_1^* = i_2^* \Phi_\mathcal{K}. \tag{7.2}$$

This is equivalent to

$$\Psi_\mathcal{K} \Phi_{\mathcal{K}^\vee} i_1^* = \Psi_\mathcal{K} T_2^{-2} i_2^* \Phi_\mathcal{K}$$

and

$$T_2^{-2} i_1^* = \Psi_\mathcal{K} T_2^{-2} i_2^* \Phi_\mathcal{K}$$

which is the same as

$$i_1^* = \Psi_\mathcal{K} i_2^* \Phi_\mathcal{K}.$$

Now formula (7.2) is equivalent to

$$T_2^2 \Phi_{\mathcal{K}^\vee} = i_2^* \Phi_{\mathcal{K}} i_1^*$$

i.e., applying $i_1^* \times i_2^*$ transforms $\mathcal{K}$ to $\mathcal{K}^\vee[2]$. □

The second item in Proposition 7.3 immediately yields:

Corollary 7.4. *Skew equivalence is an equivalence relation on K3 surfaces with involution.*

Suppose again that X_1 and X_2 are K3 surfaces and $\mathcal{K} = \mathcal{E}[1]$ for a universal vector bundle

$$\mathcal{E} \to X_1 \times X_2$$

associated with an isomorphism $X_1 = M_v(X_2)$. Then relation (7.1) (with $n = 2$) translates into

$$i_2^* \mathcal{E}_{i_1(x_1)} \simeq (\mathcal{E}_{x_1})^\vee. \tag{7.3}$$

Theorem 7.5. *Let (X_1, i_1) and (X_2, i_2) be K3 surfaces with involutions. Then the following are equivalent*

- *(X_1, i_1) and (X_2, i_2) are skew derived equivalent;*
- *there exists an orientation-preserving equivalence of Mukai lattices*

$$\phi : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}(X_2, \mathbb{Z}),$$

satisfying

$$\phi(i_1^*(v^\vee)) = (i_2^* \phi(v))^\vee. \tag{7.4}$$

As duality and pullback commute with each other, the order of these operations in (7.4) is immaterial. Furthermore, if ϕ satisfies this relation then so does $-\phi$.

Remark 7.6. We are not asserting that each cohomological equivalence satisfying (7.4) arises from a skew equivalence. Suppose that $X_1 = X_2 = X$ with the same involution i. Consider the spherical twist associated with $\mathcal{O}_X$ with kernel $\mathcal{I}_\Delta[1]$ and cohomology matrix

$$\tau_{\mathcal{O}_X} := \begin{pmatrix} 0 & 0 & -1 \\ 0 & \mathrm{I} & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{7.5}$$

Neither $\tau_{\mathcal{O}_X}$ nor $-\tau_{\mathcal{O}_X}$ is obviously realized by a kernel with the requisite self-duality property. Of course, the identity induces a skew equivalence of (X, i) with itself! Suppose now that (X_1, i_1) and (X_2, i_2) are arbitrary K3 surfaces with involution. Given ϕ satisfying (7.4), we may pre-compose or post-compose with $\tau_{\mathcal{O}_{X_1}}$ or $\tau_{\mathcal{O}_{X_2}}$ to get another matrix with the same property.

Proof of Theorem 7.5. The forward implication is clear. Indeed, the cohomological Fourier-Mukai transform $\phi_{\mathcal{K}}$ of a skew equivalence satisfies

$$(i_1, i_2)^* \phi_{\mathcal{K}} = \phi_{\mathcal{K}^\vee}$$

but $\phi_{\mathcal{K}^\vee}$ differs from $\phi_{\mathcal{K}}$ by the involution acting via $+1$ on H^0 and H^4 and -1 on H^2. Thus

$$\phi_{\mathcal{K}} : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}(X_2, \mathbb{Z})$$

satisfies relation (7.4).

For the reverse implication, we consider the cohomological Fourier-Mukai transform

$$\phi : \widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}(X_2, \mathbb{Z}).$$

Set

$$v_0 := \phi(0, 0, 1) = (r, a\ell, s),$$

where $\ell \in \mathrm{Pic}(X_2)$ is primitive and $a \in \mathbb{N}$.

- The relation (7.4) implies that $i_2^* \ell = -\ell$, which means that $\ell^2 < 0$ if $\ell \neq 0$. (The Hodge index theorem implies that the intersection form on the anti-invariant divisors is negative definite.)
- Writing $\phi(1, 0, 0) = (r', D', s')$ we have

  $$a(\ell \cdot D') - rs' - sr' = -1$$

  whence $\gcd(r, s, a\ell \cdot D') = 1$ for some anti-invariant divisor D' on X_2. Hence $\gcd(r, s, a) = 1$ as well.
- If $\ell \neq 0$ then both r and s are nonzero as v_0 is isotropic. If $\ell = 0$ then $r = 0$ or $s = 0$ but both cannot vanish. After applying a twist $\tau_{\mathcal{O}_{X_2}}$ we may assume that $r \neq 0$.
- If $r < 0$, we may replace ϕ by $-\phi$. From now on, we therefore assume $r > 0$.

We follow §4 of [Huy08] to reduce to circumstances where the wall-crossing analysis of Section 6 may be carried out.

Case I: $\mathrm{Pic}(X_2)^{i_2=-1} = 0$

This case – with $\ell = 0$ – was addressed above.

Case II: $\mathrm{Pic}(X_2)^{i_2=-1} = 1$
Taking ℓ to be the generator, all the possible equivalences are realized with Mukai vectors

$$v_0 = (r, \ell, s), \quad \gcd(r, s) = 1.$$

Indeed, this follows from Corollary 4.6: Writing $\ell^2 = -2d$ and factoring $d = \prod_{j=1}^{m} p_j^{e_j}$ into distinct primes, we see that the automorphism group of $d(\mathbb{Z}\ell)$ is C_2^m.

Thus it suffices to consider Mukai vectors with primitive first Chern class, where wall crossing applies.

Case III: $\mathrm{Pic}(X_2)^{i_2=-1} \geq 2$

(1) Suppose that $v_0 = (r, a\ell, s)$, with $\gcd(r, a) = 1$. Then there exists a anti-invariant divisor E on X_2 such that $D = rE + a\ell$ is primitive. In particular, after tensoring by $\mathcal{O}_X(E)$ the first Chern class is primitive. However, tensoring by line bundles has no impact on μ-stability.
(2) If only $\gcd(s, a) = 1$, then after applying the twist $\tau_{\mathcal{O}_{X_2}}$ we have $\gcd(r, a) = 1$.
(3) Suppose that $\gcd(r, a) = \alpha > 1$ and write

$$v_0 = (r, a\ell, s) = (\alpha r', \alpha a'\ell, s).$$

As before, choose an anti-invariant divisor E such that $a'\ell + r'E$ is primitive. Tensoring by E gives

$$\begin{aligned} \exp(E)v_0 &= (r, a\ell + rE, \tilde{s} := s + aE \cdot \ell + r\frac{E \cdot E}{2}) \\ &= (r, \alpha(a'\ell + r'E), \tilde{s}). \end{aligned}$$

Now

$$1 = \gcd(r, a, s) = \gcd(r, a, \tilde{s}) = \gcd(\alpha, \tilde{s})$$

so we are reduced to the previous case.

To summarize, up to twists by $\mathcal{O}_{X_2}$ that have no impact on our final result, for each Mukai vector v inducing a derived equivalence we may always achieve

$$M_H^{\mu s}(X_2, v) = M_H^{\mu ss}(X_2, v)$$

for polarizations H avoiding walls. This completes Case III.

We return to the situation where $v_0 = (r, a\ell, s)$ with ℓ anti-invariant, applying the wall-crossing technique of Section 6. Consider $M^{\mu s}_H(X_2, v_0)$, a K3 surface derived-equivalent to X_2, where H is a polarization on X_2 avoiding the walls. We produce an involution j on this moduli space by composing isomorphisms

$$M^{\mu s}_H(X_2, v_0) \to M^{\mu s}_H(X_2, -v_0) \to M^{\mu s}_{i_2^*H}(X_2, v_0) \to M^{\mu s}_H(X_2, v_0),$$

where the first isomorphism is induced by duality, the second is induced by i_2, and the third is $\beta_{i_2^*H,H}$ introduced in (6.2). To see that this is an involution, observe that $\beta_{H,i_2^*H} = \beta^{-1}_{i_2^*H,H}$ and use the fact that i_2 and duality are involutive and commute with each other.

We analyze how j acts on the cohomology

$$\mathrm{H}^2(M^{\mu s}_H(X_2, v_0), \mathbb{Z}) = (v_0^\vee)^\perp/\mathbb{Z}v_0^\vee. \tag{7.6}$$

The isomorphism β allows us fix these identifications as H varies, even as we cross walls. The action of i_2 on the Mukai lattice and the associated cohomology groups is given by functoriality; recall that i_2 takes v_0 to its dual. For dualization

$$M^{\mu s}_H(X_2, v_0) \to M^{\mu s}_H(X_2, -v_0)$$

the action is

$$\begin{gathered}(v_0^\vee)^\perp/\mathbb{Z}v_0^\vee \to (v_0)^\perp/\mathbb{Z}v_0 \\ \gamma \mapsto -\gamma^\vee.\end{gathered}$$

Indeed, the construction of (7.6) in [HL18, 8.1.1] and Serre duality for K3 surfaces – modulo shift by two, the cohomology of the dual of a sheaf is the dual of its cohomology – shows this is the induced mapping. To conclude j^* acts as follows:

- multiplication by $+1$ on
$$\mathrm{H}^2(X_2, \mathbb{Z})^{i_2=1} \subset \mathrm{H}^2((M^{\mu s}_H(X_2, v_0)), \mathbb{Z});$$
- multiplication by -1 on
$$(v_0^\vee)^\perp \cap \mathrm{H}^0(X_2, \mathbb{Z}) \oplus \mathrm{H}^4(X_2, \mathbb{Z});$$
- multiplication by -1 on
$$(v_0^\vee)^\perp \cap \mathrm{H}^2(X_2, \mathbb{Z})^{i_2=-1}.$$

We follow [Huy06, p. 235]. Looking at the composed cohomological Fourier-Mukai transforms

$$\widetilde{\mathrm{H}}(X_1, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(X_2, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(M^{\mu s}_H(X_2, v_0), \mathbb{Z}),$$

which takes $(0,0,1)$ to $(0,0,1)$, the Torelli theorem guarantees that $X_1 \simeq M_H^{\mu s}(X_2, v_0)$. The function relation (7.4)for ϕ guarantees that i_1 coincides with j under this isomorphism.

Our moduli space admits a universal sheaf [Huy06, Prop. 10.20]

$$\mathcal{E} \to X_2 \times M_H^{\mu s}(X_2, v_0),$$

unique up to tensor product by line bundles on the moduli space. On the other hand,

$$(i_2, I)^*\mathcal{E}^\vee \to X_2 \times M_{i_2^*H}^{\mu s}(X_2, v_0),$$

is also a universal sheaf, as is

$$(i_2, j)^*\mathcal{E}^\vee \to X_2 \times M_H^{\mu s}(X_2, v_0).$$

Applying the isomorphism with X_1, we obtain

$$(i_2, i_1)^*\mathcal{E}^\vee \simeq \mathcal{E} \otimes L_1,$$

for some line bundle L_1 on X_1. This is equivalent to

$$(i_2, i_1)^*\mathcal{E} \simeq \mathcal{E}^\vee \otimes L_1^\vee$$

and

$$\mathcal{E} \simeq (i_2, i_1)^*\mathcal{E}^\vee \otimes i_1^* L_1^\vee$$

whence L_1 is necessarily symmetric under i_1.

Rescaling $\mathcal{E} \mapsto \mathcal{E} \otimes N_1$, for N_1 a line bundle on X_1, takes

$$L_1 \mapsto L_1 \otimes N_1 \otimes i_1^* N_1.$$

A priori, the obstruction to obtaining the relation

$$(i_2, i_1)^*\mathcal{E}^\vee \simeq \mathcal{E}$$

is a cocycle in $\mathrm{H}^2(\langle i_1 \rangle, \mathrm{Pic}(X_1))$. However, any such obstruction would be visible on cohomology and thus is precluded by the relation (7.4). □

Corollary 7.7. *Under the assumptions above, the functors* $\mathrm{dual}_1 \circ i_1$ *and* $\mathrm{dual}_2 \circ i_2$ *are* C_2*-equivariantly derived equivalent.*

This motivates the formulation of Proposition 4.5.

Remark 7.8. As we recalled in Section 3, derived equivalences respect orientations on the Mukai lattice [HMS09]. The orientation reversing conjugation violates the orientation condition, in a prescribed way. Duality is the archetypal orientation-reversing Hodge isogeny.

In Sections 8 and 11 we give examples of such equivalences.

8. Rational quotients and skew equivalence

Our first task is to give examples of skew equivalences using Theorem 7.5. We remind the reader to consult Proposition 4.5 for the relevant lattice machinery.

The simplest examples are in rank two. Take (X_1, h_1) and (X_2, h_2) to be degree-two K3 surfaces, with associated involutions i_1 and i_2, such that $T(X_1) \simeq T(X_2)$. Suppose that

$$\mathrm{Pic}(X_j)^{i_j} = \mathbb{Z}\ell_j, \quad \ell_j^2 = -d;$$

note that $\mathrm{Pic}(X_j)$ is either $\langle h_j, \ell_j \rangle$ or $\left\langle h_j, \frac{h_j+\ell_j}{2} \right\rangle$, i.e., the distinguished 2-elementary subgroup is trivial or cyclic. By Corollary 4.6, possible examples correspond to isomorphisms

$$d(\mathbb{Z}\ell_1) \simeq d(\mathbb{Z}\ell_2)$$

preserving the distinguished subgroup – a vacuous condition as the discriminant group is cyclic. Thus (X_1, i_1) and (X_2, i_2) are skew equivalent.

Remark 8.1. In many examples, $M^{\mu s}_{h_j}(X_j, v_0)$ is automatically compact for $v_0 = (r, \ell, s)$, with $r < |s|$ and $\gcd(r, s) = 1$, because h_j happens not to lie on a wall.

The next group of examples arise from nontrivial stable isomorphisms. We exhibit lattice-polarized K3 surfaces with involution (X_1, i_1) and (X_2, i_2), such that the anti-invariant Picard groups are stably equivalent but inequivalent.

Specifically, we assume X_1 and X_2 are degree two K3 surfaces with

$$\mathrm{Pic}(X_j) = \mathbb{Z}h_j \oplus A_j(-1), \quad h_j^2 = 2,$$

where the involutions fix the h_j and reverse signs on A_j's. If A_1 and A_2 are stably-equivalent, inequivalent positive definite lattices then (X_1, i_1) and (X_2, i_2) are skew equivalent.

In contrast to ordinary equivalences (see 4.2) there are anti-symplectic actions with nontrivial *skew* equivalences. The resulting quotients are rational surfaces, indeed, $\mathbb{P}^2$.

Example 8.2 (Explicit matrices)**.** The matrices, in the basis p_j, q_j, for $j = 1, 2$, are given by

$$A_1 := \begin{pmatrix} 4 & 1 \\ 1 & 12 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 6 & 1 \\ 1 & 8 \end{pmatrix}.$$

We extract a stable isomorphism

$$A_1 \oplus \mathrm{U} \simeq A_2 \oplus \mathrm{U}, \quad \mathrm{U} = \langle u_1, v_1 \rangle\,, \text{ with matrix } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

First, we give an isomorphism

$$A_1 \oplus \langle e_1 \rangle \simeq A_2 \oplus \langle e_2 \rangle\,, \quad e_1^2 = -2.$$

We put

$$p_1 \mapsto p_2 + e_2,$$

and claim that the orthogonal complements to these are equivalent indefinite lattices. Indeed,

$$p_1^\perp = \langle p_1 - 4q_1, e_1 \rangle = \begin{pmatrix} 188 & 0 \\ 0 & -2 \end{pmatrix},$$

$$\begin{aligned} (p_2 + e_2)^\perp = \langle p_2 - 6q_2, 2q_2 + e_2 \rangle &= \begin{pmatrix} 282 & -94 \\ -94 & 30 \end{pmatrix} \\ = \langle p_2 + 3e_2, 2q_2 + e_2 \rangle &= \begin{pmatrix} -12 & -4 \\ -4 & 30 \end{pmatrix} \end{aligned}$$

These are equivalent via Gaussian cycles of reduced forms

$$\begin{matrix} & 0 & & 18 & & 8 & & 4 & \\ 188 & & -2 & & 26 & & -12 & & 30 \end{matrix}$$

where the indicated basis elements are

$$p_1 - 4q_1, \quad e_1, \quad p_1 - 4q_1 - 9e_1, \quad p_1 - 4q_1 - 10e_1, \quad 2(p_1 - 4q_1) - 19e_1.$$

The composed isomorphism is

$$\begin{aligned} p_1 - 4q_1 - 10e_1 &\mapsto p_2 + 3e_2, \\ 2(p_1 - 4q_1) - 19e_1 &\mapsto 2q_2 + e_2 \\ p_1 &\mapsto p_2 + e_2 \\ e_1 &\mapsto (2q_2 + e_2) - 2(p_2 + 3e_2) = 2(q_2 - p_2) - 5e_2 \\ q_1 &\mapsto 5(p_2 - q_2) + 12e_2. \end{aligned}$$

We extend the isomorphism above where $e_i = u_i - v_i$

$$\begin{aligned} u_1 + v_1 &\mapsto u_2 + v_2 \\ u_1 - v_1 &\mapsto 2(q_2 - p_2) - 5(u_2 - v_2) \\ p_1 &\mapsto p_2 + (u_2 - v_2) \\ q_1 &\mapsto 5(p_2 - q_2) + 12(u_2 - v_2) \end{aligned}$$

whence we have

$$\begin{aligned} u_1 &\mapsto (q_2 - p_2) - 2u_2 + 3v_2 \\ v_1 &\mapsto (p_2 - q_2) + 3u_2 - 2v_2. \end{aligned}$$

9. Nikulin involutions

General properties. An involution ι on a K3 surface X over $\mathbb{C}$ preserving the symplectic form is called a *Nikulin* involution. We recall basic facts concerning such involutions, following [vGS07]:

- ι has 8 isolated fixed points;
- the (resolution of singularities) $Y \to X/\iota$ is a K3 surface fitting into a diagram

$$\begin{array}{ccc} X & \stackrel{\beta}{\leftarrow} & \widetilde{X} \\ \downarrow & & \downarrow\pi \\ X/\iota & \leftarrow & Y \end{array}$$

  where β blows up the fixed points and the vertical arrows have degree two;
- the action of ι on $\mathrm{H}^2(X, \mathbb{Z})$ is uniquely determined, and there is a decomposition

$$\mathrm{H}^2(X, \mathbb{Z}) = (\mathrm{U}^{\oplus 3})_1 \oplus (\mathrm{E}_8(-1) \oplus \mathrm{E}_8(-1))_P,$$

  where the first term is invariant and the second is a permutation module for ι;
- the invariant and the anti-invariant parts of H^2 take the form:

$$\mathrm{H}^2(X, \mathbb{Z})^{\iota=1} \simeq \mathrm{U}^3 \oplus \mathrm{E}_8(-2), \quad \mathrm{H}^2(X, \mathbb{Z})^{\iota=-1} = \mathrm{E}_8(-2)$$

Let $E_1, \ldots, E_8$ denote the exceptional divisors of β and $N_1, \ldots, N_8$ the corresponding (-2)-curves on Y. The union $\cup N_i$ is the branch locus of π so there is a divisor

$$\hat{N} = (N_1 + \cdots + N_8)/2$$

saturating $\langle N_1, \ldots, N_8 \rangle \subset \mathrm{Pic}(Y)$; the minimal primitive sublattice containing these divisors is called the *Nikulin* lattice, and is denoted by N. We have [vGS07, Prop. 1.8]

$$\begin{array}{rcl} \pi_* : \mathrm{H}^2(\widetilde{X}, \mathbb{Z}) & \to & \mathrm{H}^2(Y, \mathbb{Z}) \\ \mathrm{U}^3 \oplus \mathrm{E}_8(-1) \oplus \mathrm{E}_8(-1) \oplus \langle -1 \rangle^8 & \to & \mathrm{U}(2)^3 \oplus \mathrm{N} \oplus \mathrm{E}_8(-1) \\ (u, x, y, z) & \mapsto & (u, z, x + y) \end{array}$$

and

$$\begin{array}{rcl} \pi^* : \mathrm{H}^2(Y,\mathbb{Z}) & \to & \mathrm{H}^2(\widetilde{X},\mathbb{Z}) \\ \mathrm{U}(2)^3 \oplus \mathrm{N} \oplus \mathrm{E}_8(-1) & \to & \mathrm{U}^3 \oplus \mathrm{E}_8(-1) \oplus \mathrm{E}_8(-1) \oplus \langle -1 \rangle^8 \\ (u,n,x) & \mapsto & (2u, x, x, 2\tilde{n}) \end{array}$$

where if $n = \sum n_i N_i$ then $\tilde{n} = \sum n_i E_i$. Thus we obtain a distinguished saturated sublattice

$$\mathrm{E}_8(-2) \subset \mathrm{Pic}(X)$$

that coincides with the $\iota = -1$ piece.

Proposition 9.1. *Fix a lattice* L *containing* $\mathrm{E}_8(-2)$ *as a primitive sublattice; assume* L *arises as the Picard lattice of a projective K3 surface. Then there exists a K3 surface* X *with Nikulin involution* ι *such that*

$$\mathrm{L} = \mathrm{Pic}(X) \supset \mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

Proof. Let A denote the orthogonal complement of $\mathrm{E}_8(-2)$ in L. There is a unique involution ι on L with

$$\mathrm{L}^{\iota=1} = \mathrm{A}, \quad \mathrm{L}^{\iota=-1} = \mathrm{E}_8(-2).$$

Now ι acts trivially on $d(\mathrm{L})$ – keep in mind that $d(\mathrm{E}_8(-2))$ is a two-elementary group – so we may naturally extend ι to the full K3 lattice. (It acts trivially on $\mathrm{L}^{\perp}$.) These lattice-polarized K3 surfaces form our family.

Nikulin [Nik79a, §4] explains how to get involutions for generic K3 surfaces with lattice polarization L. Choose a surface X such that $\mathrm{Pic}(X) = \mathrm{L}$ – a very general member of the family has this property. Clearly X is projective – it admits divisors with positive self-intersection. We claim there is an ample divisor $H \in \mathrm{A}$. Indeed, the ample cone of X is characterized as the chamber of the cone of positive divisors by the group generated by reflections associated with indecomposable (-2)-classes E of positive degree [LP81]. Each (-2)-class E is perpendicular to a unique ray in

$$\mathrm{A} \otimes \mathbb{R} \cap \{ \text{ cone of positive divisors } \}$$

generated by an element $a_E \in \mathrm{A}$. Note that A cannot be contained in $E^{\perp}$ as $\mathrm{E}_8(-2)$ has no (-2)-classes. We conclude that A meets each chamber in the decomposition of the positive cone – it cannot be separated from the ample cone by any of the $\mathrm{E}^{\perp}$.

Once we have the ample cone, we can extract the automorphism group of X via the Torelli Theorem: It consists of the Hodge isometries taking the ample cone to itself. In particular, any Hodge isometry fixing H is an automorphism. Thus ι is an automorphism of X. □

Proposition 9.2. *Let* L *be an even hyperbolic lattice containing* $\mathrm{E}_8(-2)$ *as a saturated sublattice. Assume that* $d(\mathrm{L})$ *has rank at most* 11*. Then* L *is unique in its genus and the homomorphism*

$$\mathrm{O}(\mathrm{L}) \to \mathrm{O}(q_{\mathrm{L}})$$

is surjective.

The condition on the rank of $d(L)$ is satisfied for Picard lattices of K3 surfaces X. We have

$$\mathrm{Pic}(X) \subset \mathrm{U}^{\oplus 3} \oplus \mathrm{E}_8(-1)^{\oplus 2}$$

which has rank 22; $d(\mathrm{Pic}(X)) \simeq d(T(X))$ so both groups are generated by ≤ 11 elements.

Proof. We apply Proposition 2.2. For odd primes p, the conditions are easily checked as the rank r of L exceeds the rank of the p-primary part $d(\mathrm{L})$. If $r \geq 12$ then the discriminant group is generated by ≤ 10 elements and we are done. Thus we focus on the $p = 2$ case with $r = 9, 10$, or 11.

Let A denote the orthogonal complement to $\mathrm{E}_8(-2)$ in L. The overlattice

$$\mathrm{L} \supset \mathrm{A} \oplus \mathrm{E}_8(-2)$$

corresponds to an isotropic subgroup

$$\mathrm{H} \subset d(\mathrm{A}) \oplus d(\mathrm{L})$$

with respect to $q_{\mathrm{A}} \oplus q_{\mathrm{L}}$. Projection maps H injectively into each summand – we may interpret these projections as kernels of the natural maps

$$d(\mathrm{A}) \to d(\mathrm{L}), \quad d(\mathrm{E}_8(-2)) \to d(\mathrm{L}).$$

Thus H is a 2-elementary group, of rank at most three. It follows that $d(\mathrm{L})$ contains at least five copies of $\mathbb{Z}/2\mathbb{Z}$. Remark 2.3 shows this validates the hypothesis of Proposition 2.2. □

The assumption on the *rank* of the discriminant groups can be replaced by bounds on its *order* [CS99, Cor. 22, p. 395] – at least for purposes of showing there is one class in each genus.

Rank nine examples. We focus on examples with Picard rank nine, following [vGS07, Prop. 2.2] which lists the possible lattices. Suppose that $\mathrm{Pic}(X)^{\iota=1} = \mathbb{Z}f$ with $f^2 = 2d$, which is necessarily ample as there are no (-2)-classes in

$$\mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

We have the lattice

$$\Lambda := (2d) \oplus \mathrm{E}_8(-2),$$

for all d. For even d we have the index-two overlattice $\widetilde{\Lambda} \supset \Lambda$, generated by

$$\frac{f+e}{2},$$

where f is a generator of $(2d)$ and $e \in \mathrm{E}_8(-2)$ is a primitive element with

$$(e,e) = \begin{cases} -4 & \text{if } d = 4m+2 \\ -8 & \text{if } d = 4m. \end{cases}$$

We are using the fact that the lattice E_8 has primitive vectors of lengths 2 and 4. Using the shorthand

$$q(v) = q_{\mathrm{E}_8(-2)}(v) \pmod{2\mathbb{Z}},$$

elements $0 \neq v \in e_8(-2) := d(\mathrm{E}_8(-2))$ are of two types

- 120 elements v with $q(v) = 1$ ($A_1 + E_7$ type),
- 135 elements v with $q(v) = 0$ (D_8 type).

Note that $\widetilde{\Lambda}$ is the unique overlattice such that $\mathrm{E}_8(-2)$ remains saturated.

Proposition 9.3. *Let (X_1, f_1) and (X_2, f_2) be polarized K3 surfaces of degree $2d$, derived equivalent via specialization of the construction in Remark 3.1. If X_1 admits a Nikulin involution fixing f_1 then*

- *X_2 admits a Nikulin involution fixing f_2;*
- *there is an isomorphism*

$$\varphi : X_1 \xrightarrow{\sim} X_2.$$

Proof. The derived equivalence induces an isomorphism of lattices with Hodge structure

$$\mathrm{H}^2(X_1, \mathbb{Z}) \supset f_1^{\perp} \simeq f_2^{\perp} \subset \mathrm{H}^2(X_2, \mathbb{Z}),$$

which means that $f_2^{\perp} \cap \mathrm{Pic}(X_2)$ contains a sublattice isomorphic to $E_8(-2)$. Thus there exists a Hodge involution

$$\iota_2^* : \mathrm{H}^2(X_2, \mathbb{Z}) \to \mathrm{H}^2(X_2, \mathbb{Z})$$

with anti-invariant summand equal to this copy of $\mathrm{E}_8(-2)$. The Torelli Theorem – see [vGS07, Prop. 2.3] – shows that X_2 admits an involution $\iota_2 : X_2 \to X_2$.

Isomorphisms of K3 surfaces specialize in families [MM64, ch. I]. This reduces us to proving the result when the X_j have Picard rank nine, putting us in the case of Proposition 9.2. The Counting Formula of [HLOY04, §2] – using the conclusions of Proposition 9.2 – implies that all Fourier-Mukai partners of X_1 are isomorphic to X_1. □

Remark 9.4. We are *not* asserting that $\varphi^* f_2 = f_1$! Suppose that X_1 and X_2 have Picard rank nine, the minimal possible rank. Then

$$\varphi^* f_2 \equiv \alpha f_1 \pmod{\mathrm{E}_8(-2)}$$

where $\alpha \pmod{4d}$ is the corresponding solution to congruence (3.1).

Thus we obtain nontrivial derived equivalence among Nikulin surfaces even in rank nine!

Rank ten examples. Turning to rank ten, we offer a generalization of [vGS07, Prop. 2.3]:

Proposition 9.5. *Fix a rank two indefinite even lattice* A *and an even extension*

$$\mathrm{L} \supset \mathrm{A} \oplus \mathrm{E}_8(-2)$$

invariant under ι; here ι fixes A *and acts by multiplication by -1 on* $\mathrm{E}_8(-2)$. *Then there exists a K3 surface X with Nikulin involution ι such that*

$$\mathrm{A} = \mathrm{Pic}(X)^{\iota=1} \subset \mathrm{Pic}(X) = \mathrm{L} \supset \mathrm{Pic}(X)^{\iota=-1} = \mathrm{E}_8(-2).$$

Proof. The lattice L embeds uniquely into the K3 lattice by Proposition 2.4. Proposition 9.1 gives the desired K3 surface with involution. □

We observed in Proposition 9.2 that the lattices L are unique in their genus and admit automorphisms realizing the full group $\mathrm{O}(d(\mathrm{L}))$. Repeating the reasoning for Proposition 9.3 we find:

Proposition 9.6. *A K3 surface X with involution ι_1, produced following Proposition 9.5 applied to* A_1, *will have a second involution ι_2 associated with* A_2. *Moreover (X, ι_1) and (X, ι_2) are not equivariantly derived equivalent.*

We elaborate on the overlattices L arising in the assumptions of Proposition 9.5. What lattices may arise from a given A? Each L arises from a 2-elementary

$$\mathrm{H} \subset d(\mathrm{A}) \oplus e_8(-2)$$

isotropic with respect to $q_{\mathrm{A}} \oplus q_{\mathrm{E}_8(-2)}$.

We consider the orbits of

$$\mathrm{H} \simeq (\mathbb{Z}/2\mathbb{Z})^2 \subset e_8(-2)$$

under automorphisms of the lattice. These reflect possible quadratic forms on $(\mathbb{Z}/2\mathbb{Z})^2$. We enumerate the possibilities, relying on description of maximal subgroups of the simple group of $\mathrm{O}_8^+(2)$ (automorphisms of $e_8(-2)$) [CCN+85, p. 85] and subgroups of $W(\mathrm{E}_8)$ (a closely related group) associated with reflections [DPR13, Table 5]. For the reader's reference, we list the root systems associated with the subgroups in parentheses:

(1) isotropic subspaces, where $q|\mathrm{H}$ is trivial – 1575 elements (D_4+D_4 type);
(2) rank one subspaces, where $q|\mathrm{H}$ has a kernel, e.g., $q(x, y) = x^2$ – $3780 = 28 \times 135$ elements ($\mathrm{A}_1 + \mathrm{A}_1 + \mathrm{D}_6$ type);
(3) "minus lines" full rank non-split subspaces, e.g., $q(x, y) = x^2 + xy + y^2$ – $1120 = 28 \cdot 120/3$ elements ($\mathrm{A}_2 + \mathrm{E}_6$ type);
(4) full rank split subspaces, e.g., $q(x, y) = xy$ – 4320 elements.

As a check, the Grassmannian $\mathrm{Gr}(2, 8)$ has Betti numbers

$$1 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 4 \quad 3 \quad 3 \quad 2 \quad 2 \quad 1 \quad 1$$

and thus, by the Weil conjectures, 10795 points of $\mathbb{F}_2$. Note that

$$10795 = 1575 + 3780 + 1120 + 4320.$$

What about arbitrary rank? Let A_1 and A_2 be indefinite lattices of rank $r \geq 2$ in the same genus. Consider overlattices

$$\mathrm{L}_1 \supset \mathrm{A}_1 \oplus \mathrm{E}_8(-2), \quad \mathrm{L}_2 \supset \mathrm{A}_1 \oplus \mathrm{E}_8(-2)$$

associated with subspaces $\mathrm{H} \subset e_8(-2)$ in the same orbit, so we have $d(\mathrm{L}_1) \simeq d(\mathrm{L}_2)$. It follows that $\mathrm{L}_1 \simeq \mathrm{L}_2$ provided the $d(\mathrm{L}_i)$ have rank at most 11 (see Proposition 9.2); this holds for Picard lattices of $K3$ surfaces. Assuming L_1 and L_2 arise as Picard lattices of K3 surfaces, we obtain results as in Propositions 9.3 and 9.6.

We conclude with one last observation:

Proposition 9.7. *The existence of a Nikulin structure for one member of a derived equivalence class induces Nikulin structures on all K3 surfaces in the equivalence class.*

Suppose X_1 and X_2 are derived equivalent and X_1 admits a Nikulin involution. Proposition 9.2 implies

$$\mathrm{Pic}(X_1) \simeq \mathrm{Pic}(X_2)$$

and we obtain a copy of $\mathrm{E}_8(-2) \subset \mathrm{Pic}(X_2)$. Proposition 9.1 guarantees X_2 admits a Nikulin involution as well.

10. Geometric application

In this section, we present a geometric application of the study of Nikulin involutions, up to derived equivalence.

Let (X_1, f_1) and (X_2, f_2) denote derived equivalent K3 surfaces of degree 12, admitting Nikulin involutions $\iota_j : X_j \to X_j$ with $\iota_j^* f_j = f_j$ for $j = 1, 2$. We assume Picard groups

$$\mathrm{Pic}(X_j) = \mathbb{Z}f_j \oplus \mathrm{E}_8(-2).$$

Note that the derived equivalence induces natural identifications between the $\mathrm{E}_8(-2)$ summands of $\mathrm{Pic}(X_1)$ and $\mathrm{Pic}(X_2)$. In particular, we obtain bijections between the fixed-point loci

$$X_1^{\iota_1} = X_2^{\iota_2}.$$

Let $Z_j \subset X_j$ denote triples of fixed points compatible with these bijections. Assuming the X_j are generic, i.e. defined by quadratic equations in $\mathbb{P}^7$, the fixed points are not collinear.

Projection from the Z_j gives surfaces

$$\mathrm{Bl}_{Z_j}(X_j) \to Y_j \subset \mathbb{P}^4$$

where the blowup normalizes the image of the projection. These constructions are compatible with the involutions on each side.

We claim that the construction of [HL18] gives a Cremona transform

$$\phi : \mathbb{P}^4 \overset{\sim}{\dashrightarrow} \mathbb{P}^4$$

such that

- the indeterminacy of ϕ is Y_1;
- the indeterminacy of ϕ^{-1} is Y_2;
- ϕ is compatible with the involutions ι_1 and ι_2 induced in the $\mathbb{P}^4$'s.

Indeed, the construction induces an isogeny of $\mathrm{H}^2(X_1,\mathbb{Z})$ and $\mathrm{H}^2(X_2,\mathbb{Z})$ induced by ϕ, restricting to an isomorphism of the primitive cohomology

$$f_1^{\perp} \xrightarrow{\sim} f_2^{\perp}.$$

The construction entails designating projection loci $Z_j' \in X_j^{[3]}$ compatible with the associated

$$X_1^{[3]} \stackrel{\sim}{\dashrightarrow} X_2^{[3]},$$

our stipulation that the Z_j consist of suitable fixed points gives compatible projection loci.

Suppose that $\phi : \mathbb{P}^n \stackrel{\sim}{\dashrightarrow} \mathbb{P}^n$ is birational and equivariant for the action of a finite group G. In this case, [KT22, Thm. 1] introduces a well-defined invariant

(10.1)
$$C_G(\phi) := \sum_{\substack{E \in \mathrm{Ex}_G(\phi^{-1}) \\ \mathrm{gen.stab}(E)=\{1\}}} [E \circlearrowleft G] - \sum_{\substack{D \in \mathrm{Ex}_G(\phi) \\ \mathrm{gen.stab}(D)=\{1\}}} [D \circlearrowleft G] \in \mathbb{Z}[\mathrm{Bir}_{G,n-1}],$$

taking values in the free abelian group on G-birational isomorphism classes of algebraic varieties of dimension $n-1$. In this case, the terms are the projectivized normal bundles of Y_1 and Y_2, taken with opposite signs. It is worth mentioning that the underlying K3 surfaces X_1 and X_2 are isomorphic by Proposition 9.3, and the group actions are conjugate under derived equivalences but not under automorphisms. The difference of classes of exceptional loci in (10.1) is *nonzero* due to Proposition 10.1 below. This gives an instance where the refinement of the invariant $c(\phi)$ in [LSZ23], [LS24] using group actions yields new information.

Proposition 10.1 (cf. Thm. 2, [LS10])**.** *Let X_1 and X_2 be smooth projective G-varieties that are not uniruled. Then any G-equivariant stable birational equivalence*

$$X_1 \times \mathbb{P}^r \stackrel{\sim}{\dashrightarrow} X_2 \times \mathbb{P}^s,$$

with trivial G-action on the second factors, arises from a G-equivariant birational equivalence

$$X_1 \stackrel{\sim}{\dashrightarrow} X_2.$$

Proof. Our assumption – that X_1 and X_2 are not uniruled – means that

$$X_1 \times \mathbb{P}^r \to X_1, \quad X_2 \times \mathbb{P}^s \to X_2$$

are maximal rationally-connected (MRC) fibrations. Since $X_1 \times \mathbb{P}^r \stackrel{\sim}{\dashrightarrow} X_2 \times \mathbb{P}^s$, the functoriality of MRC fibrations [Kol96, IV.5.5] gives a natural birational map

$$X_1 \stackrel{\sim}{\dashrightarrow} X_2.$$

When the varieties admit G-actions, the induced birational map is compatible with these actions. □

11. Enriques involutions

Let S be an Enriques surface over $\mathbb{C}$. Its universal cover is a K3 surface X with covering involution $\epsilon : X \to X$, a fixed-point-free automorphism of order two, called an *Enriques involution*.

The classification of Enriques surfaces S up to derived equivalence boils down to the classification of pairs (X, ϵ) up to C_2-equivariant derived equivalence [BM01, §6] (and [BM17] more generally). Derived equivalent Enriques surfaces are isomorphic [BM01, Prop. 6.1].

A number of authors have classified Enriques involutions on a given K3 surface X, modulo its automorphisms $\mathrm{Aut}(X)$:

- Dolgachev [Dol84] gave the first examples with finite $\mathrm{Aut}(S)$; Kondo [Kon92] offered examples of other types. See the Bibliographic Notes of [DKo23, Ch. 8] for more history, including early contributions by Fano.
- Ohashi showed that there finitely many $\mathrm{Aut}(X)$-orbits of such involutions. In the Kummer case, the possible quotients are classified by nontrivial elements of the discriminant group of the Néron-Severi group $\mathrm{NS}(X)$. There are 15 on general Kummer surfaces of product type, 31 in a general Jacobian Kummer surface, but the number is generally unbounded [Oha07], [Oha09].
- Shimada and Veniani consider *singular* (i.e. rank 20) K3 surfaces; one of their results is a parametrization of $\mathrm{Aut}(X)$-orbits on the set of Enriques involutions; the number of such orbits depends only on the genus of the transcendental lattice $T(X)$ [SV20, Thm. 3.19].

These results are based on lattice theory: two Enriques involutions on a K3 surface X are conjugate via $\mathrm{Aut}(X)$ if an only if the corresponding Enriques quotients are isomorphic [Oha07, Prop. 2.1].

Let

$$\mathrm{M} := \mathrm{U} \oplus \mathrm{E}_8(-1)$$

be the unique even unimodular hyperbolic lattice of rank 10; we have

$$\mathrm{Pic}(S)/\mathrm{torsion} \simeq \mathrm{M}$$

and

$$\mathrm{Pic}(X) \supseteq \mathrm{M}(2)$$

as a primitive sublattice. This coincides with the invariant sublattice

$$\mathrm{Pic}(X)^{\epsilon=1} \subset \mathrm{Pic}(X)$$

under the involution ϵ. Let N denote the orthogonal complement to M in $\mathrm{H}^2(X,\mathbb{Z})$, which coincides with $\mathrm{H}^2(X,\mathbb{Z})^{\epsilon=-1}$; note that $T(X) \subset \mathrm{N}$. We have

$$\mathrm{N} \simeq \mathrm{U} \oplus \mathrm{U}(2) \oplus \mathrm{E}_8(-2)$$

which has signature $(2,10)$. Thus

$$\mathrm{Pic}(X)^{\epsilon=-1} = T(X)^{\perp} \subset \mathrm{N}$$

has negative definite intersection form. The following result gives a criterion for the existence of Enriques involutions [Keu16, Thm. 1], [Oha07, Thm. 2.2], [SV20, Thm. 3.1.1]:

Proposition 11.1. *Let X be a K3 surface. Enriques involutions on X correspond to the following data: Primitive embeddings*

$$T(X) \subset \mathrm{N} \subset \mathrm{H}^2(X,\mathbb{Z})$$

such that the orthogonal complement to $T(X)$ in N *does not contain (-2)-classes.*

In particular, let X be a K3 surface with an Enriques involution. Then:

- $\mathrm{rk}\,\mathrm{Pic}(X) \geq 10$,
- if $\mathrm{rk}\,\mathrm{Pic}(X) = 10$ then there is a unique such involution,
- if $\mathrm{rk}\,\mathrm{Pic}(X) = 11$ then $\mathrm{Pic}(X)$ is isomorphic to [Oha07, Prop. 3.5]
  - $\mathrm{U}(2) \oplus \mathrm{E}_8 \oplus \langle -2n \rangle$, $n \geq 2$, or
  - $\mathrm{U} \oplus \mathrm{E}_8(2) \oplus \langle -4n \rangle$, $n \geq 1$.

Proposition 11.2. *Let X and Y be derived equivalent K3 surfaces. Assume that X admits an Enriques involution. Then X is isomorphic to Y. In particular, the existence of an Enriques involution is a derived invariant.*

Proof. In Picard rank ≥ 12, derived equivalence implies isomorphism. If X and Y and derived equivalent of rank 10 and X admits an Enriques involution, then $T(X) \simeq T(Y)$ and $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ are *stably isomorphic*. In Picard ranks 10 and 11, it suffices to show that the lattice $\mathrm{M}(2)$ is unique in its genus and all automorphisms of the discriminant group $(d(\mathrm{M}(2)), q_{\mathrm{M}(2)}))$ lift to automorphisms of $\mathrm{M}(2)$. This is implied by [Nik79b, Thm. 1.14.2]. Indeed, [SV20, Lem. 3.1.7] shows that $\mathrm{Pic}(X)$ satisfies these two conditions whenever X admits an Enriques involution. □

Corollary 4.2 implies (cf. [BM01, §6]):

Proposition 11.3. *Any C_2-equivariant derived autoequivalence*

$$(X, \epsilon_1) \sim (X, \epsilon_2)$$

arises from an automorphism of X.

We observe a corollary of Proposition 2.2: Let (X_1, ϵ_1) and (X_2, ϵ_2) denote K3 surfaces with Enriques involutions. They are orientation reversing (i.e. skew) conjugate if

- $\tau : T(X_1) \stackrel{\sim}{\to} T(X_2)$ as lattices, with compatible Hodge structures;
- $\mathrm{Pic}(X_1)^{\epsilon_1=-1}$ and $\mathrm{Pic}(X_2)^{\epsilon_2=-1}$ have the same discriminant quadratic form.

We explore this in more detail in the case of singular (rank 20) K3 surfaces. The existence of involutions on singular K3 surfaces is governed by:

Proposition 11.4. [Ser05] *Let X be a singular K3 surface with transcendental lattice $T(X)$ of discriminant d. There is no Enriques involution on X if and only if $d \equiv 3 \pmod 8$ or*

$$T(X) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \text{ or } \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}.$$

The "most algebraic example", i.e. the smallest discriminant admitting an Enriques involution, has

$$T(X) \simeq \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

In this situation there are two possibilities. We write the maximal sublattices

$$\mathrm{N} \subset \mathrm{Pic}(X)$$

such that the involution ϵ acts via -1.

We follow the notation [SV20, Table 3.1]. We consider lattices

$$N^{144}_{10,7}(2), \quad N^{242}_{10,7}(2)$$

where

$$N^{242}_{10,7}(-1) \simeq \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \oplus \mathrm{E}_8$$

with E_8 positive definite and

$$N^{144}_{10,7}(2)(-1) \simeq \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

According to `magma`, these two lattices are inequivalent but are in the same spinor genus thus are stably equivalent.

These involutions are not derived equivalent. Indeed, passing to Mukai lattices adds a hyperbolic summand U on which the involution acts trivially. However, in the case at hand we are stabilizing the (-1)-eigenspace. Thus these involutions are "skew equivalent" in the sense of Section 7.

12. Postscript on involutions in higher dimensions

There are many papers addressing the structure of involutions of higher-dimensional irreducible holomorphic symplectic varieties.

- Symplectic involutions of varieties of $K3^{[n]}$-type and their fixed loci are classified in [KMO22].
- For varieties of Kummer type – arising from an abelian surface A – involutions associated with ± 1 on A are analyzed in [HT13, Th. 4.4] and [KMO22, Th. 1.3].
- Anti-symplectic involutions on varieties of $K3^{[n]}$-type of degree two are studied in [FMOS22].
- Higher-dimensional analogs of Enriques involutions are studied in [OS11].

- Involutions on cubic fourfolds – both symplectic (see [LZ22] and [HT10]) and anti-symplectic – are studied in [Mar23]. The corresponding actions on lattices are described explicitly.
- Involutions on O'Grady type examples are considered in [MM22].

It is natural to consider whether derived equivalences of involutions on K3 surfaces X_1 and X_2 may be understood via equivalences of the induced involutions on punctual Hilbert schemes and other moduli spaces.

References

[AE22] Valery Alexeev and Philip Engel. Mirror symmetric compactifications of moduli spaces of K3 surfaces with a nonsymplectic involution, 2022. arXiv:2208.10383.

[Ale22] Valery Alexeev. Coxeter diagrams of 2-elementary K3 surfaces of genus 0, 2022. arXiv:2209.09110.

[AN06] Valery Alexeev and Viacheslav V. Nikulin. *Del Pezzo and $K3$ surfaces*, volume 15 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2006.

[BB17] Arend Bayer and Tom Bridgeland. Derived automorphism groups of K3 surfaces of Picard rank 1. *Duke Math. J.*, 166(1):75–124, 2017.

[BBHR97] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez. A Fourier-Mukai transform for stable bundles on $K3$ surfaces. *J. Reine Angew. Math.*, 486:1–16, 1997.

[BH23] Simon Brandhorst and Tommy Hofmann. Finite subgroups of automorphisms of K3 surfaces. *Forum Math. Sigma*, 11:Paper No. e54, 57, 2023.

[BM01] Tom Bridgeland and Antony Maciocia. Complex surfaces with equivalent derived categories. *Math. Z.*, 236(4):677–697, 2001.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.

[BM17] Tom Bridgeland and Antony Maciocia. Fourier-Mukai transforms for quotient varieties. *J. Geom. Phys.*, 122:119–127, 2017.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. $\mathbb{ATLAS}$ *of finite groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[DKo23] Igor Dolgachev and Shigeyuki Kondo. Enriques Surfaces II, 2023. draft book available at `https://dept.math.lsa.umich.edu/~idolga/EnriquesTwo.pdf`.

[Dol84] I. Dolgachev. On automorphisms of Enriques surfaces. *Invent. Math.*, 76(1):163–177, 1984.

[DPR13] J. Matthew Douglass, Götz Pfeiffer, and Gerhard Röhrle. On reflection subgroups of finite Coxeter groups. *Comm. Algebra*, 41(7):2574–2592, 2013.

[FMOS22] Laure Flapan, Emanuele Macrì, Kieran G. O'Grady, and Giulia Saccà. The geometry of antisymplectic involutions, I. *Math. Z.*, 300(4):3457–3495, 2022.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[HL18] Brendan Hassett and Kuan-Wen Lai. Cremona transformations and derived equivalences of K3 surfaces. *Compos. Math.*, 154(7):1508–1533, 2018.

[HLOY04] Shinobu Hosono, Bong H. Lian, Keiji Oguiso, and Shing-Tung Yau. Fourier-Mukai number of a K3 surface. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 177–192. Amer. Math. Soc., Providence, RI, 2004.

[HMS09] Daniel Huybrechts, Emanuele Macrì, and Paolo Stellari. Derived equivalences of $K3$ surfaces and orientation. *Duke Math. J.*, 149(3):461–507, 2009.

[HT10] Brendan Hassett and Yuri Tschinkel. Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces. *J. Inst. Math. Jussieu*, 9(1):125–153, 2010.

[HT13] Brendan Hassett and Yuri Tschinkel. Hodge theory and Lagrangian planes on generalized Kummer fourfolds. *Mosc. Math. J.*, 13(1):33–56, 189, 2013.

[HT17] Brendan Hassett and Yuri Tschinkel. Rational points on K3 surfaces and derived equivalence. In *Brauer groups and obstruction problems*, volume 320 of *Progr. Math.*, pages 87–113. Birkhäuser/Springer, Cham, 2017.

[HT23] Brendan Hassett and Yuri Tschinkel. Equivariant derived equivalence and rational points on K3 surfaces. *Commun. Number Theory Phys.*, 17(2):293–312, 2023.

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[Huy08] Daniel Huybrechts. Derived and abelian equivalence of $K3$ surfaces. *J. Algebraic Geom.*, 17(2):375–400, 2008.

[Keu16] JongHae Keum. Orders of automorphisms of K3 surfaces. *Adv. Math.*, 303:39–87, 2016.

[KMO22] Ljudmila Kamenova, Giovanni Mongardi, and Alexei Oblomkov. Symplectic involutions of $K3^{[n]}$ type and Kummer n type manifolds. *Bull. Lond. Math. Soc.*, 54(3):894–909, 2022.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.

[Kon92] Shigeyuki Kondō. Automorphisms of algebraic $K3$ surfaces which act trivially on Picard groups. *J. Math. Soc. Japan*, 44(1):75–98, 1992.

[KT22] Andrew Kresch and Yuri Tschinkel. Burnside groups and orbifold invariants of birational maps, 2022. arXiv:2208.05835.

[LP81] Eduard Looijenga and Chris Peters. Torelli theorems for Kähler $K3$ surfaces. *Compositio Math.*, 42(2):145–186, 1980/81.

[LS10] Qing Liu and Julien Sebag. The Grothendieck ring of varieties and piecewise isomorphisms. *Math. Z.*, 265(2):321–342, 2010.

[LS24] Hsueh-Yung Lin and Evgeny Shinder. Motivic invariants of birational maps. *Ann. of Math. (2)*, 199(1):445–478, 2024.

[LSZ23] Hsueh-Yung Lin, Evgeny Shinder, and Susanna Zimmermann. Factorization centers in dimension 2 and the Grothendieck ring of varieties. *Algebr. Geom.*, 10(6):666–693, 2023.

[LZ22] Radu Laza and Zhiwei Zheng. Automorphisms and periods of cubic fourfolds. *Math. Z.*, 300(2):1455–1507, 2022.

[Mar23] Lisa Marquand. Cubic fourfolds with an involution. *Trans. Amer. Math. Soc.*, 376(2):1373–1406, 2023.

[MM64] T. Matsusaka and D. Mumford. Two fundamental theorems on deformations of polarized varieties. *Amer. J. Math.*, 86:668–684, 1964.

[MM22] Lisa Marquand and Stevell Muller. Classification of symplectic birational involutions of manifolds of $OG10$ type, 2022. arXiv:2206.13814.

[Muk87] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[Nik79a] V. V. Nikulin. Finite groups of automorphisms of Kählerian $K3$ surfaces. *Trudy Moskov. Mat. Obshch.*, 38:75–137, 1979.

[Nik79b] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979. English translation: Math USSR-Izv. **14** (1979), no. 1, 103–167 (1980).

[Oha07] Hisanori Ohashi. On the number of Enriques quotients of a $K3$ surface. *Publ. Res. Inst. Math. Sci.*, 43(1):181–200, 2007.

[Oha09] Hisanori Ohashi. Enriques surfaces covered by Jacobian Kummer surfaces. *Nagoya Math. J.*, 195:165–186, 2009.

[Orl97] D. O. Orlov. Equivalences of derived categories and $K3$ surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Algebraic geometry, 7.

[OS11] Keiji Oguiso and Stefan Schröer. Enriques manifolds. *J. Reine Angew. Math.*, 661:215–235, 2011.

[Ser05] Ali Sinan Sertöz. Which singular $K3$ surfaces cover an Enriques surface. *Proc. Amer. Math. Soc.*, 133(1):43–50, 2005.

[Sos10] Pawel Sosna. Derived equivalent conjugate $K3$ surfaces. *Bull. Lond. Math. Soc.*, 42(6):1065–1072, 2010.

[SV20] Ichiro Shimada and Davide Cesare Veniani. Enriques involutions on singular K3 surfaces of small discriminants. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 21:1667–1701, 2020.

[vGS07] Bert van Geemen and Alessandra Sarti. Nikulin involutions on $K3$ surfaces. *Math. Z.*, 255(4):731–753, 2007.

[Yos99] Kota Yoshioka. Some examples of isomorphisms induced by Fourier-Mukai transforms, 1999. `arXiv:math/9902105v1`.

[Zha98] D.-Q. Zhang. Quotients of $K3$ surfaces modulo involutions. *Japan. J. Math. (N.S.)*, 24(2):335–366, 1998.

Department of Mathematics, Brown University Box 1917, 151 Thayer Street, Providence, RI 02912, USA
Email address: `brendan_hassett@brown.edu`

Courant Institute, New York University, New York, NY 10012, USA
Email address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA